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Orthonormality and best approximation to solve non-linear fractional delay differential equations

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Abstract

In this article, we present a numerical method for solving nonlinear fractional delay differential equations in Banach space using orthonormal Legendre polynomials. We exploit the orthonormality of the basis functions to simplify the construction of the operational matrices and reduce the computational cost of the numerical procedure. Moreover, we present a technique to convert nonlinear terms to a suitable form for applying in operational matrices based on orthonormality and best approximation. As a result, the method requires fewer computational operations while maintaining satisfactory accuracy. To demonstrate the effectiveness of the proposed method we present some examples.

Keywords: Orthonormal polynomials, Fractional delay differential equations, Riemann-Liouville integral, Caputo fractional derivative, operational matrices.
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1. Introduction

Fractional differential equations (FDEs) are an important subject in a wide range of sciences. In engineering applications such as control theory [2], bioengineering [8], electrochemistry [11], electromagnetic [9], and signal processing [12]. Some applications appear in medical and health sciences in [7], and for applying nonlinear fractional differential equations see [3]. F. A. Rihan et al. developed delay differential model for tumour-immune response with chemoimmunotherapy and optimal control [15].

Some researchers have recently investigated on the existence and uniqueness of fractional delay differential equations (FDDEs) such as, A. Kilbas et al. studied theory and applications of fractional differential equations [5]. Balachandran et al. used existence of solutions for fractional delay

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integro-differential equations [1]. H. Kumar Nashine et al. employed measure of noncompactness to solvability of fractional dynamic systems[6]. S. G. Samko et al. studied fractional integrals and derivatives[17]. But in many cases, it is difficult to obtain an exact solution for such equations, therefore, providing approximate solutions to such problems has been a focus of research. H. R. Khodabandehloi et al., used shifted Jacobi operational matrix to solve nonlinear fractional variable-order differential equation [4]. M. I. Syam et al. applied operational matrix to solve fractional delay differential equations [19]. F.R Savadkoohi et al. employed a fractional multi-wavelet basis to solve fractional delay differential equations [18]. In this research, we consider fractional delay differential equations in a Banach space. To approximate the solution, we use operational matrices and a suitable technique to approximate the nonlinear terms by a orthonormality and best approximation which lead to acceptable accuracy in numerical results.

The desired nonlinear fractional delay differential equation that is investigated in this research is as follows,

$$\begin{aligned} &D^{\theta}y(t) = y^{m}(t) + \beta y(\beta t) + q(t), \\ &y(0) = \gamma \end{aligned} \tag{1.1}$$

where D^{θ} is Caputo fractional derivative of order θ, with $n-1 \le \theta < n, 0 < \beta \le 1$, also $n, m \in \mathbb{N}$ and $\gamma \in \mathbb{R}$ where $\mathbb{N}$ and $\mathbb{R}$ are sets of natural and real numbers respectively. Moreover, q is a known function in $L^2[0,1]$ and y is unknown function.

2. Preliminaries and notations

We review some definitions and preliminaries concepts in fractional calculus (for further introduction see[14, 13]).

Definition 2.1. *Let $0 < \theta < 1$ and $t > 0$ then the Riemann–Liouville fractional integral of order θ is given by*

$$I^{\theta}y(t) = \begin{cases} \frac{1}{\Gamma(\theta)} \int_0^t \dfrac{y(s)}{(t-s)^{1-\theta}} ds, & 0 < \theta < 1 \\ y(t), \theta = 0 \end{cases} \tag{2.1}$$

and the Caputo fractional derivative of order θ is defined as

$$D^{\theta}y(t) = \frac{1}{\Gamma(n-\theta)} \int_0^t \frac{y^{n}(s)}{(t-s)^{1+\theta-n}} ds, \quad n-1 \le \theta < n, n \in N \tag{2.2}$$

provided that the integral is well-defined, and $\Gamma(\theta)$ is the Euler gamma function. Some important properties which will be used in this study are given as follows:

(i) $I^{\theta}y(t) = y(t) \quad if \quad \theta = 0.$

(ii) $I^{\theta}t^{v} = \dfrac{\Gamma(v+1)}{\Gamma(\theta+v+1)}t^{\theta+v}, \ \theta > 0, \ v+1 > 0.$

(iii) $I^{\theta+n}(D^{n}y(t)) = I^{\theta}y(t) - \displaystyle\sum_{j=0}^{n-1} \frac{D^{j}y(0)}{\Gamma(\theta+j+1)}t^{\theta+j}.$

(iv) $D^{\theta}t^{v} = \dfrac{\Gamma(v+1)}{\Gamma(v-\theta+1)}t^{v-\theta}, \ v > \theta.$

(v) $D^{\theta}c = 0,$ where c is a constant.

