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Some linear dynamics of convolution operators on Sobolev spaces

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Abstract

In this paper, we study the linear dynamical properties of a convolution operator on a Sobolev space H^s_γ in the context of locally compact abelian groups and commutative hypergroups. In particular, we provide sufficient conditions for a function to be a recurrent point for a sequence $(\widetilde{T}_{\nu_k})_k$ of convolution operators on the Sobolev space $H^s_\gamma(K)$, where K is a compact commutative Pontryagin hypergroup.

Keywords: Sobolev space, convolution algebras, locally compact groups, Pontryagin hypergroups, convolution operators, topologically recurrent operator
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1 Introduction

Sobolev spaces are important classes of function spaces that were originally introduced in the context of Euclidean spaces $\mathbb{R}^n$ [1]. During the last decades some researchers gave several versions of these spaces. For instance, Sobolev spaces on metric measure spaces, known as Hajłasz– Sobolev spaces, were introduced in [13, 14]. In 2015, P. Górka and E. Reyes in [11] introduced Sobolev spaces $H^s_\gamma(G)$ on a general locally compact abelian group G, and investigated its basic properties, where γ is a function on the dual group $\hat{G}$ with $s > 0$. Also, recently some version of Sobolev spaces have been defined in the context of hypergroup Gelfand pairs in [2]. In the last decade study of linear dynamical systems has been so much active and many researches have been done on this field; see papers [5, 6, 7, 8, 17, 18].

In this paper, we intend to study some linear dynamical properties of the Sobolev space $H^s_\gamma(G)$. For more details and notions regarding linear dynamics of operators see the monographs [3, 12]. Precisely, by some bounded regular measure τ on a compact abelian group G, we consider a convolution operator $\widetilde{T}_\nu : f \mapsto \nu * f$, where $f \in H^s_\gamma(G)$, and we prove that if $(\nu_n)_n$ is a sequence in $M(G)$, and $(\widetilde{T}_{\nu_n})_n$ is asymptotically recurrent on $H^s_\gamma(G)$, then, there is a strictly increasing sequence (n_k) in $\mathbb{N}$ such that $\lim_{k\to\infty} \nu_{n_k}(G) = 1$. Also, for every compact commutative Pontryagin hypergroup K, we show that if $\varnothing \neq S \subseteq \widehat{K}$ is finite and $\gamma \in \Gamma_K$ with $\gamma(\xi) = 0$ for all $\xi \in S$, then for any sequence $(\nu_k)_k \subseteq M(K)$ such that $1 \in \overline{\{\hat{\nu_k}(\xi) : k \in \mathbb{N}\}}$ for all $\xi \in S$, every function f in $H^s_\gamma(K)$ with $\hat{f}(\xi) = 0$ for all $\xi \notin S$, is a recurrent point for the sequence $(\widetilde{T}_{\nu_k})_k$.

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2 Preliminaries

In this paper, G is a locally compact abelian group with a Haar measure μ, the dual of G is denoted by $\hat{G}$ and m is the Plancherel measure on $\hat{G}$ corresponding to μ. In fact, for every $f \in L^2(G)$ we have

$$\int_{\hat{G}} |\hat{f}(\xi)|^2 \, dm(\xi) = \int_G |f(x)|^2 \, d\mu(x),$$

where $\hat{f}$ denotes the Fourier transform of f. For each $p > 0$, simply we denote $L^p(G) := L^p(G, \mu)$ and $L^p(\hat{G}) := L^p(\hat{G}, m)$.

The set of all functions $\gamma : \hat{G} \to [0, \infty)$ for which there is some $C_\gamma > 0$ such that

$$\gamma(\xi\eta) \leq C_\gamma \left(\gamma(\xi) + \gamma(\eta)\right) \tag{2.1}$$

for all $\xi, \eta \in \hat{G}$, is denoted by Γ_G.

Example 2.1. Let $G := \mathbb{R}$ be the additive group of real line. Then, $\widehat{G} \cong \mathbb{R}$ via $\xi \mapsto a_\xi$, where $\xi(x) = e^{2\pi i a_\xi x}$ for all $x \in \mathbb{R}$. We can show that the function $\gamma : \widehat{G} \to [0, \infty)$ defined by

$$\gamma(\xi) := \log(e + |\xi|), \qquad (\xi \in \widehat{G})$$

belongs to Γ. For this, note that for every $\xi, \eta \in \widehat{G}$ we have

$$\begin{aligned} \gamma(\xi + \eta) &= \log(e + |\xi + \eta|) \\ &\leq \log(e + |\xi| + |\eta|) \\ &\leq \log\left((e + |\xi|)(e + |\eta|)\right) \\ &= \log(e + |\xi|) + \log(e + |\eta|) \\ &= \gamma(\xi) + \gamma(\eta). \end{aligned}$$

