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Poisson distribution on a subclass of analytic functions

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Abstract

In this paper, we find a necessary and sufficient condition, an inclusion relation for the Poisson distribution series $F(m,z)$ to be in the class $\mathcal{G}(\alpha,\beta,\epsilon)$ of analytic functions with negative coefficients. Further, we consider the integral operator $G(m,z)=\int_0^z \frac{F(m,t)}{t}dt$ to be in the above class. Several corollaries and consequences of the main results are also considered.

Keywords: Analytic functions, Hadamard product, Poisson distribution series
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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{s=2}^{\infty}a_s z^s, \tag{1.1}$$

which are analytic in the open unit disk $U=\{z:|z|<1\}$ and $\mathcal{T}$ be the class of functions of the form

$$f(z)=z-\sum_{s=2}^{\infty}a_s z^s \qquad (a_s\geq 0) \tag{1.2}$$

which are analytic in U.

Let $\mathcal{G}(\alpha,\beta,\epsilon)$ denote the class of functions $f(z)\in\mathcal{T}$ which satisfy the condition

$$\operatorname{Re}\left\{\beta\frac{f\ (z)}{z}+(1-\beta)f\ '(z)+\alpha zf\ ''(z)\right\}>1-|\epsilon| \tag{1.3}$$

for some $\alpha(\alpha\geq 0)$, $-1\leq\beta\leq 0$ and $\epsilon\in\mathbb{C}$, and for all $z\in U$. This class was introduced and studied by Atshan and Kulkarni [5]. We note that the class $\mathcal{G}(\alpha,0,1-\gamma)=\mathcal{G}(\alpha,1-\gamma)$ was introduced by Altintas [2] who obtained several results concerning this class. The class $\mathcal{G}(\alpha,0,\epsilon)=\mathcal{G}(\alpha,\epsilon)$ was introduced by Srivastava and Owa [18]. For the class $\mathcal{G}(0,\beta,\epsilon)=\mathcal{G}(\beta,\epsilon)$, see [6].

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The distributions of random variables have generated a great deal of interest in recent years. Their probability density functions, in a real variable x and a complex variable z, have played an important role in statistics and probability theory. For this reason, distributions have been studied extensively. Many kinds of distributions appeared from real life situations like Binomial distribution, Poisson distribution, geometric distribution, hypergeometric distribution and negative binomial distribution.

A variable X is said to be Poisson distributed if it takes the values $0, 1, 2, 3, ...$ with probabilities e^{-m}, $m\frac{e^{-m}}{1!}$, $m^2\frac{e^{-m}}{2!}$, $m^3\frac{e^{-m}}{3!}$, ...,respectively, where m is called the parameter. Thus

$$P(X = r) = \frac{m^r e^{-m}}{r!},\ r = 0, 1, 2, 3, \cdots .$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathfrak{R}^\tau(A, B)$ if it satisfies the inequality

$$\left| \frac{f\,'(z) - 1}{(A - B)\tau - B\left[f\,'(z) - 1\right]} \right| < 1$$

where $z \in U$, $\tau \in \mathbb{C}\backslash \{0\}, -1 \leq B < A \leq 1$. The class $\mathfrak{R}^\tau(A, B)$ was introduced earlier by Dixit and Pal [10]. If we put

$$\tau = 1, A = \alpha \text{ and } B = -\alpha\ (0 < \alpha \leq 1),$$

we obtain the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left| \frac{f\,'(z) - 1}{f\,'(z) + 1} \right| < \alpha \quad (z \in U; 0 < \alpha \leq 1),$$

which was studied by Padmanabhan [16] and Caplinger and Causey [8].