To approximate the solution of the fractional delay differential equation (1.1) in Banach space $L^2[0,1]$, we use operational matrices with Legendre orthonormal polynomials as a basis functions. The Legendre polynomials via Rodriguez formula on $[-1,1]$ are in the form,

$$\varphi_i(t) = \frac{d^i}{dt^i}[(t^2-1)^i],$$

and shifted Legendre orthonormal polynomials on $[0,1]$ can be given as,

$$\begin{aligned}
&\varphi_0(t) = 1\\
&\varphi_1(t) = \sqrt{3}(2t-1)\\
&\varphi_2(t) = \sqrt{5}(6t^2-6t+1)\\
&\varphi_3(t) = \sqrt{7}(20t^3-30t^2+12t-1)\\
&\varphi_4(t) = 3(70t^4-140t^3+90t^2-20t+1)\\
&\varphi_5(t) = \sqrt{11}(252t^5-630t^4+560t^3-210t^2+30t-1)
\end{aligned}$$

Furthermore, the orthonormal Legendre polynomials on the interval $[0,1]$ can be expressed using the binomial expansion as follows [16],

$$\varphi_i(t) = \sqrt{2i+1}\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!(k!)^2}t^k, \quad t\in[0,1] \tag{2.3}$$

3. Error estimation

In this section, we introduce two upper bounds for errors arising in the best approximation.

Lemma 1. Let $y:[0,1]\longrightarrow R$ and $y\in C^n[0,1]$ and $Q^n y$ denote the orthogonal projection of y onto X_n, then we have,

$$\|Q^n y - y\| \leq \frac{2}{4^n n!}\sup_{0\leq t\leq 1}|y^{(n)}(t)|,$$

it is evident that, if number of basis function n increases, then $Q^n y \longrightarrow y$.

Proof. Orthogonal projection operator Q^n is defined as,

$$\begin{aligned}
&Q^n : L^2[0,1] \longrightarrow X_n \qquad (3.1)\\
&(Q^n y)(t) = \sum_{i=1}^{n} c_i\varphi_i(t), \quad 0\leq t\leq 1,
\end{aligned}$$

where X_n is a finite dimension subspace of $L^2[0,1]$ and $\{\varphi_i(t)\}_{i=1}^n$ are orthonormal basis element of space X_n. Suppose that the residual function is defined as $R(t) = (Q^n y)(t) - y(t)$, by applying the inner product of residual function and basis functions equal to zero, we compute the coefficients of the approximation,

$$< R(t), \varphi_j(t) >= 0, \quad j = 1,2,\ldots,n.$$

Thus,

$$< (Q^n y)(t), \varphi_j(t) >=< y(t), \varphi_j(t) >, \quad j = 1, 2, \ldots, n \tag{3.2}$$

from (3.1) and orthonormality of functions $\{\varphi_i(t)\}_{i=1}^n$ concludes that,

$$< (Q^n y(t)), \quad \varphi_j(t) >=< \textstyle\sum_{i=1}^n c_i \varphi_i(t), \quad \varphi_j(t) >= c_j, \tag{3.3}$$

From (3.2) and (3.3) we have,

$$c_j =< y(t), \varphi_j(t) >, \quad j = 1, 2, \ldots, n. \tag{3.4}$$

Substituting (3.4) into (3.1),

$$(Q^n y)(t) = \sum_{i=1}^n < y(t), \varphi_i(t) > \varphi_i(t), \quad i = 1, 2, \ldots, n.$$

Thus, $(Q^n y)(t)$ is the best approximation of function $y(t)$.