Hence, $\gamma \in \Gamma_{\mathbb{R}}$. Now, fix some $s > 0$, and consider $\gamma_s : \widehat{G} \to [0, \infty)$ which is defined by:

$$\gamma_s(\xi) = (1 + |\xi|)^s, \qquad (\xi \in \widehat{G})$$

for all $\xi \in \widehat{G}$. Note that for every $a, b \geq 0$ and $p > 0$, if $0 < s \leq 1$, then the function $t \mapsto t^s$ is concave, which implies $(a + b)^s \leq a^s + b^s$, and whenever $s > 1$, the function $t \mapsto t^s$ is convex, which implies $(\frac{a+b}{2})^s \leq \frac{a^s+b^s}{2}$, so $(a + b)^s \leq 2^{s-1}(a^s + b^s)$. Then,

$$\begin{aligned} \gamma_s(\xi + \eta) &= (1 + |\xi + \eta|)^s \\ &\leq (1 + |\xi| + |\eta|)^s \\ &< \left((1 + |\xi|) + (1 + |\eta|)\right)^s \\ &\leq C_s\left((1 + |\xi|)^s + (1 + |\eta|)^s\right) \\ &= C_s(\gamma_s(\xi) + \gamma_s(\eta)), \end{aligned}$$

where $C_s = 1$ if $0 < s \leq 1$, and $C_s = 2^{s-1}$ if $s > 1$. Hence, $\gamma_s \in \Gamma$.

Now, we recall the following concept from [11].

Definition 2.2. Let $\gamma \in \Gamma_G$ and $s > 0$. Then, the set of all functions $f \in L^2(G)$ with

$$\|f\|_{H^s_\gamma(G)} := \left(\int_{\hat{G}} \left(1 + \gamma(\xi)^2\right)^s \left|\hat{f}(\xi)\right|^2 dm(\xi)\right)^{\frac{1}{2}} < \infty$$

is called the Sobolev space and is denoted by $H^s_\gamma(G)$.

Remark 2.3. In fact,

$$H^s_\gamma(G) = \left\{ f \in L^2(G) : \left(1 + \gamma^2\right)^{\frac{s}{2}} \hat{f} \in L^2(\hat{G}) \right\}.$$

Note that by [11, Theorem 2], for each $s > 0$, and $\gamma \in \Gamma_G$ with $\frac{1}{(1+\gamma^2)^s} \in L^1(\hat{G})$, $(H^s_\gamma(G), \|\cdot\|_{H^s_\gamma(G)})$ is a Banach algebra with pointwise product.

3 Main Results

Let $\mathcal{M}(G)$ be the space of all complex Radon measures on a locally compact group G, and let $L^1(G)$ be the space of integrable functions with respect to the left Haar measure μ on G. Then, for every $\mu \in \mathcal{M}(G)$ and $f \in L^1(G)$ we define

$$(\mu * f)(x) := \int_G f(y^{-1}x)\, d\mu(y), \qquad (f * \mu)(x) := \int_G \Delta_G(y^{-1}) f(xy^{-1})\, d\mu(y)$$

for all $x \in G$, where Δ_G is the modular function of the group G. We refer to [9] for more details regarding locally compact groups. Note that if G is abelian, then $\Delta_G \equiv 1$ and $\mu * f = f * \mu$. In particular,

$$(\delta_a * f)(x) = \int_G f(y^{-1}x)\, d\delta_a(y) = f(a^{-1}x).$$

Also, the convolution of each $f, g \in L^1(G)$ is defined by

$$(g * f)(x) = \int_G g(y) f(y^{-1}x)\, dy$$

for all $x \in G$. For every $\mu \in \mathcal{M}(G)$ and $f \in L^1(G)$, we have $\mu * f \in L^1(G)$ and

$$\|\mu * f\|_1 \leq \|\mu\| \cdot \|f\|_1,$$

where $\|\mu\|$ is the total variation norm of the measure μ. For an abelian group G,

$$\widehat{\mu * f}(\chi) = \hat{\mu}(\chi) \cdot \hat{f}(\chi) \quad \text{for } \chi \in \hat{G},$$

where $\hat{\mu}$ is the Fourier-Stieltjes transform of the measure μ.

Definition 3.1. Let G be an abelian compact group and $\nu \in M(G)$. Then, the *convolution operator* $\widetilde{T}_\nu : H^s_\gamma(G) \to H^s_\gamma(G)$ is defined by

$$\widetilde{T}_\nu(f) := \nu * f$$

for all $f \in H^s_\gamma(G)$.

Lemma 3.2. Let G be an abelian compact group, and fix an element $\nu \in M(G)$. Then, $\widetilde{T}_\nu : H^s_\gamma(G) \to H^s_\gamma(G)$ is a bounded linear operator.

