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Nonnegative solutions of discrete resonant second-order boundary value problems

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Abstract

This article examines a second-order difference equation's boundary value problem at resonance. First, we build a typical fixed point operator using a shift parameter. Next, in order to provide sufficient conditions for the existence of nonnegative solutions to the problem under consideration, we use an appropriate fixed point theorem. In conclusion, we offer a pair of instances to demonstrate the relevance of established findings.

Keywords: Discrete boundary value problem, resonance, fixed point, existence, nonnegative solution
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1 Introduction

Boundary value problems are used to model many physical and engineering systems in heat transfer to determine steady-state temperature distributions, in mechanics to analyze the bending of beams and the vibration of strings, and in fluid mechanics to study the flow between surfaces. Boundary value problems are also important in electrostatics, quantum mechanics, chemical diffusion, and structural engineering, as they help describe systems in equilibrium and accurately predict real-world behavior [9, 11, 20, 21, 22].

Numerous novel techniques have been developed by researchers to prove necessary conditions on the existence of nonnegative solutions for boundary value problems (BVPs) connected to ordinary differential equations at resonance. In order to apply an appropriate fixed point theorem, Han [13] adjusted the BVP at resonance and evaluated a regular BVP (a procedure known as shift argument). In Webb et al. [23], analogous nonresonant perturbed problems with the identical boundary conditions were considered in order to establish the existence and multiplicity of positive solutions for non-perturbed BVPs at resonance. Using the shift argument, Infante et al. [15] conducted a comprehensive analysis of BVPs associated with the Neumann boundary conditions. Standard fixed point theorems were utilized by Eloe et al. [3, 4] to derive adequate conditions on the presence of positive solutions for nonlinear BVPs at resonance. Using the shift argument, Neugebauer et al. [10] investigated the existence of solutions for the second-order BVP at resonance. According to this method, the author [1] handled a second-order differential equation with periodic boundary conditions in order to address a resonant BVP. Positive solutions for discrete second-order BVPs at resonance with this method have not, as far as we know, been investigated [2, 5, 6, 7, 14, 16, 17, 18, 19]. Inspired by these studies, we examine BVPs for second-order difference equations of the following type in this article.

$$\begin{cases} \left(\Delta^2 y\right)(\zeta-1) = f(\zeta, y(\zeta)), & \zeta \in \mathbb{N}_1^T, \\ \left(\Delta y\right)(0) = 0, \quad \left(\Delta y\right)(T) = 0, \end{cases} \tag{1.1}$$

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where $T \in \mathbb{N}_3$, $y : \mathbb{N}_0^{T+1} \to \mathbb{R}$ and $f : \mathbb{N}_1^T \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Since there are nontrivial constant solutions for the equivalent homogeneous problem

$$\begin{cases} \left(\Delta^2 y\right)(\zeta - 1) = 0, & \zeta \in \mathbb{N}_1^T, \\ (\Delta y)(0) = 0, & (\Delta y)(T) = 0, \end{cases}$$

the BVP (1.1) is considered to be at resonance. To apply the Guo–Krasnoselskii fixed point theorem, we therefore create an equivalent nonresonant BVP and modify (1.1) using a shift argument. We express this theorem as follows for convenience:

Theorem 1.1. [12] Let $Y = (Y, \|\cdot\|)$ be a Banach space and $\mathcal{M} \subset Y$ be a cone in Y. Assume Λ_1 and Λ_2 are bounded open subsets of Y with $0 \in \Lambda_1$ and $\overline{\Lambda}_1 \subset \Lambda_2$. If $\mathfrak{T} : \mathcal{M} \cap \left(\overline{\Lambda}_2 \backslash \Lambda_1\right) \to \mathcal{M}$ be a completely continuous operator such that either

1. $\|\mathfrak{T}y\| \le \|y\|$, $y \in \mathcal{M} \cap \partial\overline{\Lambda}_1$ and $\|\mathfrak{T}y\| \ge \|y\|$, $y \in \mathcal{M} \cap \partial\overline{\Lambda}_2$, or
2. $\|\mathfrak{T}y\| \ge \|y\|$, $y \in \mathcal{M} \cap \partial\overline{\Lambda}_1$ and $\|\mathfrak{T}y\| \le \|y\|$, $y \in \mathcal{M} \cap \partial\overline{\Lambda}_2$.

Then, $\mathfrak{T}$ has a fixed point in $\mathcal{M} \cap \left(\overline{\Lambda}_2 \backslash \Lambda_1\right)$.

The following discrete calculus principles [8] will be used throughout the article. Denote by $\mathbb{N}_e = \{e, e+1, e+2, \ldots\}$ and $\mathbb{N}_e^d = \{e, e+1, e+2, \ldots, d\}$ for any real numbers e and d such that $d - e \in \mathbb{N}_1$.