Therefore, the error of $(Q^n y)(t)$ is less than or equal to the error of Chebyshev interpolation polynomial $T^n y(t)$ with degree n,

$$\|Q^n y - y\|^2 = \int_0^1 |Q^n y(t) - y(t)|^2 dt \leq \int_0^1 |T^n y(t) - y(t)|^2 dt,$$

also, upper bound of error of interpolation is given as,

$$|T^n y(t) - y(t)| \leq max|(t - t_1)...(t - t_n)\frac{y^{(n)}(\varepsilon)}{n!}|,$$

where $t_1, t_2, ..., t_n$ are zeros of Chebyshev polynomials in the interval $[0, 1]$. By linear transformation $t = \frac{1}{2}u + \frac{1}{2}$, concludes that,

$$= \left|\frac{1}{2}((u + 1) - (u_1 + 1))\frac{1}{2}((u + 1) - (u_2 + 1))...\frac{1}{2}((u + 1) - (u_n + 1))\right|$$

$$= \left(\frac{1}{2}\right)^n |(u - u_1)...(u - u_n)| \leq \frac{1}{2^n} \times \frac{|T_n(u)|}{2^{n-1}} = \frac{2}{4^n}.$$

Thus,

$$\|Q^n y - y\|^2 \leq \int_0^1 |T^n y(t) - y(t)|^2 dt \leq \int_0^1 \left(\frac{2}{4^n n!} \sup_{0 \leq t \leq 1} |y^{(n)}(t)|\right)^2 dt$$

$$= \left(\frac{2}{4^n n!} \sup_{0 \leq t \leq 1} |y^{(n)}(t)|\right)^2$$

equivalently

$$\|Q^n y - y\| \leq \frac{2}{4^n n!} \sup_{0 \leq t \leq 1} |y^{(n)}(t)|. \qquad \square$$

Lemma 2. Let function $y : [0, 1] \longrightarrow R$ and $y \in C^n[0, 1]$ and $Q^n y$ denote the orthogonal projection of function y onto X_n, then using the Riemann-Liouville fractional integral, the following inequality holds:

$$\|I^\theta(Q^n y) - I^\theta(y)\| \leq \frac{2}{4^k n! \Gamma(\theta + 1)} \sup_{0 \leq t \leq 1} |y^{(n)}(t)|.$$

Proof: with respect to the Riemann-Liouville fractional integral (2.1) we have,

$$|I^{\theta}(Q^n y) - I^{\theta}(y)| = |\frac{1}{\Gamma(\theta)}\int_0^t \frac{y(s)-(Q^n y)(s)}{(t-s)^{1-\theta}}ds| \le \frac{1}{\Gamma(\theta)}\int_0^t \frac{|y(s)-(Q^n y)(s)|}{|(t-s)^{1-\theta}|}ds,$$

since $0 \le s \le t \le 1$ and $(t-s)^{\theta-1} \ge 0$ for $0 < \theta \le 1$, and similar to the process of the proof of Lemma.1 we can write,

$$\begin{aligned}|I^{\theta}(Q^n y) - I^{\theta}(y)| &\le \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}|y(s)-(Q^n y)(s)|ds \\ &\le \frac{2}{4^n n!\Gamma(\theta)}\sup_{0\le t\le 1}|y^{(n)}(t)|\int_0^t (t-s)^{\theta-1}ds \\ &= \frac{2}{4^n n!\Gamma(\theta)}\sup_{0\le t\le 1}|y^{(n)}(t)|\frac{t^{\theta}}{\theta} \\ &\le \frac{2}{4^n n!\Gamma(\theta+1)}\sup_{0\le t\le 1}\left|y^{(n)}(t)\right| \end{aligned} \tag{3.5}$$

therefore, inequality (3.5) implies that,

$$\|I^{\theta}(Q^n y) - I^{\theta}(y)\| \le \frac{2}{4^n n!\Gamma(\theta+1)}\sup_{0\le t\le 1}\left|y^{(n)}(t)\right|$$

for sufficiently large n, the right-hand side tends to zero, hence

$$\|I^{\theta}(Q^n y) - I^{\theta}(y)\| \to 0.$$

As a result

$$I^{\theta}(Q^n y) \to I^{\theta}(y),$$

that is the approximation $I^{\theta}(Q^n y)$ provides a satisfactory approximation of $I^{\theta}(y)$. □

4. Using operational matrices to solve equation (1.1)

In the following discussion, we consider Eq.(1.1) in two cases $m = 1$ and $m \ge 2$, which are linear and non-linear fractional delayed differential equations, respectively.

