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Improvement of some Turán-type inequalities for polynomials

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Abstract

For a polynomial $P(z) = a_n \prod_{j=1}^{n} (z - z_j)$ of degree n having all zeros in $|z| \leq k, k \geq 1$, it is known that

$$|P'(z)| \geq \frac{2}{1+k^n} \sum_{j=1}^{n} \frac{k}{k+|z_j|} \max_{|z|=1} |P(z)|.$$

In this paper, we prove an improvement of this inequality and also extend the same to the polar derivative of a polynomial, thereby generalizing some Turán-type inequalities for polynomials.

Keywords: Polynomial, Inequalities, Polar derivative, Zeros.
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1 Introduction

For each positive integer n, let $\mathcal{P}_n$ denote the linear space of all polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree atmost n over the field $\mathbb{C}$ of complex numbers. If $P \in \mathcal{P}_n$, then concerning the estimate of $|P'(z)|$ in terms of $|P(z)|$ on $|z| = 1$, we have for all $z \in \mathbb{C}$, the following famous inequality due to Bernstein [4].

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

The result is sharp and equality holds only when $P(z)$ is a constant multiple of z^n. This inequality can be sharpened if there is a restriction on the zeros of $P(z)$. In fact, *if* $P(z) \neq 0$ *in* $|z| < 1$, *then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|, \tag{1.2}$$

whereas in reverse direction, *if* $P(z) \neq 0$ *in* $|z| > 1$, *then*

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.3}$$

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Both the inequalities are sharp and equality in each holds for the polynomials $P(z) = \alpha z^n + \beta$, where $|\alpha| = |\beta|$. Inequality (1.2) was conjectured by Erdös and later verified by Lax [7], whereas inequality (1.3) is due to Turán [12]. Inequality (1.3) has been generalized in various ways and among others, (for references see [1], [8], [9], [10]). Aziz [2] proved the following generalization of this inequality.

If $P(z) = a_n \prod_{j=1}^{n}(z - z_j)$ *is a complex polynomial of degree* n *with* $|z_j| \leq k, k \geq 1$, *then*

$$\max_{|z|=1}|P'(z)| \geq \frac{2}{1+k^n}\sum_{j=1}^{n}\frac{k}{k+|z_j|}\max_{|z|=1}|P(z)|. \tag{1.4}$$

2 Basic Lemmas

Lemma 2.1. Let $P \in \mathcal{P}_n$ has all zeros in $|z| \leq 1$ and $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$, then for $|z| = 1$

$$|Q'(z)| \leq |P'(z)|.$$

The above lemma follows from a result due to Bernstein [5].

Lemma 2.2. If $P(z)$ is a polynomial of degree $n \geq 1$, then for $R \geq 1$

$$\max_{|z|=R}|P(z)| \leq R^n \max_{|z|=1}|P(z)| - (R^n - R^{n-2})|P(0)|, \text{ if } \ n > 1 \tag{2.1}$$

and

$$\max_{|z|=R}|P(z)| \leq R\max_{|z|=1}|P(z)| - (R-1)|P(0)|, \text{ if } \ n = 1. \tag{2.2}$$

The above lemma is due to Frappier et al.[6].

Lemma 2.3. If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k, k \geq 1$, then for $0 \leq l < 1$

$$\begin{aligned}\max_{|z|=k}|P(z)| \geq{}& \frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)| + l\left(\frac{k^n-1}{k^n+1}\right)\min_{|z|=k}|P(z)| \\ &+ \frac{2k^{n-1}}{1+k^n}|a_{n-1}|\left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right), \text{ if } \ n > 2\end{aligned} \tag{2.3}$$

and

$$\max_{|z|=k}|P(z)| \geq \frac{2k^2}{1+k^2}\max_{|z|=1}|P(z)| - l\left(\frac{k^2-1}{k^2+1}\right)\min_{|z|=k}|P(z)| + \frac{k(k-1)^2}{1+k^2}|a_1|, \text{ if } \ n = 2 \tag{2.4}$$

The above lemma is due to Rather et al.[11].