Recently, Porwal [17] introduce a power series whose coefficients are probabilities of Poisson distribution

$$K(m, z) = z + \sum_{s=2}^{\infty} \frac{m^{s-1}}{(s-1)!} e^{-m} z^s, \qquad (z \in U)$$

and we note that, by ratio test the radius of convergence of above series is infinity. In [17], Porwal also defined the series

$$F(m, z) = 2z - K(m, z) = z - \sum_{s=2}^{\infty} \frac{m^{s-1}}{(s-1)!} e^{-m} z^s, \qquad (z \in U).$$

Consider the linear operator

$$\mathcal{I}(m) : \mathcal{A} \to \mathcal{A}$$

defined by

$$\mathcal{I}(m)f = K(m, z) * f(z) = z + \sum_{s=2}^{\infty} \frac{m^{s-1}}{(s-1)!} e^{-m} a_s z^s. \tag{1.4}$$

Motivated by results on connections between various subclasses of analytic univalent functions using special functions and distribution series (see for example, [1, 3, 4, 7, 9, 12, 13, 14, 15]) we determine a necessary and sufficient condition for $F(m, z)$ to be in the class $\mathcal{G}(\alpha, \beta, \epsilon)$. Furthermore, we give sufficient condition for $\mathfrak{R}^\tau(A, B) \subset \mathcal{G}(\alpha, \beta, \epsilon)$. Finally, we give conditions for the integral operator $G(m, z) = \int\limits_0^z \frac{F(m,t)}{t} dt$ to be in the class $\mathcal{G}(\alpha, \beta, \epsilon)$.

To establish our main results, we will require the following Lemmas.

Lemma 1.1. [5] A function f of the form (1.2) is in the class $\mathcal{G}(\alpha, \beta, \epsilon)$ if and only if

$$\sum_{s=2}^{\infty} \left[\beta + s(1 - \beta + \alpha s - \alpha)\right] a_s \leq |\epsilon| . \tag{1.5}$$

where $\alpha(\alpha \geq 0), -1 \leq \beta \leq 0$ and $\epsilon \in \mathbb{C}$.The result (1.5) is sharp.

Lemma 1.2. [10]If $f \in \mathfrak{R}^{\tau}(A, B)$ is of the form , then

$$|a_s| \leq (A-B)\frac{|\tau|}{s}, \qquad s \in \mathbb{N}-\{1\}. \tag{1.6}$$

The result (1.6) is sharp.

Unless otherwise mentioned, we shall assume in this paper that $\alpha(\alpha \geq 0), -1 \leq \beta \leq 0$, $\epsilon \in \mathbb{C}$ and $z \in U$.

2 A necessary and sufficient condition

Firstly, we obtain a necessary and sufficient condition for $F(m, z)$ to be in the class $\mathcal{G}(\alpha, \beta, \epsilon)$.

Theorem 2.1. If $m > 0$, then $F(m, z)$ is in the class $\mathcal{G}(\alpha, \beta, \epsilon)$ if and only if

$$\alpha m^2 + (2\alpha + 1 - \beta)m + 1 - e^{-m} \leq |\epsilon|\,, \tag{2.1}$$

Proof . Since

$$F(m, z) = z - \sum_{s=2}^{\infty} \frac{m^{s-1}}{(s-1)!} e^{-m} z^s$$

and by virtue of Lemma 1.1, it suffices to show that

$$\sum_{s=2}^{\infty} (\alpha s^2 + s(1-\beta-\alpha) + \beta)\frac{m^{s-1}}{(s-1)!} e^{-m} \leq |\epsilon|\,.$$

Let

$$\mathcal{L}_1(m, \beta, \alpha) = \sum_{s=2}^{\infty} (\alpha s^2 + s(1-\beta-\alpha) + \beta)\frac{m^{s-1}}{(s-1)!} e^{-m}.$$

Writing

$$s^2 = (s-1)(s-2) + 3(s-1) + 1$$

and

$$s = (s-1) + 1.$$

By simple computation, we get

$$\begin{aligned}
\mathcal{L}_1(m, \beta, \alpha) &= \sum_{s=2}^{\infty} \alpha(s-1)(s-2)\frac{m^{s-1}}{(s-1)!} e^{-m} \\
&\quad + (1-\beta+2\alpha)\sum_{s=2}^{\infty}(s-1)\frac{m^{s-1}}{(s-1)!} e^{-m} + \sum_{s=2}^{\infty}\frac{m^{s-1}}{(s-1)!} e^{-m} \\
&= \alpha\sum_{s=3}^{\infty}\frac{m^{s-1}}{(s-3)!} e^{-m} + (1-\beta+2\alpha)\sum_{s=2}^{\infty}\frac{m^{s-1}}{(s-2)!} e^{-m} + \sum_{s=2}^{\infty}\frac{m^{s-1}}{(s-1)!} e^{-m} \\
&= e^{-m}\left[\alpha m^2 e^m + (1-\beta+2\alpha) m e^m + (e^m - 1)\right] \\
&= \alpha m^2 + (1-\beta+2\alpha)m + (1-e^{-m}).
\end{aligned}$$