Proof . First, we note that clearly the mapping $\widetilde{T}_\nu : H^s_\gamma(G) \to H^s_\gamma(G)$ is linear. Let $f \in H^s_\gamma(G)$, i.e. $f \in L^2(G)$, and

$$\|f\|_{H^s_\gamma(G)} = \left(\int_{\widehat{G}} (1 + \gamma(\zeta)^2)^s |\widehat{f}(\zeta)|^2 dm(\zeta) \right)^{\frac{1}{2}} < \infty.$$

This is well-known that $\widetilde{T}_\nu(f) \in L^2(G)$. In fact,

$$\begin{aligned} \|\widetilde{T}_\nu(f)\|_2 &= \|\nu * f\|_2 \\ &= \left(\int_G |(\nu * f)(x)|^2\, d\mu(x) \right)^{\frac{1}{2}} \\ &\leq \left(\int_G \left(\int |f(y^{-1}x)|\, d|\nu|(y) \right)^2 d\mu(x) \right)^{\frac{1}{2}} \end{aligned}$$

Now, by the Minkowski's inequality for integrals (see [10, Theorem 6.19]) we have

$$\begin{aligned} \|\widetilde{T}_\nu(f)\|_2 &\leq \int_G \left(\int_G |f(y^{-1}x)|^2 d\mu(x) \right)^{\frac{1}{2}} d|\nu|(y) \\ &= \int \left(\int |f(x)|^2\, d\mu(x) \right)^{\frac{1}{2}} d|\nu|(y) \\ &= \int_G \|f\|_2\, d|\nu|(y) \\ &= \|f\|_2\, |\nu|(G) < \infty, \end{aligned}$$

so $\widetilde{T}_\nu(f) \in L^2(G)$. Also,

$$\begin{aligned}\|\widetilde{T}_\nu(f)\|^2_{H^s_\gamma(G)} &= \int_{\widehat{G}} (1+\gamma(\xi)^2)^s |(\widehat{\widetilde{T}_\nu(f)})(\xi)|^2 dm(\xi) \\ &= \int_{\widehat{G}} (1+\gamma(\xi)^2)^s |(\widehat{\nu * f})(\xi)|^2 dm(\xi) \\ &= \int_{\widehat{G}} (1+\gamma(\xi)^2)^s |\widehat{\nu}(\xi).\widehat{f}(\xi)|^2 dm(\xi).\end{aligned}$$

By [9, page 94], we have $\hat{\nu} \in C_b(\hat{G})$, so $N := \sup_{\xi\in\hat{G}} |\widehat{\nu}(\xi)| < \infty$, and then

$$\begin{aligned}\|\widetilde{T}_\nu(f)\|^2_{H^s_\gamma(G)} &\leq N^2 \int_{\widehat{G}} (1+\gamma(\xi)^2)^s |\widehat{f}(\xi)|^2 dm(\xi) \\ &= N^2 \|f\|^2_{H^s_\gamma(G)}.\end{aligned}$$

This implies that

$$\|\widetilde{T}_\nu(f)\|_{H^s_\gamma(G)} \leq N\|f\|_{H^s_\gamma(G)},$$

hence, $\widetilde{T}_\nu : H^s_\gamma(G) \to H^s_\gamma(G)$ is well-defined and bounded. □

Definition 3.3. Let X be a Banach space, and T be a bounded linear operator on X.

1. $(T_n)_n \subseteq B(X)$ is called *topologically recurrent* if for each non-empty open subset U of X there is some $n \in \mathbb{N}$ such that
$$U \cap T_n^{-1}(U) \neq \varnothing,$$
where $T_n^{-1}(U) := \{x \in X : T_n(x) \in U\}$.
2. $(T_n)_n \subseteq B(X)$ is called *asymptotically recurrent* if for each non-empty open subset U of X and any $N \in \mathbb{N}$ there is some $n \geq N$ with
$$U \cap T_n^{-1}(U) \neq \varnothing.$$
3. An element $v \in X$ is called a *recurrent point* for the sequence $(T_n)_n \subseteq B(X)$ if for every $\varepsilon > 0$ there is some large enough $n \in \mathbb{N}$ such that $\|T_n(f) - f\| < \varepsilon$. If $v \in X$ is a recurrent point for the sequence $(T^n)_n$, then we say that v is a recurrent point for the operator T.

Example 3.4. Consider the compact abelian unit circle group $G := \mathbb{T}$. Then, $\widehat{G} \cong \mathbb{Z}$ via $\xi \mapsto n_\xi$, where $\xi(z) = z^{n_\xi}$ for all $z \in \mathbb{T}$. Fix some number $N \in \mathbb{N}$. Now, we define the function $\gamma : \mathbb{Z} \to \mathbb{C}$ with

$$\gamma(n) := \begin{cases} 0 & \text{if } n = 0, \pm1, \pm2, \ldots, \pm N \\ \log(e + |n|) & \text{otherwise.} \end{cases}$$