3 Lemmas and Proofs of Lemmas

Lemma 3.1. Let $P \in \mathcal{P}_n$ has all zeros in $|z| \leq 1$ and $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$, then for $|z| = 1$

$$|Q'(z)| \leq |P'(z)|.$$

The above lemma follows from a result due to Bernstein [5].

Lemma 3.2. If $P(z)$ is a polynomial of degree $n \geq 1$, then for $R \geq 1$

$$\max_{|z|=R}|P(z)| \leq R^n \max_{|z|=1}|P(z)| - (R^n - R^{n-2})|P(0)|, \text{ if } \ n > 1 \tag{3.1}$$

and

$$\max_{|z|=R}|P(z)| \leq R \max_{|z|=1}|P(z)| - (R-1)|P(0)|, \text{ if } \ n = 1. \tag{3.2}$$

The above lemma is due to Frappier et al.[6].

Lemma 3.3. If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k, k \geq 1$, then for $0 \leq l < 1$

$$\max_{|z|=k}|P(z)| \geq \frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)| + l\left(\frac{k^n-1}{k^n+1}\right)\min_{|z|=k}|P(z)| + \frac{2k^{n-1}}{1+k^n}|a_{n-1}|\left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right), \text{ if } \ n > 2 \tag{3.3}$$

and

$$\max_{|z|=k}|P(z)| \geq \frac{2k^2}{1+k^2}\max_{|z|=1}|P(z)| - l\left(\frac{k^2-1}{k^2+1}\right)\min_{|z|=k}|P(z)| + \frac{k(k-1)^2}{1+k^2}|a_1|, \text{ if } \ n = 2 \tag{3.4}$$

The above lemma is due to Rather et al.[11].

4 Main Results

In this paper, we first prove the following result which generalize a result due to Aziz [2].

Theorem 4.1. If $P(z) = a_n \prod_{j=1}^{n}(z - z_j)$ be such that all its zeros lie in $|z| \leq k, k \geq 1$, then for $0 \leq l < 1$

$$\max_{|z|=1}|P'(z)| \geq \psi(k)|a_1| + \frac{1}{k^{n-1}}\sum_{j=1}^{n}\frac{1}{k+|z_j|}\left[\frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)| + \frac{2k^{n-1}}{1+k^n}|a_{n-1}|\phi(k)\right], \tag{4.1}$$

where $\phi(k) = \begin{cases} \dfrac{k^n-1}{n} - \dfrac{k^{n-2}-1}{n-2}, & for \quad n > 2 \\ \dfrac{(k-1)^2}{2}, & for \quad n = 2 \end{cases}$

and

$\psi(k) = \begin{cases} 1 - \dfrac{1}{k^2}, & for \quad n > 2 \\ 1 - \dfrac{1}{k}, & for \quad n = 2. \end{cases}$

Proof . Since all zeros of $P(z)$ lie in $|z| \leq k, k \geq 1$. Therefore $F(z) = P(kz)$ has all zeros in $|z| \leq 1$. Now $F(z) = P(kz) = a_n k^n \prod_{j=1}^{n}(z - \frac{z_j}{k})$, therefore for $F(z) \neq 0$, we have

$$\frac{zF'(z)}{F(z)} = \sum_{j=1}^{n}\frac{z}{z - \frac{z_j}{k}}.$$

Hence for the points $z = e^{i\theta}, 0 \leq \theta < 2\pi$ with $F(z) \neq 0$, we have

$$\begin{aligned} Re\left(\frac{e^{i\theta}F'(e^{i\theta})}{F(e^{i\theta})}\right) &= Re\left(\sum_{j=1}^{n}\frac{e^{i\theta}}{e^{i\theta} - z_j/k}\right) \\ &= Re\left(\sum_{j=1}^{n}\frac{1}{1 - e^{-i\theta}z_j/k}\right). \end{aligned}$$

Since $|z_j| \leq k$ and for $|w| = |e^{-i\theta}t_j| \leq 1, t_j = z_j/k, k \geq 1$, we have

$$\begin{aligned} Re\left(\frac{1}{1-w}\right) &= Re\left(\frac{1}{1-e^{-i\theta}t_j}\right) \\ &= Re\left(\frac{1}{1-|t_j|e^{-i(\theta-\phi)}}\right) \\ &= Re\left(\frac{1}{1-|t_j|cos(\theta-\phi)+i|t_j|sin(\theta-\phi)}\right) \\ &= \frac{1-|t_j|cos(\theta-\phi)}{1+|t_j|^2-2|t_j|cos(\theta-\phi)} \\ &\geq \frac{1}{1+|t_j|} \\ &= \frac{1}{1+|z_j|/k}. \end{aligned}$$