But, this last expression is bounded above by $|\epsilon|$ if and only if (2.1) is satisfied. □

Corollary 2.2. If $m > 0$, then $F(m, z)$ is in the class $\mathcal{G}(\alpha, \epsilon)$ if and only if

$$\alpha m^2 + (2\alpha + 1)m + 1 - e^{-m} \leq |\epsilon|\,, \tag{2.2}$$

Corollary 2.3. If $m > 0$, then $F(m, z)$ is in the class $\mathcal{G}(\beta, \epsilon)$ if and only if

$$(1-\beta)m + 1 - e^{-m} \leq |\epsilon|\,, \tag{2.3}$$

3 Inclusion properties

Making use of Lemma 1.2, we will give the following inclusion relation.

Theorem 3.1. Let $m > 0$. If $f \in \Re^{\tau}(A, B)$, then $\mathcal{I}(m)f \in \mathcal{G}(\alpha, \beta, \epsilon)$ if

$$(A - B)\,|\tau| \left[\alpha m + (1 - \beta)(1 - e^{-m}) + \beta \left(\frac{1}{m} - \frac{1}{m} e^{-m} - e^{-m}\right)\right] \leq |\epsilon| \tag{3.1}$$

where $\tau \in \mathbb{C}\backslash\{0\}$, $-1 \leq B < A \leq 1$.

Proof . Let $f \in \Re^{\tau}(A, B)$ then in view of Lemma 1.1, it suffices to show that

$$\sum_{s=2}^{\infty} (\alpha s^2 + s(1 - \beta - \alpha) + \beta) \frac{m^{s-1}}{(s-1)!} e^{-m} \, |a_s| \leq |\epsilon| \,.$$

Let

$$\mathcal{L}_2(m, \beta, \alpha) = \sum_{s=2}^{\infty} (\alpha s^2 + s(1 - \beta - \alpha) + \beta) \frac{m^{s-1}}{(s-1)!} e^{-m} \, |a_s| \,,$$

since $f \in \Re^{\tau}(A, B)$ by Lemma 1.2, we have

$$|a_s| \leq (A - B) \frac{|\tau|}{s}, s \in \mathbb{N} - \{1\} \,.$$

Now

$$\mathcal{L}_2(m, \beta, \alpha) \leq (A - B)\,|\tau|\, e^{-m} \left[\sum_{s=2}^{\infty} (\alpha s + (1 - \beta - \alpha) + \frac{\beta}{s}) \frac{m^{s-1}}{(s-1)!}\right].$$

Writing

$$s = (s - 1) + 1,$$

we get

$$\begin{aligned}
\mathcal{L}_2(m, \beta, \alpha) &\leq (A - B)\,|\tau|\, e^{-m} \left[\alpha \sum_{s=2}^{\infty} \frac{(s-1)m^{s-1}}{(s-1)!} + (1 - \beta) \sum_{s=2}^{\infty} \frac{m^{s-1}}{(s-1)!} + \beta \sum_{s=2}^{\infty} \frac{m^{s-1}}{s!}\right] \\
&= (A - B)\,|\tau|\, e^{-m} \left[\alpha \sum_{s=2}^{\infty} \frac{m^{s-1}}{(s-2)!} + (1 - \beta) \sum_{s=2}^{\infty} \frac{m^{s-1}}{(s-1)!} + \beta \sum_{s=2}^{\infty} \frac{m^{s-1}}{s!}\right] \\
&= (A - B)\,|\tau|\, e^{-m} \left[\alpha m e^{m} + (1 - \beta)(e^{m} - 1) + \beta \left(\frac{1}{m} e^{m} - \frac{1}{m} - 1\right)\right] \\
&= (A - B)\,|\tau| \left[\alpha m + (1 - \beta)(1 - e^{-m}) + \beta \left(\frac{1}{m} - \frac{1}{m} e^{-m} - e^{-m}\right)\right].
\end{aligned}$$