Then, $\gamma \in \Gamma_{\mathbb{T}}$, and for every $f \in H^s_\gamma(\mathbb{T})$ we have

$$\|f\|^2_{H^s_\gamma(\mathbb{T})} = \sum_{|n|\leq N} |\hat{f}(n)|^2 + \sum_{|n|>N} (1 + \log^2(e + |n|))^s |\hat{f}(n)|^2.$$

Pick some $a \in \mathbb{T}$ with $a^j \neq 1$ for all $j \in \mathbb{N}$, and consider some measure $\nu \in \mathcal{M}(\mathbb{T})$ such that $\hat{\nu}(n) = a^n$ for all $n = 0, \pm1, \pm2, \ldots, \pm N$. Then, for every $k \in \mathbb{N}$ and $f \in H^s_\gamma(\mathbb{T})$ we have

$$\widehat{\widetilde{T}^k_\nu(f)}(n) = \hat{\nu}(n)^k \hat{f}(n) = a^{nk} \hat{f}(n).$$

Assume that

$$f \in \{f \in H^s_\gamma(\mathbb{T}) : \hat{f}(n) = 0 \text{ for all } n \text{ with } |n| > N\}.$$

Then,

$$\begin{aligned}\|\widetilde{T}_\nu^k(f)-f\|^2_{H^s_\gamma(\mathbb{T})} &= \sum_{|n|\le N}(1+\gamma(n)^2)^s|\hat{\nu}(n)^k\hat{f}(n)-\hat{f}(n)|^2+\sum_{|n|>N}(1+\log^2(e+|n|))^s|\hat{\nu}(n)^k\hat{f}(n)-\hat{f}(n)|^2\\ &= \sum_{|n|\le N}(1+0^2)^s|(a^n)^k-1|^2|\hat{f}(n)|^2 \quad (\text{since } \hat{f}(n)=0\\ &= \sum_{|n|\le N}|(a^k)^n-1|^2|\hat{f}(n)|^2.\end{aligned}$$

The set $\{a^n : n\in\mathbb{N}\}$ is dense in $\mathbb{T}$ because a is not a root of unity. So, there are $n_1<n_2<n_3<\ldots$ in $\mathbb{N}$ such that $\lim_{k\to\infty} a^{nn_k}=1$ for all $n=0,\pm1,\pm2,\ldots,\pm N$, hence

$$\lim_{k\to\infty}\|\widetilde{T}_\nu^{n_k}(f)-f\|_{H^s_\gamma(\mathbb{T})}=0.$$

This implies that every $f\in\{f\in H^s_\gamma(\mathbb{T}) \mid \hat{f}(n)=0 \text{ for all } n \text{ with } |n|>N\}$ is a recurrent point for the operator $\widetilde{T}_\nu$.

Recall that if μ is the normalized Haar measure on a compact abelian group, then the corresponding Plancherel measure will be the counting measure.

In the following result, we give some necessary condition for that a sequence of convolution operators to be asymptotically recurrent on a Sobolev space. The orthogonality of characters of a compact group is basic in this proof.

Theorem 3.5. Let G be a compact abelian group with a normalized Haar measure μ. Assume that $(\nu_n)_n$ is a sequence in $M(G)$, and $(\widetilde{T}_{\nu_n})_n$ is asymptotically recurrent on $H^s_\gamma(G)$. Then, there is a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$ such that $\lim_{k\to\infty}\widehat{\nu_{n_k}}(1)=1$, i.e. $\lim_{k\to\infty}\nu_{n_k}(G)=1$.

Proof . Let $(\widetilde{T}_{\nu_n})_n$ be asymptotically recurrent on $H^s_\gamma(G)$. Note that $\chi_G\in H^s_\gamma(G)$. Indeed, since G is compact we have $\chi_G\in L^2(G)$, and since $\hat{G}$ is an orthogonal subset of $L^2(G)$,

$$\widehat{1}(\xi)=\int_G\overline{\xi(x)}\,d\mu(x)=\langle 1,\xi\rangle=\begin{cases}1, & \xi=1\\ 0, & \xi\neq 1,\end{cases}$$

and then

$$\begin{aligned}\|1\|^2_{H^s_\gamma(G)} &= \int_{\widehat{G}}(1+\gamma(\xi)^2)^s\chi_{\{1\}}(\xi)\,dm(\xi)\\ &= (1+\gamma(1)^2)^s<\infty,\end{aligned}$$

where 1 is the constant function one on G. For every $\epsilon>0$ denote

$$B(1;\epsilon):=\{f\in H^s_\gamma(G):\|1-f\|_{H^s_\gamma(G)}<\epsilon\}.$$

Then, for every $k\in\mathbb{N}$, there is some $n_k\in\mathbb{N}$ (with $n_1<n_2<\ldots$) such that

$$B(1;\frac{1}{4^k})\bigcap\widetilde{T}^{-1}_{\nu_{n_k}}\left(B(1;\frac{1}{4^k})\right)\neq\varnothing.$$