Definition 1.2. [8] The forward jump operator $\varsigma : \mathbb{N}_e \to \mathbb{N}_{e+1}$ is

$$\varsigma(\zeta) = \zeta + 1, \quad \zeta \in \mathbb{N}_e.$$

Definition 1.3. [8] Let $y : \mathbb{N}_a^b \to \mathbb{R}$ and $M \in \mathbb{N}_1$. The first-order forward difference of y is

$$(\Delta y)(\zeta) = y(\varsigma(\zeta)) - y(\zeta), \quad \zeta \in \mathbb{N}_a^{b-1},$$

and the M^{th}-order forward difference of y is defined recursively by

$$\left(\Delta^M y\right)(\zeta) = \left(\Delta\left(\Delta^{M-1} y\right)\right)(\zeta), \quad \zeta \in \mathbb{N}_a^{b-M}.$$

Denote by

$$\mathcal{B} = \left\{ y : \mathbb{N}_0^{T+1} \to \mathbb{R} \mid (\Delta y)(0) = 0,\ (\Delta y)(T) = 0 \right\} \subseteq \mathbb{R}^T.$$

Clearly, $\mathcal{B}$ is a Banach space equipped with the maximum norm defined by

$$\|y\| = \max_{\zeta \in \mathbb{N}_1^T} |y(\zeta)|,$$

for any $y \in \mathcal{B}$.

2 Existence of Positive Solutions

Let λ be a real number and consider the equivalent BVP

$$\begin{cases} \left(\Delta^2 y\right)(\zeta - 1) + \lambda^2 y(\zeta) = f(\zeta, y(\zeta)) + \lambda^2 y(\zeta) = g(\zeta, y(\zeta)), & \zeta \in \mathbb{N}_1^T, \\ (\Delta y)(0) = 0, & (\Delta y)(T) = 0. \end{cases} \tag{2.1}$$

Clearly, the BVP (2.1) is not at resonance because the unique solution of the corresponding homogeneous problem

$$\begin{cases} \left(\Delta^2 y\right)(\zeta - 1) + \lambda^2 y(\zeta) = 0, & \zeta \in \mathbb{N}_1^T, \\ (\Delta y)(0) = 0, & (\Delta y)(T) = 0, \end{cases}$$

is $y \equiv 0$. Throughout this section, we assume that

(A 1) $f(\zeta, y) \geq -\lambda^2 y$ for all $(\zeta, y) \in \mathbb{N}_1^T \times [0, \infty)$;
(A 2) $0 < \lambda^2 < 4\sin^2 \dfrac{\pi}{4T}$.

In Lemmas 2.1 and 2.2, we construct the Green's function associated with (2.1) and state its properties. Denote by

$$\theta = \arccos\left(1 - \frac{\lambda^2}{2}\right), \quad 0 < \theta < \frac{\pi}{2T},$$
$$m = \frac{\cos^2 \frac{2T-1}{2}\theta}{\sin\theta \sin\theta T} \quad \text{and} \quad M = \frac{\cos^2 \frac{\theta}{2}}{\sin\theta \sin\theta T}.$$

Lemma 2.1. [6] Assume $k : \mathbb{N}_1^T \to \mathbb{R}$ is a continuous function. Then, the linear boundary value problem

$$\begin{cases} \left(\Delta^2 y\right)(\zeta - 1) + \lambda^2 y(\zeta) = k(\zeta), & \zeta \in \mathbb{N}_1^T, \\ (\Delta y)(0) = 0, \quad (\Delta y)(T) = 0, \end{cases}$$

has a unique solution

$$y(\zeta) = \sum_{s=1}^{T} G(\zeta, s) k(s), \quad \zeta \in \mathbb{N}_0^{T+1},$$

where

$$G(\zeta, s) = \frac{1}{(\sin\theta)(\sin\theta T)} \begin{cases} \left(\cos\frac{2s-1}{2}\theta\right)\left(\cos\frac{2T+1-2\zeta}{2}\theta\right), & 1 \leq s \leq \zeta \leq T+1, \\ \left(\cos\frac{2\zeta-1}{2}\theta\right)\left(\cos\frac{2T+1-2s}{2}\theta\right), & 0 \leq \zeta \leq s \leq T. \end{cases} \tag{2.2}$$

Lemma 2.2. The Green's function $G(\zeta, s)$ given by (2.2) satisfies the following properties:

1. $G(\zeta, s) > 0$ for all $(\zeta, s) \in \mathbb{N}_0^{T+1} \times \mathbb{N}_1^T$;
2. $m \leq G(\zeta, s) \leq M$ for all $(\zeta, s) \in \mathbb{N}_0^{T+1} \times \mathbb{N}_1^T$.

Proof . Since $G(0, s) = G(1, s)$ and $G(T, s) = G(T+1, s)$ for all $s \in \mathbb{N}_1^T$, we consider

$$G(\zeta, s) = \frac{1}{(\sin\theta)(\sin\theta T)} \begin{cases} \left(\cos\frac{2s-1}{2}\theta\right)\left(\cos\frac{2T+1-2\zeta}{2}\theta\right), & 1 \leq s \leq \zeta \leq T, \\ \left(\cos\frac{2\zeta-1}{2}\theta\right)\left(\cos\frac{2T+1-2s}{2}\theta\right), & 1 \leq \zeta \leq s \leq T. \end{cases}$$

For $1 \leq s \leq \zeta \leq T$, we have

$$\frac{1}{2} \leq \frac{2s-1}{2} \leq \frac{2T-1}{2}, \quad \frac{1}{2} \leq \frac{2T+1-2\zeta}{2} \leq \frac{2T-1}{2}.$$

Since $0 < \theta < \frac{\pi}{2T}$, we get

$$0 < \frac{2s-1}{2}\theta < \frac{\pi}{2}, \quad 0 < \frac{2T+1-2\zeta}{2}\theta < \frac{\pi}{2}.$$