State $m = 1$.

We approximate the solution in subspaces of $L^2[0,1]$ such as spaces X_n so that it can use by orthonormal Legendre polynomials in the interval [0,1]. This projection operator was given by (3.1), and operational matrix form of it, is given as,

$$y(t) = C_n^T \Phi_n(t) \tag{4.1}$$

where $C_n = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$, $\Phi_n = \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \vdots \\ \varphi_n(t) \end{pmatrix}$ and

$$c_i = \frac{\langle y(t), \varphi_i(t)\rangle}{\langle \varphi_i(t), \varphi_i(t)\rangle} = \int_0^1 y(t)\varphi_i(t)dt, \qquad i = 1, 2, ..., n \tag{4.2}$$

also,

$$y(0^+) = \sum_{i=1}^{n} c_{0i}\varphi_i(t) = (c_{01}, c_{02}, \cdots, c_{0n})\Phi_n(t) = C_{0n}^T \Phi_n(t) \tag{4.3}$$

$$c_{0i} = y(0^+)\int_0^1 \varphi_i(t)dt, \qquad i = 1, 2, ..., n.$$

Moreover, to approximate $y(\beta t)$ we use orthonormality of basis elements $\{\varphi_i(\beta t)\}_{i=1}^n$ on interval $[0, \frac{1}{\beta}]$,

$$y(\beta t) = \sum_{i=1}^{n} c_{i\beta}\varphi_i(\beta t) = C_{n\beta}^T \Phi_n(\beta t) \tag{4.4}$$

where the coefficients of the above relation are as follows,

$$c_{i\beta} = \frac{\langle y(\beta t), \varphi_i(\beta t)\rangle}{\langle \varphi_i(\beta t), \varphi_i(\beta t)\rangle} = \frac{\int_0^{\frac{1}{\beta}} y(\beta t)\varphi_i(\beta t)dt}{\int_0^{\frac{1}{\beta}} \varphi_i(\beta t)\varphi_i(\beta t)dt} = \frac{\frac{1}{\beta}\int_0^1 y(t)\varphi_i(t)dr}{\frac{1}{\beta}\int_0^1 \varphi_i(t)\varphi_i(t)dr} = c_i,$$

in which $c_i,\ i = 1, ..., n$ are computed by (4.2) and applying it in (4.4), concludes that,

$$y(\beta t) = C_n^T \Phi_n(\beta t). \tag{4.5}$$

In (4.5), to project $\Phi_n(\beta t)$ into X_n, we compute $\Phi_n(\beta t)$ in terms of $\Phi_n(t)$ as follows,

$\Phi_n(\beta t) =$

$$\begin{pmatrix} \varphi_1(\beta t) \\ \varphi_2(\beta t) \\ \vdots \\ \varphi_n(\beta t) \end{pmatrix} = \begin{pmatrix} \sum\limits_{i=1}^{n} e_{1i}\varphi_i(t) \\ \sum\limits_{i=1}^{n} e_{2i}\varphi_i(t) \\ \vdots \\ \sum\limits_{i=1}^{n} e_{ni}\varphi_i(t) \end{pmatrix} = \begin{pmatrix} e_{11} & \cdots & e_{1n} \\ e_{21} & \cdots & e_{2n} \\ & \vdots & \\ e_{n1} & \cdots & e_{nn} \end{pmatrix} \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \vdots \\ \varphi_n(t) \end{pmatrix} = E\Phi_n(t),$$

where,

$$E = [e_{ij}]_{i,j=1}^n, e_{ij} = \frac{\langle \varphi_j(\beta t), \varphi_i(t)\rangle}{\langle \varphi_i(t), \varphi_i(t)\rangle} = \int_0^1 \varphi_j(\beta t)\varphi_i(t)dt, \tag{4.6}$$

Also,

$$q(t) = \sum_{i=1}^{n} q_i\varphi_i(t) = Q_n^T \Phi_n(t), \text{ where, } q_i = \int_0^1 q(t)\varphi_i(t)dt. \tag{4.7}$$

By applying Riemann-Liouville integral operator to $\Phi_n(t)$, elements of operational matrix are computed in the following form,