For each $k\in\mathbb{N}$ pick some

$$f_k\in B(1;\frac{1}{4^k})\bigcap\widetilde{T}^{-1}_{\nu_{n_k}}\left(B(1;\frac{1}{4^k})\right).$$

Then, for each $k\in\mathbb{N}$,

$$\|f_k-1\|_{H^s_\gamma(G)}<\frac{1}{4^k}, \tag{3.1}$$

and

$$\left\|\widetilde{T}_{\nu_{n_k}}(f_k)-1\right\|_{H^s_\gamma(G)}<\frac{1}{4^k}. \tag{3.2}$$

By the inequality (3.1),

$$
\begin{aligned}
(\frac{1}{4^k})^2 &> \int_{\widehat{G}} \left(1+\gamma(\xi)^2\right)^s |\widehat{f_k - 1}(\xi)|^2 \, dm(\xi) \\
&\geq \int_{\widehat{G}} |\widehat{f_k - 1}(\xi)|^2 \, dm(\xi) \\
&= \sum_{\xi \in \widehat{G}} |\widehat{f}_k(\xi) - \hat{1}(\xi)|^2 \\
&= |\widehat{f}_k(1) - 1|^2 + \sum_{1 \neq \xi \in \widehat{G}} |\widehat{f}_k(\xi)|^2 \\
&\geq |\widehat{f}_k(1) - 1|^2.
\end{aligned}
$$

Hence, $\lim_{k\to\infty} \widehat{f}_k(1) = 1$. Also, for every $k \in \mathbb{N}$,

$$
\begin{aligned}
(\frac{1}{4^k})^2 &> \left\| \widetilde{T}_{\nu_{n_k}}(f_k) - 1 \right\|^2_{H^s_\gamma(G)} \\
&= \|\nu_{n_k} * f_k - 1\|^2_{H^s_\gamma(G)} \\
&= \sum_{\xi \in \widehat{G}} \left(1+\gamma(\xi)^2\right)^s \left| \left[\nu_{n_k} * f_k - 1\right]\hat{}(\xi) \right|^2 \\
&= \sum_{\xi \in \widehat{G}} \left(1+\gamma(\xi)^2\right)^s \left| \left[\nu_{n_k} * f_k\right]\hat{}(\xi) - \widehat{1}(\xi) \right|^2 \\
&= \left(1+\gamma(1)^2\right)^s \left| \left[\nu_{n_k} * f_k\right]\hat{}(1) - 1 \right|^2 + \sum_{1 \neq \xi \in \widehat{G}} \left(1+\gamma(\xi)^2\right)^s \left| \left[\nu_{n_k} * f_k\right]\hat{}(\xi) \right|^2 \\
&= \left(1+\gamma(1)^2\right)^s \left| \widehat{\nu_{n_k}}(1)\widehat{f}_k(1) - 1 \right|^2 + \sum_{1 \neq \xi \in \widehat{G}} \left(1+\gamma(\xi)^2\right)^s \left| \widehat{\nu_{n_k}}(\xi)\widehat{f}_k(\xi) \right|^2 \\
&\geq \left(1+\gamma(1)^2\right)^s \left| \widehat{\nu_{n_k}}(1)\widehat{f}_k(1) - 1 \right|^2,
\end{aligned}
$$

So, $\lim_{k\to\infty} \widehat{\nu_{n_k}}(1)\widehat{f}_k(1) = 1$. Therefore, $\lim_{k\to\infty} \widehat{\nu_{n_k}}(1) = 1$. □

Corollary 3.6. Let G be a compact abelian group with a normalized Haar measure μ. Assume that $\nu \in M(G)$, and $\widetilde{T}_\nu$ is asymptotically recurrent on $H^s_\gamma(G)$. Then, $\nu(G) = 1$.

Proof . For every $n \in \mathbb{N}$ put $\nu_n := \nu * \nu * \ldots * \nu$ (n-times). Then, thanks to the last theorem there are $n_1 < n_2 < \ldots$ in $\mathbb{N}$ such that

$$1 = \lim_{k\to\infty} \nu * \widehat{\nu * \ldots} * \nu(1) = \lim_{k\to\infty} \hat{\nu}(1)^{n_k},$$

and so

$$1 = \hat{\nu}(1) = \int_G \overline{1(x)} \, d\nu(x) = \nu(G).$$

□

Definition 3.7. Let X be a Banach space, and let $T : X \to X$ be a bounded linear operator.

- The operator T is called *topologically mixing* if for every pair of non-empty open subsets $U, V \subseteq X$ there exists $N \in \mathbb{N}$ such that
$$U \cap T^{-n}(V) \neq \varnothing \quad \text{for all } n \geq N.$$
- The operator T is called *topologically transitive* if for every pair of non-empty open subsets $U, V \subseteq X$ there exists some $n \in \mathbb{N}$ such that
$$U \cap T^{-n}(V) \neq \varnothing.$$

Recall that any function $f \in L^1(G)$ induces a regular measure $f\mu \in M(G)$. We denote $T_f := T_{f\mu}$.