Consequently, we obtain

$$0 < \frac{\cos^2 \frac{2T-1}{2}\theta}{(\sin\theta)(\sin\theta T)} \leq G(\zeta, s) \leq \frac{\cos^2 \frac{\theta}{2}}{(\sin\theta)(\sin\theta T)}.$$

For $1 \leq \zeta \leq s \leq T$, we have

$$\frac{1}{2} \leq \frac{2\zeta-1}{2} \leq \frac{2T-1}{2}, \quad \frac{1}{2} \leq \frac{2T+1-2s}{2} \leq \frac{2T-1}{2}.$$

Since $0 < \theta < \frac{\pi}{2T}$, we get

$$0 < \frac{2\zeta-1}{2}\theta < \frac{\pi}{2}, \quad 0 < \frac{2T+1-2s}{2}\theta < \frac{\pi}{2}.$$

Consequently, we obtain

$$0 < \frac{\cos^2 \frac{2T-1}{2}\theta}{(\sin\theta)(\sin\theta T)} \leq G(\zeta, s) \leq \frac{\cos^2 \frac{\theta}{2}}{(\sin\theta)(\sin\theta T)}.$$

□

Theorem 2.3. y is a solution of (2.1) if, and only if, y is a solution of the Fredholm summation equation

$$y(\zeta) = \sum_{s=1}^{T} G(\zeta, s) g(s, y(s)), \quad \zeta \in \mathbb{N}_0^{T+1}.$$

Clearly, y is a solution of (1.1) if, and only if, y is a solution of (2.1). So, Theorem 2.3 can be restated as follows.

Theorem 2.4. y is a solution of (1.1) if, and only if, y is a solution of the Fredholm summation equation

$$y(\zeta) = \sum_{s=1}^{T} G(\zeta, s) g(s, y(s)), \quad \zeta \in \mathbb{N}_0^{T+1}.$$

Define the operator $S : \mathcal{B} \to \mathcal{B}$ by

$$(Sy)(\zeta) = \sum_{s=1}^{T} G(\zeta, s) g(s, y(s)), \quad \zeta \in \mathbb{N}_0^{T+1}.$$

It follows from Theorem 2.4 that y is a fixed point of S if, and only if, y is a solution of (1.1). Since S is a summation operator on a discrete finite set, it is trivially completely continuous. Define the cone $\mathcal{K}$ in $\mathcal{B}$ by

$$\mathcal{K} = \left\{ y \in \mathcal{B} \mid y(\zeta) \geq 0 \text{ for } \zeta \in \mathbb{N}_1^T, \text{ and } \min_{\zeta \in \mathbb{N}_1^T} y(\zeta) \geq \frac{m}{M} \|y\| \right\}.$$

Here

$$\frac{m}{M} = \frac{\cos^2 \frac{2T-1}{2}\theta}{\cos^2 \frac{\theta}{2}}.$$

Clearly,

$$0 < \frac{m}{M} < 1.$$

Lemma 2.5. $S : \mathcal{K} \to \mathcal{K}$.

Proof . The continuity of g implies the continuity of S. Let $y \in \mathcal{K}$. Then, by (A 1) and Lemma 2.2, we have $(Sy)(\zeta) \geq 0$ for all $\zeta \in \mathbb{N}_1^T$. Also,

$$\begin{aligned}
\min_{\zeta \in \mathbb{N}_1^T} (Sy)(\zeta) &= \min_{\zeta \in \mathbb{N}_1^T} \left[\sum_{s=1}^{T} G(\zeta, s) g(s, y(s)) \right] \\
&\geq \sum_{s=1}^{T} \left[\min_{\zeta \in \mathbb{N}_1^T} G(\zeta, s) \right] g(s, y(s)) \\
&= \frac{m}{M} \sum_{s=1}^{T} \left[\max_{\zeta \in \mathbb{N}_1^T} G(\zeta, s) \right] g(s, y(s)) \\
&\geq \frac{m}{M} \max_{\zeta \in \mathbb{N}_1^T} \left[\sum_{s=1}^{T} G(\zeta, s) g(s, y(s)) \right] \\
&= \frac{m}{M} \|Sy\|.
\end{aligned}$$

Therefore, $S\mathcal{K} \subset \mathcal{K}$. □

Theorem 2.6. Assume the following condition holds.

(A 3) There exist numbers $0 < r < R < \infty$ such that

(a) $g(\zeta, z) \leq \dfrac{r}{MT}, \quad (\zeta, z) \in \mathbb{N}_1^T \times [0, r];$

(b) $g(\zeta, z) \geq \dfrac{MR}{m^2T}, \quad (\zeta, z) \in \mathbb{N}_1^T \times \left[R, \dfrac{M}{m}R\right].$

Then, (1.1) has at least one nonnegative solution y such that

$$\frac{m}{M}r \leq y(\zeta) \leq \frac{M}{m}R, \quad \zeta \in \mathbb{N}_1^T. \tag{2.3}$$

Proof . Take $y \in \mathcal{K}$ with $\|y\| = r$. Then, for all for $s \in \mathbb{N}_1^T$, $0 \leq y(s) \leq r$ and

$$(Sy)(\zeta) = \sum_{s=1}^{T} G(\zeta, s) g(s, y(s)) \leq M \sum_{s=1}^{T} g(s, y(s)) \leq M \left(\frac{r}{MT}\right) T = \|y\|. \tag{2.4}$$