$I^\theta \Phi_n(t) =$

$$\begin{pmatrix} I^\theta\varphi_1(t) \\ I^\theta\varphi_2(t) \\ \vdots \\ I^\theta\varphi_n(t) \end{pmatrix} = \begin{pmatrix} \sum\limits_{i=1}^{n} \delta_{1i}\varphi_i(t) \\ \vdots \\ \sum\limits_{i=1}^{n} \delta_{ni}\varphi_i(t) \end{pmatrix} = \begin{pmatrix} \delta_{11} & \cdots & \delta_{1n} \\ \delta_{21} & \cdots & \delta_{2n} \\ & \vdots & \\ \delta_{n1} & \cdots & \delta_{nn} \end{pmatrix} \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \vdots \\ \varphi_n(t) \end{pmatrix} = \Delta\Phi_n(t), \tag{4.8}$$

where $\Delta = [\delta_{ij}]_{i,j=1}^{n}$,

$$\delta_{ij} = \frac{\langle I^{\theta}\varphi_i(t), \varphi_j(t)\rangle}{\langle \varphi_j(t), \varphi_j(t)\rangle} = \frac{1}{\Gamma(\theta)} \int_0^1 \int_0^t \frac{\varphi_i(s)}{(t-s)^{1-\theta}} \varphi_j(t) ds dt.$$

$$= \int_0^1 \left(\frac{1}{\Gamma(\theta)} \int_0^t \frac{\varphi_i(s)}{(t-s)^{1-\theta}} ds \right) \varphi_j(t) dt = \int_0^1 I^{\theta}\varphi_i(t)\varphi_j(t) dt. \tag{4.9}$$

Now, instead directly evaluating the double integral in (4.9) and decrease computational operations, the orthonormality of Legendre polynomial is used in the closed form of (2.3). In other words, effecting of Riemann-Liouville integral operator on $\varphi_i(t)$ is expressed by (2.3) in this form,

$$I^{\theta}(\varphi_i(t)) = \sqrt{2i+1} \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!(k!)^2} I^{\theta}(t^k)$$

$$= \sqrt{2i+1} \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!(k!)^2} \frac{\Gamma(k+1)}{\Gamma(\theta+k+1)} t^{\theta+k}, \tag{4.10}$$

by substituting (4.10) into (4.9) and applying (2.3) again leads to,

$$\begin{aligned}
\delta_{ij} &= \int_0^1 I^{\theta}\varphi_i(t)\varphi_j(t) dt \\
&= \sqrt{2i+1} \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!(k!)^2} \frac{\Gamma(k+1)}{\Gamma(\theta+k+1)} \int_0^1 \varphi_j(t) t^{\theta+k} dt, \\
&= \sqrt{2i+1} \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!(k!)^2} \frac{\Gamma(k+1)}{\Gamma(\theta+k+1)} \times \\
&\qquad \sqrt{2j+1} \sum_{L=0}^{j} (-1)^{j+L} \frac{(j+L)!}{(j-L)!(L!)^2} \int_0^1 t^L t^{\theta+k} dt \\
&= \sqrt{2i+1} \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!(k!)^2} \frac{\Gamma(k+1)}{\Gamma(\theta+k+1)} \times \\
&\qquad \sqrt{2j+1} \sum_{L=0}^{j} (-1)^{j+L} \frac{(j+L)!}{(j-L)!(L!)^2} \frac{1}{(L+\theta+k+1)}.
\end{aligned} \tag{4.11}$$

The entries of matrix Δ in (4.11), can be computed more conveniently and with less computational operations in comparison with directly evaluating the double integral in (4.9).

Applying the Riemann-Liouville fractional integral operator to both sides of Eq. (1.1), we obtain,

$$y(t) - y(0^+) = I^{\theta}y(t) + \beta I^{\theta}y(\beta t) + I^{\theta}q(t)$$

According Eqs. (4.1), (4.3) and (4.5)-(4.7) we have,

$$C_n^T \Phi_n(t) - C_{0n}^T \Phi_n(t) = C_n^T I^\theta \Phi_n(t) + \beta C_n^T E I^\theta \Phi_n(t) + Q_n^T I^\theta \Phi_n(t),$$

therefore, from(4.8),

$$C_n^T \Phi_n(t) - C_{0n}^T \Phi_n(t) = C_n^T \Delta \Phi_n(t) + \beta C_n^T E \Delta \Phi_n(t) + Q_n^T \Delta \Phi_n(t),$$

$$(C_n^T - C_n^T \Delta - \beta C_n^T E \Delta - C_{0n}^T - Q_n^T \Delta)\Phi_n(t) = 0.$$