That is

$$(1-|t_j|\cos(\theta-\phi))(1+|t_j|) \geq 1+|t_j|^2-2|t_j|\cos(\theta-\phi).$$

If $(|t_j|-|t_j|^2)\cos(\theta-\phi) \geq |t_j|^2-|t_j|$. That is $\cos(\theta-\phi) \geq -1$, which is true. Therefore for $|z|=1$,

$$Re\left(\frac{zF'(z)}{F(z)}\right) \geq \sum_{j=1}^{n} \frac{1}{1+\frac{|z_j|}{k}} = \sum_{j=1}^{n} \frac{k}{k+|z_j|}.$$

This implies for $|z|=1$

$$\left|\frac{zF'(z)}{F(z)}\right| \geq Re\left(\frac{zF'(z)}{F(z)}\right) \geq \sum_{j=1}^{n} \frac{k}{k+|z_j|}.$$

Hence

$$|F'(z)| \geq \sum_{j=1}^{n} \frac{k}{k+|z_j|}|F(z)|.$$

Equivalently, we get

$$k\max_{|z|=1}|P'(kz)| \geq \sum_{j=1}^{n} \frac{k}{k+|z_j|}|P(kz)|.$$

That is

$$\max_{|z|=k}|P'(z)| \geq \sum_{j=1}^{n} \frac{1}{k+|z_j|}\max_{|z|=k}|P(z)|. \tag{4.2}$$

Applying Lemma 3.2 for $P'(z)$, where $P'(z)$ being a polynomial of degree $n-1$, we get for $R=k\geq 1$ and for $n>2$

$$\max_{|z|=k}|P'(z)| \leq k^{n-1}\max_{|z|=1}|P'(z)| - (k^{n-1}-k^{n-3})|a_1|. \tag{4.3}$$

Combining (4.2) and (4.3), we get

$$k^{n-1}\max_{|z|=1}|P'(z)| - (k^{n-1}-k^{n-3})|a_1| \geq \sum_{j=1}^{n} \frac{1}{k+|z_j|}\max_{|z|=k}|P(z)|. \tag{4.4}$$

Now using (3.3) of Lemma 3.3 with $l=0$ and $R=k\geq 1$ in (4.4), we get for $n>2$

$$\max_{|z|=1}|P'(z)| \geq \left(1-\frac{1}{k^2}\right)|a_1| + \frac{1}{k^{n-1}}\sum_{j=1}^{n}\frac{1}{k+|z_j|}\left[\frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)| + \frac{2k^{n-1}|a_{n-1}|}{k^n+1}\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)\right].$$

Equivalently, we get

$$\max_{|z|=1}|P'(z)| \geq \psi(k)|a_1| + \frac{1}{k^{n-1}}\sum_{j=1}^{n}\frac{1}{k+|z_j|}\left[\frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)| + \frac{2k^{n-1}}{1+k^n}|a_{n-1}|\phi(k)\right],$$

where $\phi(k)$ and $\psi(k)$ are defined in the statement of theorem. This proves the theorem for the case $n > 2$. For the case $n = 2$, the result follows on similar lines in view of part second of Lemma 3.2 and Lemma 3.3. Hence proof of Theorem 4.1 is completed. □

Remark 4.2. Inequality due to Turán [12] can be obtained as a special case from Theorem 4.1 if we put $k = 1$.

For a polynomial $P(z)$ of degree n, the polar derivative $D_\alpha : \mathcal{P}_n \to \mathcal{P}_n$ is a function defined as

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z).$$

It is to be observed that

$$\lim_{|\alpha|\to\infty}\frac{D_\alpha P(z)}{\alpha} = P'(z).$$

Aziz and Rather [3] established the following inequality for the polar derivative of a polynomial.