Now, if we let $\Lambda_1 = \{y \in \mathcal{B} \mid \|y\| < r\}$, then from (2.4), we have

$$\|Sy\| \leq \|y\|, \quad y \in \mathcal{K} \cap \partial\Lambda_1.$$

Further, let

$$R_1 = \frac{M}{m}R \quad \text{and} \quad \Lambda_2 = \{y \in \mathcal{B} \mid \|y\| < R_1\}.$$

Then, $y \in \mathcal{K}$ and $\|y\| = R_1$ implies

$$\min_{\zeta \in \mathbb{N}_1^T} y(\zeta) \geq \frac{m}{M}\|y\| = \frac{m}{M}R_1 = R,$$

and hence

$$R \leq y(\zeta) \leq \frac{M}{m}R, \quad \zeta \in \mathbb{N}_1^T.$$

Therefore, for all $\zeta \in \mathbb{N}_1^T$, we obtain

$$(Sy)(\zeta) = \sum_{s=1}^{T} G(\zeta, s) g(s, y(s)) \geq m \sum_{s=1}^{T} g(s, y(s)) \geq m \left(\frac{MR}{m^2T}\right) T = \|y\|.$$

Hence,

$$\|Sy\| \geq \|y\|, \quad y \in \mathcal{K} \cap \partial\Lambda_2.$$

Consequently, by the first part of Theorem 1.1, it follows that S has a fixed point y in $\mathcal{K} \cap \left(\overline{\Lambda}_2 \backslash \Lambda_1\right)$. We have $r \leq \|y\| \leq R_1$. Since $y \in \mathcal{K}$, we have

$$y(\zeta) \geq \frac{m}{M}\|y\|, \quad \zeta \in \mathbb{N}_1^T,$$

implying that (2.3) holds. □

Theorem 2.7. Assume the following condition holds.

(A 4) There exist numbers $0 < r < R < \infty$ such that

(a) $g(\zeta, z) \geq \dfrac{r}{mT}, \quad (\zeta, z) \in \mathbb{N}_1^T \times [0, r];$

(b) $g(\zeta, z) \leq \dfrac{R}{mT}, \quad (\zeta, z) \in \mathbb{N}_1^T \times \left[R, \dfrac{M}{m}R\right].$

Then, (1.1) has at least one nonnegative solution y such that (2.3) holds.

Proof . The proof is analogous to that of Theorem 2.6 and uses the second part of Theorem 1.1. So, we omit it. □

Theorem 2.8. Assume the following condition holds.

(A 5) There exist numbers $0 < r < p < R < \infty$ such that

(a) $g(\zeta, z) \geq \dfrac{r}{mT}, \quad (\zeta, z) \in \mathbb{N}_1^T \times [0, r]$;

(b) $g(\zeta, z) \geq \dfrac{MR}{m^2 T}, \quad (\zeta, z) \in \mathbb{N}_1^T \times \left[R, \dfrac{M}{m}R\right]$;

(c) $g(\zeta, z) < \dfrac{p}{MT}, \quad (\zeta, z) \in \mathbb{N}_1^T \times \left[\dfrac{m}{M}p, p\right]$.

Then, BVP (1.1) has at least two positive solutions y_1 and y_2 such that $\|y_1\| < p < \|y_2\|$ and

$$\frac{m}{M}r \leq y_1(\zeta) < p, \quad \frac{m}{M}p < y_2(\zeta) \leq \frac{M}{m}R, \quad \zeta \in \mathbb{N}_1^T.$$

Proof . Take $y \in \mathcal{K}$ with $\|y\| = r$. Then, for all $s \in \mathbb{N}_1^T$, $0 \leq y(s) \leq r$ and

$$(Sy)(\zeta) = \sum_{s=1}^{T} G(\zeta, s) g(s, y(s)) \geq m \sum_{s=1}^{T} g(s, y(s)) \geq m \left(\frac{r}{mT}\right) T = \|y\|.$$

Thus, we may let $\Lambda_1 = \{y \in \mathcal{B} \mid \|y\| < r\}$ so that

$$\|Sy\| \geq \|y\|, \quad y \in \mathcal{K} \cap \partial\Lambda_1.$$

Further, for $y \in \mathcal{K}$ with $\|y\| = p$, we have

$$\frac{m}{M}p \leq y(s) \leq p, \quad s \in \mathbb{N}_1^T,$$

and, therefore,

$$(Sy)(\zeta) = \sum_{s=1}^{T} G(\zeta, s) g(s, y(s)) \leq M \sum_{s=1}^{T} g(s, y(s)) < M \left(\frac{p}{MT}\right) T = \|y\|.$$

Hence, if we set $\Lambda_2 = \{y \in \mathcal{B} \mid \|y\| < p\}$, then

$$\|Sy\| < \|y\|, \quad y \in \mathcal{K} \cap \partial\Lambda_2. \tag{2.5}$$

Consequently, by the second part of Theorem 1.1, it follows that S has a fixed point y_1 in $\mathcal{K} \cap \left(\overline{\Lambda}_2 \backslash \Lambda_1\right)$. We have $r \leq \|y_1\| \leq p$. Because by (2.5) $Sy \neq y$ for $y \in \mathcal{K}$ with $\|y\| = p$, hence $r \leq \|y_1\| < p$. For $y \in \mathcal{K}$, we have

$$y(\zeta) \geq \frac{m}{M}\|y\|, \quad \zeta \in \mathbb{N}_1^T.$$

Therefore,

$$\frac{m}{M}r \leq y_1(\zeta) < p, \quad \zeta \in \mathbb{N}_1^T.$$

Let

$$R_1 = \frac{M}{m}R \text{ and } \Lambda_3 = \{y \in \mathcal{B} \mid \|y\| < R_1\}.$$