From the linear independence of the basis functions $\{\varphi_i(t)\}_{i=1}^n$, it follows that,

$$C_n^T - C_n^T \Delta - \beta C_n^T E \Delta = C_{0n}^T + Q_n^T \Delta.$$

Moreover, we can get an algebraic linear system as follows,

$$(I_n - \Delta^T - \beta \Delta^T E^T) C_n = C_{0n} + \Delta^T Q_n, \tag{4.12}$$

where $I_{n\times n}$ is identity matrix and arrays of C_{0n}, E and Δ are given by(4.3),(4.6) and (4.11) respectively. By solving algebraic linear system (4.12), we can find an approximate solution of linear fractional delay differential equation.

State $m \geq 2,\ m \in \mathbb{N}$.

To solve the nonlinear form of fractional delay differential Eq.(1.1), we use the operational matrix (4.1) to approximate the nonlinear term in the suitable form with the help of ortonormality and best approximation as follows,

$$y^m(t) = \left(\sum_{i=1}^n c_i \varphi_i(t)\right)^m = \sum_{j=1}^n \hat{c}_{jm} \varphi_j(t). \tag{4.13}$$

For nonlinear second term of (4.13), we project it in the generated space by $\{\varphi_j(t)\}_{j=1}^n$, as the last term in (4.13). Thus coefficients $\hat{c}_{jm}$'s can be approximated in terms of coefficients c_j's in a nonlinear form,

$$\hat{c}_{jm} = \int_0^1 (\sum_{i=1}^n c_i \varphi_i(t))^m \varphi_j(t) dt,\ j = 1, 2, ..., n. \tag{4.14}$$

Regarding (4.13) the matrix form of nonlinear term can be given as,

$$y^m(t) = \hat{C}_{nm}^T \Phi_n(t), \qquad \hat{C}_{nm}^T = (\hat{c}_{1m}, \hat{c}_{2m}, ..., \hat{c}_{nm}). \tag{4.15}$$

In the following, by applying the Rimann-Lioville integral operator on the both sides of equation (1.1) for $m \geq 2$ and using (4.1)-(4.7) and (4.15) we have,

$$y(t) - y(0^+) = I^\theta y^m(t) + \beta I^\theta y(\beta t) + I^\theta q(t),$$
$$C_n^T \Phi_n(t) - C_{0n}^T \Phi_n(t) = \hat{C}_{nm}^T I^\theta \Phi_n(t) + \beta C_n^T I^\theta \Phi_n(t) + Q_n^T I^\theta \Phi_n(t),$$

from (4.8) and linear independence of basis functions $\{\varphi_i(t)\}_{i=1}^n$ we obtain the following algebraic nonlinear system,

$$(C_n^T - \hat{C}_{nm}^T \Delta - \beta C_n^T E \Delta - C_{0n}^T - Q_n^T \Delta)\Phi_n(t) = 0,$$

$$C_n - \Delta^T \hat{C}_{nm} - \beta \Delta^T E^T C_n = C_{0n} + \Delta^T Q_n. \tag{4.16}$$

For further explanation of the vector arrays $\hat{C}^T_{n,m}$, we use (4.14) and (4.15) for $n = 2, 3$ and $m = 2, 3$.

$$
\begin{aligned}
&\hat{C}^T_{22} = (\hat{c}_{12}, \hat{c}_{22}) = (c_1^2 + c_2^2,\ 2c_1c_2),\\
&\hat{C}^T_{23} = (\hat{c}_{13}, \hat{c}_{23}) = (c_1^3 + 3c_1c_2^2,\ 3c_1^2c_2 + \frac{9}{5}c_2^3),\\
&\hat{C}^T_{32} = (\hat{c}_{12}, \hat{c}_{22}, \hat{c}_{32}) =\\
&\qquad \Big(c_1^2 + c_2^2 + c_3^2,\ 2c_1c_2 + \frac{4}{\sqrt{5}}c_2c_3,\ \frac{2}{\sqrt{5}}c_2^2 + \frac{2}{7}c_3(7c_1 + \sqrt{5}c_3)\Big), \qquad (4.17)\\
&\hat{C}^T_{33} = (\hat{c}_{13}, \hat{c}_{23}, \hat{c}_{33}) =\\
&\qquad\quad \Big(c_1^3 + 3c_1(c_2^2 + c_3^2) + (\frac{2}{7\sqrt{5}}c_3(21c_2^2 + 5c_3^2)),\\
&\qquad\quad\ \ \frac{3}{35}c_2(35c_1^2 + 21c_2^2 + 28\sqrt{5}c_1c_3 + 55c_3^2),\\
&\qquad\qquad\quad 3c_1^2c_3 + (\frac{6}{7\sqrt{5}}c_1(7c_2^2 + 5c_3^2)) + \frac{3}{7}(11c_2^2c_3 + 5c_3^3)\Big).
\end{aligned}
$$