Theorem 3.8. For every $w \in L^1(\mathbb{T})$, satisfying one of the following conditions:

- $\sup_{n\in\mathbb{Z}} |\hat{w}(n)| < 1$,
- $A := \{t \in \mathbb{Z} : |\hat{w}(t)| > 1\} \neq \varnothing$ and $\sup_{n\notin A} |\hat{w}(n)| < 1$,
- for some $j \in \mathbb{Z}$, $|\hat{w}(j)| = 1$ with $\hat{w}(j) \neq 1$,

the operator $\widetilde{T}_w$ is not topologically mixing on $H^s_\gamma(\mathbb{T})$.

Proof . Assume that $w \in L^1(\mathbb{T})$.

Case 1: $\delta := \sup_{n\in\mathbb{Z}} |\hat{w}(n)| < 1$. Pick some non-zero function $g \in H^s_\gamma(\mathbb{T})$, denote $r := \frac{1}{2}\|g\|_{H^s_\gamma(\mathbb{T})}$, and put $U := B(g; r)$, as an open ball in $H^s_\gamma(\mathbb{T})$. Then, $0 \notin U$. Let $f \in U$. For every $n \in \mathbb{N}$ we have

$$\begin{aligned}
\|\widetilde{T}^n_w(f)\|^2_{H^s_\gamma(\mathbb{T})} &= \sum_{k\in\mathbb{Z}} (1+\gamma(k)^2)^s |\widehat{\widetilde{T}^n_w(f)}(k)|^2 \\
&= \sum_{k\in\mathbb{Z}} (1+\gamma(k)^2)^s |\hat{w}(k)|^{2n}\, |\hat{f}(k)|^2 \\
&\leq \delta^{2n} \sum_{k\in\mathbb{Z}} (1+\gamma(k)^2)^s\, |\hat{f}(k)|^2 \\
&= \delta^{2n}\, \|f\|^2_{H^s_\gamma(\mathbb{T})},
\end{aligned}$$

so,

$$\|\widetilde{T}^n_w(f)\|_{H^s_\gamma(\mathbb{T})} < \delta^n\ \|f\|_{H^s_\gamma(\mathbb{T})}$$

for all $n \in \mathbb{N}$. Therefore, for every $f \in U$ there is some N_f such that for every $n \geq N_f$ we have $\|\widetilde{T}^n_w(f)\|_{H^s_\gamma(\mathbb{T})} < r$. On the other hand, note that for every $h \in U$ since $\|h-g\|_{H^s_\gamma(\mathbb{T})} < r$ we have

$$\|h\|_{H^s_\gamma(\mathbb{T})} \geq \|g\|_{H^s_\gamma(\mathbb{T})} - \|h-g\|_{H^s_\gamma(\mathbb{T})} > \|g\|_{H^s_\gamma(\mathbb{T})} - r = r.$$

This implies that for every $f \in U$ there is some number $N_f > 0$ such that for every $n \geq N_f$, $f \notin \widetilde{T}_w^{-n}(U)$.

Case 2: $A := \{t \in \mathbb{Z} : |\hat{w}(t)| > 1\} \neq \varnothing$ and $\theta := \sup_{n\notin A} |\hat{w}(n)| < 1$. Same as the case one, consider some $0 \neq g \in H^s_\gamma(\mathbb{T})$, put $r := \frac{1}{2}\|g\|_{H^s_\gamma(\mathbb{T})}$, $U := B(g; r)$, and let $f \in U$. If $\hat{f}(j) \neq 0$ for some $j \in A$, then $\lim_{n\to\infty} \|\widetilde{T}^n_w(f)\|_{H^s_\gamma(\mathbb{T})} = \infty$ because

$$\|\widetilde{T}^n_w(f)\|_{H^s_\gamma(\mathbb{T})} \geq \|\hat{w}(j)|^{2n}\, |\hat{f}(k)|^2$$

for all $n \in \mathbb{N}$. Also, if for every $j \in A$ we have $\hat{f}(j) = 0$, then $\lim_{n\to\infty} \|\widetilde{T}^n_w(f)\|_{H^s_\gamma(\mathbb{T})} = 0$ because for every $n \in \mathbb{N}$,

$$\begin{aligned}
\|\widetilde{T}^n_w(f)\|^2_{H^s_\gamma(\mathbb{T})} &= \sum_{k\notin A} (1+\gamma(k)^2)^s |\hat{w}(k)|^{2n}\, |\hat{f}(k)|^2 \\
&\leq \theta^{2n} \sum_{k\notin A} (1+\gamma(k)^2)^s\, |\hat{f}(k)|^2 \leq \theta^{2n}\, \|f\|^2_{H^s_\gamma(\mathbb{T})}.
\end{aligned}$$