Then, for $y \in \mathcal{K}$ with $\|y\| = R_1$, we have

$$\min_{\zeta \in \mathbb{N}_1^T} y(\zeta) \geq \frac{m}{M}\|y\| = \frac{m}{M}R_1 = R.$$

Hence,

$$R \leq y(\zeta) \leq \frac{M}{m}R, \quad \zeta \in \mathbb{N}_1^T.$$

Therefore,

$$(Sy)(\zeta) = \sum_{s=1}^{T} G(\zeta, s) g(s, y(s)) \geq m \sum_{s=1}^{T} g(s, y(s)) \geq m \left(\frac{MR}{m^2 T}\right) T = \|y\|.$$

Hence,

$$\|Sy\| \geq \|y\|, \quad y \in \mathcal{K} \cap \partial\Lambda_3.$$

Thus, S has a fixed point y_2 in $\mathcal{K} \cap \left(\overline{\Lambda}_3 \backslash \Lambda_2\right)$ according to the first part of Theorem 1.1. We have $p < \|y_2\| \leq R_1$. Because for $y \in \mathcal{K}$, we have

$$y(\zeta) \geq \frac{m}{M}\|y\|, \quad \zeta \in \mathbb{N}_1^T.$$

Hence, it follows that

$$\frac{m}{M}p < y_2(\zeta) \leq \frac{M}{m}R, \quad \zeta \in \mathbb{N}_1^T.$$

□

Theorem 2.9. Assume the following condition holds.

(A 6) There exist numbers $0 < r < p < R < \infty$ such that
  (a) $g(\zeta, z) \leq \dfrac{r}{MT}, \quad (\zeta, z) \in \mathbb{N}_1^T \times [0, r];$
  (b) $g(\zeta, z) \leq \dfrac{R}{mT}, \quad (\zeta, z) \in \mathbb{N}_1^T \times \left[R, \dfrac{M}{m}R\right];$
  (c) $g(\zeta, z) > \dfrac{p}{mT}, \quad (\zeta, z) \in \mathbb{N}_1^T \times \left[\dfrac{m}{M}p, p\right].$

Then, (1.1) has at least two nonnegative solutions y_1 and y_2 such that $\|y_1\| < p < \|y_2\|$ and

$$\frac{m}{M}r \leq y_1(\zeta) < p, \quad \frac{m}{M}p < y_2(\zeta) \leq \frac{M}{m}R, \quad \zeta \in \mathbb{N}_1^T.$$

Proof . The proof is analogous to that of Theorem 2.8. So, we omit it. □

3 Using a Negative Shift Argument

Let λ be a real number and consider the equivalent BVP

$$\begin{cases} \left(\Delta^2 y\right)(\zeta - 1) - \lambda^2 y(\zeta) = f(\zeta, y(\zeta)) - \lambda^2 y(\zeta) = h(\zeta, y(\zeta)), \quad \zeta \in \mathbb{N}_1^T, \\ (\Delta y)(0) = 0, \quad (\Delta y)(T) = 0. \end{cases} \tag{3.1}$$

Clearly, the BVP (3.1) is not at resonance because the unique solution of the corresponding homogeneous problem

$$\begin{cases} \left(\Delta^2 y\right)(\zeta - 1) - \lambda^2 y(\zeta) = 0, \quad \zeta \in \mathbb{N}_1^T, \\ (\Delta y)(0) = 0, \quad (\Delta y)(T) = 0, \end{cases}$$

is $y \equiv 0$. Throughout this section, we assume that

(B 1) $f(\zeta, y) \geq \lambda^2 y$ for all $(\zeta, y) \in \mathbb{N}_1^T \times [0, \infty)$;
(B 2) $\lambda^2 > 0$.

In Lemmas 3.1 and 3.2, we construct the Green's function associated with (3.1) and state its properties. Denote by

$$\eta = \frac{1}{2}\left(\lambda^2 + 2 + \sqrt{\lambda^4 + 4\lambda^2}\right),$$

$$n = \frac{\eta^2 - 1}{(\eta^T - \eta^{-T})}\left(\eta + \eta^{1-T}\right)^2 \quad \text{and} \quad N = \frac{\eta^2 - 1}{(\eta^T - \eta^{-T})}\left(\eta^T + 1\right)^2.$$

Lemma 3.1. [6, 7] Assume $k : \mathbb{N}_1^T \to \mathbb{R}$ is a continuous function. Then, the linear boundary value problem

$$\begin{cases} \left(\Delta^2 y\right)(\zeta - 1) - \lambda^2 y(\zeta) = k(\zeta), \quad \zeta \in \mathbb{N}_1^T, \\ (\Delta y)(0) = 0, \quad (\Delta y)(T) = 0, \end{cases}$$

has a unique solution

$$y(\zeta) = \sum_{s=1}^{T} H(\zeta, s)k(s), \quad \zeta \in \mathbb{N}_0^{T+1},$$

where

$$H(\zeta, s) = \frac{\eta^2 - 1}{(\eta^T - \eta^{-T})} \begin{cases} \left(\eta^s + \eta^{1-s}\right)\left(\eta^{\zeta-T} + \eta^{T-\zeta+1}\right), & 1 \le s \le \zeta \le T+1, \\ \left(\eta^\zeta + \eta^{1-\zeta}\right)\left(\eta^{s-T} + \eta^{T-s+1}\right), & 0 \le \zeta \le s \le T. \end{cases} \tag{3.2}$$