But, the last expression is bounded above by $|\epsilon|$ if and only if (3.1) is satisfied. Hence the proof is completed $\square$

Corollary 3.2. Let $m > 0$. If $f \in \Re^{\tau}(A, B)$, then $\mathcal{I}(m)f \in \mathcal{G}(\alpha, \epsilon)$ if

$$(A - B)\,|\tau| \left[\alpha m + (1 - e^{-m})\right] \leq |\epsilon| \,, \tag{3.2}$$

where $\tau \in \mathbb{C}\backslash\{0\} - 1 \leq B < A \leq 1$.

Corollary 3.3. Let $m > 0$. If $f \in \Re^{\tau}(A, B)$, then $\mathcal{I}(m)f \in \mathcal{G}(\beta, \epsilon)$ if

$$(A - B)\,|\tau| \left[(1 - \beta)(1 - e^{-m}) + \beta \left(\frac{1}{m} - \frac{1}{m} e^{-m} - e^{-m}\right)\right] \leq |\epsilon| \,, \tag{3.3}$$

where $\tau \in \mathbb{C}\backslash\{0\} - 1 \leq B < A \leq 1$.

4 An integral operator

Theorem 4.1. Let $m > 0$, then

$$G(m,z) = \int_0^z \frac{F(m,t)}{t}dt$$

belongs to the class $\mathcal{G}(\alpha,\beta,\epsilon)$ if and only if

$$\alpha m + (1-\beta)(1-e^{-m}) + \beta\left(\frac{1}{m} - \frac{1}{m}e^{-m} - e^{-m}\right) \leq |\epsilon|\,. \tag{4.1}$$

Proof . Since

$$\begin{aligned} G(m,z) &= z - \sum_{s=2}^{\infty} \frac{e^{-m}m^{s-1}}{(s-1)!}\frac{z^s}{s} \\ &= z - \sum_{s=2}^{\infty} \frac{e^{-m}m^{s-1}}{s!}z^s. \end{aligned}$$

By Lemma 1.1,we need only to show that

$$\mathcal{L}_3(m,\beta,\alpha) = \sum_{s=2}^{\infty}(\alpha s^2 + s(1-\beta-\alpha)+\beta)\frac{m^{s-1}}{s!}e^{-m} \leq |\epsilon|$$

or

$$\mathcal{L}_3(m,\beta,\alpha) = \sum_{s=2}^{\infty}(\alpha s + (1-\beta-\alpha)+\frac{\beta}{s})\frac{m^{s-1}}{(s-1)!}e^{-m} \leq |\epsilon|\,.$$

Now by writing $s = (s-1)+1$ and proceeding as in Theorem 3.1, we get

$$\mathcal{L}_3(m,\beta,\alpha) = \alpha m + (1-\beta)(1-e^{-m}) + \beta\left(\frac{1}{m} - \frac{1}{m}e^{-m} - e^{-m}\right)$$

which is bounded above by $|\epsilon|$ if and only if (4.1) holds. □

Corollary 4.2. Let $m > 0$, then $G(m,z) = \int_0^z \frac{F(m,t)}{t}dt$ belongs to the class $\mathcal{G}(\alpha,\epsilon)$ if and only if

$$\alpha m + (1-e^{-m}) \leq |\epsilon|\,. \tag{4.2}$$

Corollary 4.3. Let $m > 0$, then $G(m,z) = \int_0^z \frac{F(m,t)}{t}dt$ belongs to the class $\mathcal{G}(\beta,\epsilon)$ if and only if

$$(1-\beta)(1-e^{-m}) + \beta\left(\frac{1}{m} - \frac{1}{m}e^{-m} - e^{-m}\right) \leq |\epsilon|\,. \tag{4.3}$$

5 Conclusions

In the present paper , we find a necessary and sufficient condition and inclusion relation for Poisson distribution series to be in the class $\mathcal{G}(\alpha,\beta,\epsilon)$ of analytic functions with negative coefficients. Further, we consider an integral operator related to Poisson distribution series. Some interesting corollaries and applications of the results are also discussed.

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