In [10, 18], an operational matrix technique was employed to compute $y^2(t)$ as the product of a coefficient vector and $\Phi(t)$ in the form,

$$
\begin{aligned}
y^2(t) &= (C^T\Phi_n(t))(C^T\Phi_n(t))^T \qquad (4.18)\\
&= C^T\Phi_n(t)\Phi_n(t)^TC\\
&= C^T\tilde{C}\Phi_n(t) \cong \tilde{\tilde{C}}^T\Phi_n(t)
\end{aligned}
$$

where $\Phi_n(t)$ is given by (4.1) and for details see [10, 18]. For comparing the proposed method with the mentioned references, we consider $\hat{C}^T_{32}$ and $\tilde{\tilde{C}}^T$ in (4.17) and (4.19) respectively,

$$
\begin{aligned}
\tilde{\tilde{C}}^T &= \Big(c_1^2 - \sqrt{3}(c_1 - 2c_2)c_2 + \sqrt{3}c_3(c_1 - 6c_2 + 6c_3), \qquad (4.19)\\
&\frac{c_1^2}{2} + (-\frac{1}{2} + \sqrt{3})c_2^2 + \frac{1}{30}(60\sqrt{3} - 78\sqrt{5} + 5\sqrt{15})c_2c_3 + (-2\sqrt{3} + 3\sqrt{5})c_3^2\\
&+c_1((1 - \frac{1}{\sqrt{3}})c_2 + \frac{\sqrt{5}}{2}(1 - 2\sqrt{3})c_3),\\
&\frac{c_1^2}{3} + c_1(\frac{-1}{6}(-6 + \sqrt{3})c_2 + (1 + \frac{11}{6\sqrt{5}} - \sqrt{15})c_3) + \frac{1}{210}(21(-5 + 8\sqrt{3})c_2^2\\
&+28(15\sqrt{3} - 15\sqrt{5} + \sqrt{15})c_2c_3 + 5(7 - 84\sqrt{3} + 96\sqrt{5})c_3^2)\Big).
\end{aligned}
$$

As demonstrated, the approach based on the orthogonality concept is considerably easier and requires significantly fewer computational operations.

5. Numerical results

In this section the method is used for solving some numerical examples.

Example 1. Consider the nonlinear fractional delay differential equation with initial value,

$$
\begin{cases}
D^\theta y(t) = y^2(t) + \beta y(\beta t) + q(t), & 0 < t \leq 1, \quad 0 < \beta, \theta < 1\\
y(0^+) = 0,
\end{cases}
$$

where

$$q(t) = -t^{2\theta}\beta^2 - \beta^2(t\beta)^\theta + \frac{\pi\beta\theta csc(\pi\theta)}{\Gamma(1-\theta)},$$

and exact solution is $y(t) = \beta t^\theta$. To compute the approximate solution, we use algebraic non-

t	$\theta = \frac{1}{9}$ $\beta = 0.3$	$\theta = \frac{3}{4}$ $\beta = 0.1$	$\theta = \frac{7}{8}$ $\beta = 0.01$
0.0	2.4×10^{-1}	1.0×10^{-2}	4.7×10^{-4}
0.1	2.0×10^{-2}	2.1×10^{-3}	1.1×10^{-4}
0.2	9.1×10^{-3}	6.1×10^{-4}	3.1×10^{-5}
0.3	5.10×10^{-3}	1.8×10^{-3}	1.0×10^{-3}
0.4	4.03×10^{-4}	2.2×10^{-3}	1.3×10^{-4}
0.5	4.7×10^{-3}	1.9×10^{-3}	1.2×10^{-4}
0.6	6.4×10^{-3}	1.2×10^{-3}	1.0×10^{-4}
0.7	9.0×10^{-3}	2.2×10^{-4}	5.7×10^{-5}
0.8	1.2×10^{-2}	1.1×10^{-3}	5.2×10^{-6}
0.9	1.5×10^{-2}	2.7×10^{-3}	8.1×10^{-5}
1.0	1.9×10^{-2}	4.5×10^{-3}	1.7×10^{-4}