If $P(z)$ is a polynomial of degree n, having all zeros in $|z| \leq k, k \geq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$,

$$\max_{|z|=1}|D_\alpha P(z)| \geq n\left(\frac{|\alpha|-k}{1+k^n}\right)\max_{|z|=1}|P(z)|.$$

We next prove the following result on the polar derivative of a polynomial having all zeros in $|z| \leq k, k \geq 1$.

Theorem 4.3. If $P \in \mathcal{P}_n$, be such that $P(z)$ has all zeros in $|z| \leq k, k \geq 1$, then for $0 \leq l < 1$ and for any complex α with $|\alpha| \geq k$

$$\max_{|z|=1}|D_\alpha P(z)| \geq \psi(k)|na_0 + \alpha a_1| + \frac{(|\alpha|-k)}{k^{n-1}}\sum_{j=1}^{n}\frac{1}{k+|z_j|}\left\{\frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)| + \frac{2k^{n-1}}{1+k^n}|a_{n-1}|\phi(k)\right\}, \tag{4.5}$$

where $\phi(k)$ and $\psi(k)$ for both the cases $n > 2$ and $n = 2$ are defined in the statement of Theorem 4.1.

Proof . By given hypothesis all zeros of $P(z)$ lie in $|z| \leq k, k \geq 1$. Also

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z) = \alpha P'(z) + nP(z) - zP'(z).$$

Therefore

$$\begin{aligned}|D_\alpha P(z)| &\geq |\alpha||P'(z)| - |nP(z) - zP'(z)| \\ &= |\alpha||P'(z)| - |Q'(z)|.\end{aligned} \tag{4.6}$$

By using Lemma 3.1, we get for $|z| = 1$

$$|D_\alpha P(z)| \geq (|\alpha| - 1)|P'(z)|.$$

Now suppose $g(z) = P(kz)$, where all zeros of $g(z)$ lie in $|z| \leq 1$. Therefore

$$|D_{\alpha/k}P(kz)| \geq (|\alpha| - k)|P'(kz)|,$$

where

$$\max_{|z|=1}|D_{\alpha/k}P(kz)| = \max_{|z|=k}|D_\alpha P(z)| = \max_{|z|=k}|nP(z) + (\alpha - z)P'(z)|.$$

Hence

$$\max_{|z|=k}|D_\alpha P(z)| \geq (|\alpha| - k)\max_{|z|=k}|P'(z)|. \tag{4.7}$$

As $D_\alpha P(z)$ is a polynomial of degree at most $n-1$, therefore using (3.1) of Lemma 3.2, we get for $n>2$

$$\max_{|z|=k}|D_\alpha P(z)| \leq k^{n-1}\max_{|z|=1}|D_\alpha P(z)| - (k^{n-1}-k^{n-3})|na_0+\alpha a_1|. \tag{4.8}$$

Using (4.2) and (4.8) in (4.7), we get

$$k^{n-1}\max_{|z|=1}|D_\alpha P(z)| - (k^{n-1}-k^{n-3})|na_0+\alpha a_1| \geq (|\alpha|-k)\left(\sum_{j=1}^{n}\frac{1}{k+|z_j|}\max_{|z|=k}|P(z)|\right). \tag{4.9}$$

Again using 3.3 of Lemma 3.3 with $l=0$ and $R=k\geq 1$ in (4.9), we get for $n>2$

$$\max_{|z|=1}|D_\alpha P(z)| \geq \left(1-\frac{1}{k^2}\right)|na_0+\alpha a_1| + \frac{(|\alpha|-k)}{k^{n-1}}\sum_{j=1}^{n}\frac{1}{k+|z_j|}\left\{\frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)| + \frac{2k^{n-1}}{1+k^n}|a_{n-1}|\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)\right\}.$$

This proves the theorem for the case $n>2$. For the case $n=2$, the result follows on similar lines in view of part second of Lemma 3.2 and Lemma 3.3. Thus Theorem 4.3 is completely proved. □ If we divide both sides of (4.5) by $|\alpha|$ and letting $|\alpha|\to\infty$, it reduces to Theorem 4.1.

Remark 4.4. For $k=1$, we get an extension of a result due to Aziz [2] to a polar derivative of a polynomial.

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