Case 3: for some $j \in \mathbb{Z}$, $|\hat{w}(j)| = 1$ and $\hat{w}(j) \neq 1$. In this case, consider some $g \in H^s_\gamma(\mathbb{T})$ that $\hat{g}(j) = 1$ and for every $k \neq j$, $\hat{g}(k) = 0$. Denote $U := B(g; r)$, where $r := \frac{1}{2}\, \|g\|_{H^s_\gamma(\mathbb{T})}$. Let $f \in U$. Note that

$$4r^2 = \|g\|^2_{H^s_\gamma(\mathbb{T})} = (1+\gamma(j)^2)^s\, |\hat{g}(j)|^2 = (1+\gamma(j)^2)^s,$$

and also

$$r^2 > \|f-g\|^2_{H^s_\gamma(\mathbb{T})} \geq (1+\gamma(j)^2)^s\, |\hat{f}(j) - \hat{g}(j)|^2 = (1+\gamma(j)^2)^s\, |\hat{f}(j) - 1|^2,$$

so,

$$|\hat{f}(j) - 1| < \frac{r}{(1+\gamma(j)^2)^{s/2}} = \frac{1}{2}.$$

On the other hand, for every $n \in \mathbb{N}$ we have

$$
\begin{aligned}
\|\widetilde{T}_w^n(f) - g\|_{H^s_\gamma(\mathbb{T})} &= \left(\sum_{k\in\mathbb{Z}} (1+\gamma(k)^2)^s |\widehat{\widetilde{T}_w^n(f)}(k) - \hat{g}(k)|^2\right)^{1/2} \\
&\geq (1+\gamma(j)^2)^{s/2} |\hat{w}(j)^n \hat{f}(j) - \hat{g}(j)| \\
&\geq (1+\gamma(j)^2)^{s/2} |\hat{w}(j)^n \hat{f}(j) - 1|.
\end{aligned}
$$

Since $|\hat{f}(j) - 1| < \frac{1}{2}$ and $1 \neq \hat{w}(j) \in \mathbb{T}$, there exists some N large enough with $|\hat{w}(j)^n \hat{f}(j) - 1| \geq \frac{1}{2}$, and so

$$
\|\widetilde{T}_w^N(f) - g\|_{H^s_\gamma(\mathbb{T})} \geq (1+\gamma(j)^2)^{s/2} \frac{1}{2} = r,
$$

thus, $\widetilde{T}_w^N(f) \notin U$. $\square$

3.1 Stability of $\widetilde{T}_\nu$ in the context of hypergroups

In this subsection, we intend to present some fact regarding the more general situation i.e. $\widetilde{T}_\nu$ on $H^s_\gamma(K)$, where K is a compact commutative hypergroup. Recall that hypergroups are generalization of locally compact groups. For convenience of readers we recall the definition of hypergroups. Next, $M(Z)$ is the set of complex Radon measures on a locally compact Hausdorff space Z, and $\mathrm{supp}(\mu)$ is the support of a measure μ. Also, δ_x denotes the Dirac measure.

Definition 3.9. A locally compact Hausdorff space K with $*$ as a convolution on $M(K)$ and $x \mapsto x^-$ as an involutive function on K is called a hypergroup if the followings hold:

(i) $(M(K), +, *)$ is an algebra;
(ii) for every $x, y \in K$, $\delta_x * \delta_y$ is a probability measure with compact support;
(iii) $(x, y) \mapsto \delta_x * \delta_y$ from $K \times K$ to $M^+(K)$ is continuous;
(iv) $(x, y) \mapsto \mathrm{supp}(\delta_x * \delta_y)$ from $K \times K$ to the set of all non-empty compact subsets of K is continuous;
(v) $x \mapsto x^-$ is a homeomorphism from K onto K and for any $x, y \in K$, $(\delta_x * \delta_y)^- = \delta_{y^-} * \delta_{x^-}$. Also, There is some $e \in K$ that for every $x \in K$, $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$. Moreover, for any $x, y \in K$, $e \in \mathrm{supp}(\delta_x * \delta_y)$ if and only if $y = x^-$.

Double coset spaces, polynomial hypergroups and locally compact groups are famous examples of hypergroups. For more examples and details regarding locally compact hypergroups we refere to [4, 15, 16]. In the sequel, K will be a hypergroup equipped with a convolution * and involution $x \mapsto x^-$. A hypergroup is called *Pontryagin* whenever it is commutative, strong, and $\widehat{K}$ is a hypergroup too. In general, if K is compact, then $\widehat{K}$ is discrete. Also, if π_K is a normalized Haar measure on K, then the Plancherel measure $\pi_{\widehat{K}}$ on $\widehat{K}$ corresponding to π_K is as

$$
\pi_{\widehat{K}}(\{\xi\}) = \frac{1}{\delta_\xi * \delta_{\bar{\xi}}(\{1\})}, \qquad (\xi \in \widehat{G}).
$$