Lemma 3.2. The Green's function $H(\zeta, s)$ given by (3.2) satisfies the following properties:

1. $H(\zeta, s) > 0$ for all $(\zeta, s) \in \mathbb{N}_0^{T+1} \times \mathbb{N}_1^T$;
2. $n \le H(\zeta, s) \le N$ for all $(\zeta, s) \in \mathbb{N}_0^{T+1} \times \mathbb{N}_1^T$.

Proof . Since $H(0, s) = H(1, s)$ and $H(T, s) = H(T + 1, s)$ for all $s \in \mathbb{N}_1^T$, we consider

$$H(\zeta, s) = \frac{\eta^2 - 1}{(\eta^T - \eta^{-T})} \begin{cases} \left(\eta^s + \eta^{1-s}\right)\left(\eta^{\zeta-T} + \eta^{T-\zeta+1}\right), & 1 \le s \le \zeta \le T, \\ \left(\eta^\zeta + \eta^{1-\zeta}\right)\left(\eta^{s-T} + \eta^{T-s+1}\right), & 1 \le \zeta \le s \le T. \end{cases} \tag{3.3}$$

For $1 \le s \le \zeta \le T$, we have

$$1 - T \le 1 - s \le 0, \quad 1 - T \le \zeta - T \le 0, \quad 1 \le T - \zeta + 1 \le T,$$

implying that

$$0 < \frac{\eta^2 - 1}{(\eta^T - \eta^{-T})}\left(\eta + \eta^{1-T}\right)^2 \le H(\zeta, s) \le \frac{\eta^2 - 1}{(\eta^T - \eta^{-T})}\left(\eta^T + 1\right)^2.$$

For $1 \le \zeta \le s \le T$, we have

$$1 - T \le 1 - \zeta \le 0, \quad 1 - T \le s - T \le 0, \quad 1 \le T - s + 1 \le T,$$

implying that

$$0 < \frac{\eta^2 - 1}{(\eta^T - \eta^{-T})}\left(\eta + \eta^{1-T}\right)^2 \le H(\zeta, s) \le \frac{\eta^2 - 1}{(\eta^T - \eta^{-T})}\left(\eta^T + 1\right)^2.$$

□

Theorem 3.3. *y is a solution of (3.1) if, and only if, y is a solution of the Fredholm summation equation*

$$y(\zeta) = \sum_{s=1}^{T} H(\zeta, s)h(s, y(s)), \quad \zeta \in \mathbb{N}_0^{T+1}.$$

Clearly, y is a solution of (1.1) if, and only if, y is a solution of (3.1). So, Theorem 3.3 can be restated as follows.

Theorem 3.4. *y is a solution of (1.1) if, and only if, y is a solution of the Fredholm summation equation*

$$y(\zeta) = \sum_{s=1}^{T} H(\zeta, s)h(s, y(s)), \quad \zeta \in \mathbb{N}_0^{T+1}.$$

Define the operator $\mathfrak{S} : \mathcal{B} \to \mathcal{B}$ by

$$\left(\mathfrak{S}y\right)(\zeta) = \sum_{s=1}^{T} H(\zeta, s)h(s, y(s)), \quad \zeta \in \mathbb{N}_0^{T+1}.$$

It follows from Theorem 3.4 that y is a fixed point of $\mathfrak{S}$ if, and only if, y is a solution of (1.1). Since $\mathfrak{S}$ is a summation operator on a discrete finite set, it is trivially completely continuous. Define the cone $\mathcal{L}$ in $\mathcal{B}$ by

$$\mathcal{L} = \left\{y \in \mathcal{B} \mid y(\zeta) \ge 0 \text{ for } \zeta \in \mathbb{N}_1^T\right\}.$$

Lemma 3.5. $\mathfrak{S} : \mathcal{L} \to \mathcal{L}$.

Proof . The continuity of h implies the continuity of $\mathfrak{S}$. Let $y \in \mathcal{L}$. Then, by (B 1) and Lemma 3.2, we have $(\mathfrak{S}y)(\zeta) \geq 0$ for all $\zeta \in \mathbb{N}_1^T$. Therefore, $\mathfrak{S}\mathcal{L} \subset \mathcal{L}$. □

Theorem 3.6. Assume the following condition holds.

(B 3) There exist numbers $0 < r < R < \infty$ such that
  (a) $h(\zeta, z) \leq \dfrac{r}{NT}, \quad (\zeta, z) \in \mathbb{N}_1^T \times [0, r]$;
  (b) $h(\zeta, z) \geq \dfrac{NR}{n^2T}, \quad (\zeta, z) \in \mathbb{N}_1^T \times \left[0, \dfrac{NR}{n}\right]$.