Table 1: Numerical results of example 1

linear system (4.16) by iterative Newton method in subspace X_n of $L^2[0,1]$. The absolute errors are presented in Table1, for $n = 2$ and the parameters are chosen arbitrarily as $\theta = \frac{1}{9}, \frac{3}{4}, \frac{7}{8}$ and $\beta = 0.3, 0.1, 0.01$ respectively. As a sample, for $\theta = \frac{7}{8}, \beta = 0.01$, and start point $C^T = (0,0)$, the approximate solution obtained at the second stage of Newton method is given by $y(t) = 0.000474893 + 0.00969584t$, which it's absolute errors are shown in $4th$ column of Table1.

Example 2. Consider the nonlinear fractional delay differential equation with initial value

$$\begin{cases} D^\theta y(t) = y^3(t) + \beta y(\beta t) + q(t), & 0 < t \leq 1, \quad 0 < \beta, \theta < 1 \\ y(0^+) = 0, \end{cases}$$

where

$$q(t) = -t^{6\theta}\beta^3 - \beta^2(t\beta^{2\theta}) + \frac{2t^\theta\theta\beta\Gamma(2\theta)}{\Gamma(1+\theta)},$$

the exact solution is $y(t) = \beta t^{2\theta}$. To find the approximate solution, we use algebraic nonlinear system (4.16) by iterative Newton method in subspace X_n of $L^2[0,1]$. The absolute errors are shown in Table2, for $n = 2$ and some arbitrary parameters as $\theta = \frac{2}{7}, \frac{4}{5}, \frac{8}{9}$ and $\beta = 0.2, 0.01, 0.001$ respectively. As an example, for $\theta = \frac{8}{9}, \beta = 0.001$, and the start point $C^T = (0,0)$, the approximate solution obtained at the fourth stage of Newton method is given by $y(t) = -0.000143848 + 0.00100717t$, which it's absolute errors are given in $4th$ column of Table2. Clearly, greater accuracy in the above examples can be achieved by increasing n.

6. Conclusion

We proposed a numerical method based on orthonormal Legendre polynomials and operational matrices to solve nonlinear fractional delay differential equations. By employing the orthonormality

t	$\theta = \frac{2}{7}$ $\beta = 0.2$	$\theta = \frac{4}{5}$ $\beta = 0.01$	$\theta = \frac{8}{9}$ $\beta = 0.001$
0.0	4.8×10^{-2}	1.2×10^{-3}	1.4×10^{-4}
0.1	1.0×10^{-2}	4.5×10^{-4}	5.9×10^{-5}
0.2	9.4×10^{-4}	4.6×10^{-5}	3.8×10^{-7}
0.3	3.5×10^{-3}	3.6×10^{-4}	4.0×10^{-5}
0.4	5.1×10^{-3}	5.1×10^{-4}	6.2×10^{-5}
0.5	4.9×10^{-3}	5.3×10^{-4}	6.8×10^{-5}
0.6	3.4×10^{-3}	4.2×10^{-4}	5.7×10^{-5}
0.7	8.3×10^{-4}	1.9×10^{-4}	3.0×10^{-5}
0.8	2.5×10^{-3}	1.3×10^{-4}	1.0×10^{-5}
0.9	6.6×10^{-3}	5.8×10^{-4}	6.6×10^{-5}
1.0	1.1×10^{-2}	1.1×10^{-3}	1.3×10^{-4}

Table 2: Numerical results of example 2

of the basis functions and best approximation a suitable technique for nonlinear term to reduce the computational operations was used. The numerical results are shown the acceptable accuracy. It is expected that this approach can also be extended to solve other classes of fractional differential and delay differential equations in future studies.

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