See the paper [2] which gives a generalization of Sobolev spaces on hypergroups. In this situation, the relation (2.1) will be changed to

$$
\int_{\widehat{K}} \gamma(\zeta)\, d(\delta_\xi * \delta_\eta)(\zeta) \leq C_\gamma \left(\gamma(\xi) + \gamma(\eta)\right). \tag{3.3}
$$

Theorem 3.10. Assume that K is a compact commutative Pontryagin hypergroup with a normalized Haar measure μ. Let $\varnothing \neq S \subseteq \widehat{K}$ be finite, and consider some $\gamma \in \Gamma_K$ with $\gamma(\xi) = 0$ for all $\xi \in S$. Let $(\nu_k)_k \subseteq M(K)$ such that $1 \in \overline{\{\hat{\nu_k}(\xi) : k \in \mathbb{N}\}}$ for all $\xi \in S$. Then, any

$$
f \in \Omega := \{f \in H^s_\gamma(K) : \hat{f}(\xi) = 0 \text{ for all } \xi \notin S\}
$$

is a recurrent point for the sequence $(\widetilde{T}_{\nu_k})_k$.

Proof . First, note that for any $f \in H^s_\gamma(K)$,

$$\|f\|^2_{H^s_\gamma(K)} = \sum_{\xi\in S} |\hat{f}(\xi)|^2 \frac{1}{\delta_\xi * \delta_{\bar{\xi}}(\{1\})} + \sum_{\xi\notin S} (1+\gamma(\xi)^2)^s |\hat{f}(\xi)|^2 \frac{1}{\delta_\xi * \delta_{\bar{\xi}}(\{1\})},$$

and by [4, Proposition 2.2.26] for every $k \in \mathbb{N}$ we have

$$\widehat{\widetilde{T}_{\nu_k}(f)}(\xi) = \hat{\nu_k}(\xi)\hat{f}(\xi), \qquad (\xi \in \widehat{K}).$$

Assume that $f \in \Omega$. Then,

$$\begin{aligned}\|\widetilde{T}_{\nu_k}(f) - f\|^2_{H^s_\gamma(K)} &= \sum_{\xi\in S} (1+\gamma(\xi)^2)^s |\hat{\nu_k}(\xi)\hat{f}(\xi) - \hat{f}(\xi)|^2 \frac{1}{\delta_\xi * \delta_{\bar{\xi}}(\{1\})} + \sum_{\xi\notin S} (1+\gamma^2(\xi))^s |\hat{\nu_k}(\xi)\hat{f}(\xi) - \hat{f}(n)|^2 \frac{1}{\delta_\xi * \delta_{\bar{\xi}}(\{1\})} \\ &= \sum_{n\in S} |\hat{\nu_k}(\xi) - 1|^2 |\hat{f}(\xi)|^2 \frac{1}{\delta_\xi * \delta_{\bar{\xi}}(\{1\})}.\end{aligned}$$

Since $1 \in \overline{\{\hat{\nu_k}(\xi) : \xi \in S, k \in \mathbb{N}\}}$ and S is finite, there are $n_1 < n_2 < n_3 < \dots$ in $\mathbb{N}$ such that $\lim_{k\to\infty} \hat{\nu_{n_k}}(\xi) = 1$ for all $\xi \in S$, hence

$$\lim_{k\to\infty} \|\widetilde{T}_{\nu_{n_k}}(f) - f\|_{H^s_\gamma(K)} = 0.$$

This completes the proof. □

Corollary 3.11. Assume that K is a compact commutative Pontryagin hypergroup with a normalized Haar measure μ. Let $\varnothing \neq S \subseteq \widehat{K}$ be finite, and consider some $\gamma \in \Gamma_K$ with $\gamma(\xi) = 0$ for all $\xi \in S$. Let $\nu \in M(K)$ such that $1 \in \overline{\{\hat{\nu}(\xi)^k : k \in \mathbb{N}\}}$ for all $\xi \in S$. Then, any

$$f \in \{f \in H^s_\gamma(K) : \hat{f}(\xi) = 0 \text{ for all } \xi \notin S\}$$

is a recurrent point for the operator $\widetilde{T}_\nu$.

Corollary 3.12. Assume that K is a compact commutative Pontryagin hypergroup with a normalized Haar measure μ. Let $\varnothing \neq S \subseteq \widehat{K}$ be finite, and consider some $\gamma \in \Gamma_K$ with $\gamma(\xi) = 0$ for all $\xi \in S$. Let $(a_n)_n \subseteq K$ such that $1 \in \overline{\{\bar{\xi}(a_k) : k \in \mathbb{N}\}}$ for all $\xi \in S$. Then, setting $T_n := \widehat{T}_{\delta_{a_n}}$ for all $n \in \mathbb{N}$, any

$$f \in \{f \in H^s_\gamma(K) : \hat{f}(\xi) = 0 \text{ for all } \xi \notin S\}$$

is a recurrent point for the operator $(T_n)_n$.

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