Then, (1.1) has at least one nonnegative solution y such that

$$r \leq y(\zeta) \leq \frac{N}{n}R, \quad \zeta \in \mathbb{N}_1^T. \tag{3.4}$$

Proof . Take $y \in \mathcal{L}$ with $\|y\| = r$. Then, for all for $s \in \mathbb{N}_1^T$, $0 \leq y(s) \leq r$ and

$$(\mathfrak{S}y)(\zeta) = \sum_{s=1}^{T} H(\zeta, s)h(s, y(s)) \leq N \sum_{s=1}^{T} h(s, y(s)) \leq N\left(\frac{r}{NT}\right)T = \|y\|. \tag{3.5}$$

Now, if we let $\Lambda_1 = \{y \in \mathcal{B} \mid \|y\| < r\}$, then from (3.5), we have

$$\|\mathfrak{S}y\| \leq \|y\|, \quad y \in \mathcal{L} \cap \partial\Lambda_1.$$

Further, let

$$R_2 = \frac{N}{n}R \quad \text{and} \quad \Lambda_2 = \{y \in \mathcal{B} \mid \|y\| < R_2\}.$$

Then, $y \in \mathcal{L}$ and $\|y\| = R_2$ implies

$$0 \leq y(\zeta) \leq R_2, \quad \zeta \in \mathbb{N}_1^T.$$

Therefore, for all $\zeta \in \mathbb{N}_1^T$, we obtain

$$(\mathfrak{S}y)(\zeta) = \sum_{s=1}^{T} H(\zeta, s)h(s, y(s)) \geq n \sum_{s=1}^{T} h(s, y(s)) \geq n\left(\frac{NR}{n^2T}\right)T = \|y\|.$$

Hence, $\|\mathfrak{S}y\| \geq \|y\|$ for all $y \in \mathcal{L} \cap \partial\Lambda_2$. Thus, $\mathfrak{S}$ has a fixed point y in $\mathcal{L} \cap \left(\overline{\Lambda}_2 \backslash \Lambda_1\right)$, according to the first part of Theorem 1.1. Given $r \leq \|y\| \leq R_2$, (3.4) must hold. □

Theorem 3.7. Assume the following condition holds.

(B 4) There exist numbers $0 < r < R < \infty$ such that
  (a) $h(\zeta, z) \geq \dfrac{r}{nT}, \quad (\zeta, z) \in \mathbb{N}_1^T \times [0, r]$;
  (b) $h(\zeta, z) \leq \dfrac{R}{nT}, \quad (\zeta, z) \in \mathbb{N}_1^T \times \left[0, \dfrac{NR}{n}\right]$.

Then, (1.1) has at least one nonnegative solution y such that (3.4) holds.

Proof . The proof is analogous to that of Theorem 3.6 and uses the second part of Theorem 1.1. So, we omit it. □

Theorem 3.8. Assume the following condition holds.

(B 5) There exist numbers $0 < r < p < R < \infty$ such that
  (a) $h(\zeta, z) \geq \dfrac{r}{nT}, \quad (\zeta, z) \in \mathbb{N}_1^T \times [0, r]$;

(b) $h(\zeta, z) \geq \dfrac{NR}{n^2T}, \quad (\zeta, z) \in \mathbb{N}_1^T \times \left[0, \dfrac{NR}{n}\right];$

(c) $h(\zeta, z) < \dfrac{p}{NT}, \quad (\zeta, z) \in \mathbb{N}_1^T \times [0, p].$

Then, (1.1) has at least two nonnegative solutions y_1 and y_2 such that $\|y_1\| < p < \|y_2\|$ and

$$r \leq y_1(\zeta) < p < y_2(\zeta) \leq \frac{N}{n} R, \quad \zeta \in \mathbb{N}_1^T.$$

Proof . Take $y \in \mathcal{L}$ with $\|y\| = r$. Then, for all $s \in \mathbb{N}_1^T$, $0 \leq y(s) \leq r$ and

$$(\mathfrak{S}y)(\zeta) = \sum_{s=1}^{T} H(\zeta, s) h(s, y(s)) \geq n \sum_{s=1}^{T} h(s, y(s)) \geq n \left(\frac{r}{nT}\right) T = \|y\|.$$

Thus, we may let $\Lambda_1 = \{y \in \mathcal{B} \mid \|y\| < r\}$ so that

$$\|\mathfrak{S}y\| \geq \|y\|, \quad y \in \mathcal{L} \cap \partial\Lambda_1.$$

Further, for $y \in \mathcal{L}$ with $\|y\| = p$, we have

$$0 \leq y(s) \leq p, \quad s \in \mathbb{N}_1^T,$$

and, therefore,

$$(\mathfrak{S}y)(\zeta) = \sum_{s=1}^{T} H(\zeta, s) h(s, y(s)) \leq N \sum_{s=1}^{T} h(s, y(s)) < N \left(\frac{p}{NT}\right) T = \|y\|.$$

Hence, if we set $\Lambda_2 = \{y \in \mathcal{B} \mid \|y\| < p\}$, then

$$\|\mathfrak{S}y\| < \|y\|, \quad y \in \mathcal{L} \cap \partial\Lambda_2. \tag{3.6}$$

Consequently, by the second part of Theorem 1.1, it follows that $\mathfrak{S}$ has a fixed point y_1 in $\mathcal{L} \cap \left(\overline{\Lambda}_2 \backslash \Lambda_1\right)$. We have $r \leq \|y_1\| \leq p$. Because by (3.6), $\mathfrak{S}y \neq y$ for $y \in \mathcal{L}$ with $\|y\| = p$, hence $r \leq \|y_1\| < p$. For $y \in \mathcal{L}$, we have

$$r \leq y_1(\zeta) < p, \quad \zeta \in \mathbb{N}_1^T.$$

Let

$$R_2 = \frac{N}{n} R \text{ and } \Lambda_3 = \{y \in \mathcal{B} \mid \|y\| < R_2\}.$$

Then, for $y \in \mathcal{L}$ with $\|y\| = R_2$, we have

$$0 \leq y(\zeta) \leq R_2, \quad \zeta \in \mathbb{N}_1^T.$$

Therefore,

$$(\mathfrak{S}y)(\zeta) = \sum_{s=1}^{T} H(\zeta, s) h(s, y(s)) \geq n \sum_{s=1}^{T} h(s, y(s)) \geq n \left(\frac{NR}{n^2T}\right) T = \|y\|.$$

Hence,

$$\|\mathfrak{S}y\| \geq \|y\|, \quad y \in \mathcal{L} \cap \partial\Lambda_3.$$

Thus, $\mathfrak{S}$ has a fixed point y_2 in $\mathcal{L} \cap \left(\overline{\Lambda}_3 \backslash \Lambda_2\right)$, according to the first part of Theorem 1.1, and we have $p < \|y_2\| \leq R_2$. □

Theorem 3.9. Assume the following condition holds.

(B 6) There exist numbers $0 < r < p < R < \infty$ such that

(a) $h(\zeta, z) \leq \dfrac{r}{NT}, \quad (\zeta, z) \in \mathbb{N}_1^T \times [0, r];$

(b) $h(\zeta, z) \leq \dfrac{R}{nT}, \quad (\zeta, z) \in \mathbb{N}_1^T \times \left[0, \dfrac{NR}{n}\right]$;

(c) $h(\zeta, z) > \dfrac{p}{nT}, \quad (\zeta, z) \in \mathbb{N}_1^T \times [0, p]$.

Then, (1.1) has at least two nonnegative solutions y_1 and y_2 such that $\|y_1\| < p < \|y_2\|$ and

$$r \leq y_1(\zeta) < p < y_2(\zeta) \leq \frac{N}{n}R, \quad \zeta \in \mathbb{N}_1^T.$$

Proof . The proof is analogous to that of Theorem 3.8. So, we omit it. □

4 Examples

We give two examples in this section to show how known results can be applied.

Example 4.1. Consider (1.1) with $T = 4$ and

$$f(\zeta, \xi) = \frac{\zeta}{2} - \frac{9\xi}{100}, \quad (\zeta, \xi) \in \mathbb{N}_1^4 \times [0, \infty).$$

Choose $\lambda^2 = \frac{1}{10}$ such that

$$g(\zeta, \xi) = \frac{\zeta}{2} + \frac{\xi}{100}, \quad (\zeta, \xi) \in \mathbb{N}_1^4 \times [0, \infty).$$

Clearly, the assumptions (A 1) and (A 2) hold. We obtain $\theta = 0.3176$, $m = 0.6584$ and $M = 3.2685$. If we take $r = 1$ and $R = 10$ so that the conditions

1. $g(\zeta, \xi) \geq 0.5 \geq 0.3797, \quad 0 \leq \xi \leq 1$;
2. $g(\zeta, \xi) \leq 2.9964 \leq 3.797, \quad 10 \leq \xi \leq 49.643$,

hold for all $\zeta \in \mathbb{N}_1^4$ implying that the assumption (A 4) holds. Then, by Theorem 2.7, the BVP (1.1) has at least one nonnegative solution y such that

$$0.2014 \leq y(\zeta) \leq 49.643, \quad \zeta \in \mathbb{N}_1^4.$$

Example 4.2. Consider (1.1) with $T = 4$ and

$$f(\zeta, \xi) = \frac{\zeta}{2} + \frac{100001\xi}{100000}, \quad (\zeta, \xi) \in \mathbb{N}_1^4 \times [0, \infty).$$

Choose $\lambda^2 = 1$ such that

$$h(\zeta, \xi) = \frac{\zeta}{2} + \frac{\xi}{100000}, \quad (\zeta, \xi) \in \mathbb{N}_1^4 \times [0, \infty).$$

Clearly, the assumptions (B 1) and (B 2) hold. We obtain $\eta = 2.6180$, $n = 0.8913$ and $N = 286.9571$. If we take $r = 1$ and $R = 10$ so that the conditions

1. $h(\zeta, \xi) \geq 0.5 \geq 0.2805, \quad 0 \leq \xi \leq 1$;
2. $h(\zeta, \xi) \leq 2.5322 \leq 2.805, \quad 0 \leq \xi \leq 3219.535$,

hold for all $\zeta \in \mathbb{N}_1^4$ implying that the assumption (B 4) holds. Then, by Theorem 3.7, the BVP (1.1) has at least one nonnegative solution y such that

$$1 \leq y(\zeta) \leq 3219.535, \quad \zeta \in \mathbb{N}_1^4.$$

5 Conclusion

In this article, we built a typical fixed point operator using a shift parameter for a second-order difference equation's boundary value problem at resonance, and provided sufficient conditions for the existence of nonnegative solutions using an appropriate fixed point theorem. The shift-argument technique can be extended to problems with indefinite weight functions or nonlocal boundary conditions, systems of discrete equations at resonance, higher-order difference equations, or the study of sign-changing solutions.

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