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Existence of a sequence of solutions for a mixed boundary value problem for a complete Sturm-Liouville equation

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Abstract

This paper establishes the existence of infinitely many weak solutions for a mixed boundary value problem for a complete Sturm-Liouville equation. Our method relies on critical point theory, and we extend and enhance several recent findings. Additionally, we provide an example to illustrate the main results.

Keywords: Sturm-Liouville, Weak solutions, Infinitely many solutions, Variational methods
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1 Introduction

The aim of this paper is to investigate the existence of infinitely many weak solutions for the following problem

$$\begin{cases} -z'' + \gamma(\varsigma)z' + \delta(\varsigma)z = \lambda h(\varsigma, z(\varsigma)), & \varsigma \neq \varsigma_j,\ \varsigma \in (a,b), \\ \Delta(z'(\varsigma_j) = I_j(z(\varsigma_j)), & j = 1, 2, \ldots, d, \\ z(a) = z'(b) = 0 \end{cases} \tag{P^h}$$

where $\lambda > 0$, h is an L^1-Carathéodory function, and $\gamma, \delta \in L^\infty([a,b])$ are such that

$$\operatorname{ess}\inf_{\varsigma\in[a,b]} \delta(\varsigma) > -\left(\frac{\pi}{2(b-a)}\right)^2, \tag{1.1}$$

$a = \varsigma_0 < \varsigma_1 < \varsigma_1 < \ldots < \varsigma_{d-1} < \varsigma_d = b$, $\Delta(z'(\varsigma_j) = z'(\varsigma_j^+) - z'(\varsigma_j^-) = \lim_{s\to\varsigma_j^+} z'(s) - \lim_{s\to\varsigma_j^-} z'(s)$, $j = 1, 2, \ldots, d-1$ and the functions $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \ldots, d$ are continuous functions.

Authors often impose extra conditions on impulsive functions when investigating the existence of solutions for impulsive differential equations. However, many results established for equations without impulses rely on conditions that are not directly transferable, making the specific effect of the impulses less clear. As a result, the effects of impulses may not always be clearly evident. To address this gap, recent years have seen researchers examining the existence of solutions influenced by impulses [6, 7].

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In recent years, extensive research has been conducted on the existence of solutions for Sturm-Liouville problems with mixed boundary conditions. This field has produced many significant findings, and readers are encouraged to consult [3, 5, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18] and their associated references for more information. For example, D'Aguì in [8] employed variational methods to prove the existence of nontrivial solutions for a mixed boundary value problem related to the Sturm-Liouville equation. Furthermore, Averna et al. in [3] used multiple critical point theorems to establish the existence of three solutions for a Sturm-Liouville mixed boundary value problem. In this paper, we build on the results presented in [4], where, in contrast to other studies, the coefficients γ and δ are permitted to change signs. Bonanno et al. in [4] employed critical point theory to demonstrate the existence of infinitely many distinct positive solutions for the equation (P^h) when $I_j = 0$.

In this paper, we utilize critical theorems obtained in [14, Theorem 9.12], which we will recall in the next section (see Theorem 2.9), to discuss the existence of infinitely many weak solutions for the problem described in (P^h) (see Theorem 3.1). We present Example 3.5, where the hypotheses of Theorem 3.1 are satisfied. Additionally, Theorem 3.6 follows as a consequence of Theorem 3.1. Finally, as a special case of Theorem 3.1, we derive Theorem 3.8 for the case where h does not depend on ς.

2 Preliminaries

This section is devoted to introduce some basic notations and results which will be used in the proofs of our main results. Take the Sobolev space

$$\mathrm{E} = \left\{ z \in W^{1,2}([a,b]) : \ z(a) = 0 \right\}$$

endowed with the following norm:

$$\|z\| = \left(\int_a^b |z(s)|^2 \mathrm{d}s \right)^{\frac{1}{2}} + \left(\int_a^b |z'(s)|^2 \mathrm{d}s \right)^{\frac{1}{2}}.$$

Moreover, for all $z \in \mathrm{E}$, put

$$\|z\|_0 = \|z'\|_{L^2} = \left(\int_a^b |z'(s)|^2 \mathrm{d}s \right)^{\frac{1}{2}}.$$

The following proposition holds true.

Proposition 2.1. [4, Proposition 2.1] The norms $\|\cdot\|_0$ and $\|\cdot\|$ are equivalent on E.

Here, we point out the following result.

Proposition 2.2. [4, Proposition 2.2] (Poincaré inequalities) For all $z \in \mathrm{E}$, one has

(1) $\max_{\varsigma \in [a,b]} |z(\varsigma)| \leq (b-a)^{\frac{1}{2}} \|z\|_0$,

(2) $\|z\|_{L^2} \leq \dfrac{2(b-a)}{\pi} \|z'\|_{L^2}$.

Remark 2.3. [4, Remark 2.1] We observe that the Poincaré inequalities hold true in the Sobolev space $W^{1,2}([a,b])$, as given in [2], with different constants.

Now, let us introduce another norm in the space E, given by

$$\|z\|_{\mathrm{E}} = \left(\int_a^b e^{-\Phi(\varsigma)} |z'(\varsigma)|^2 \mathrm{d}\varsigma + \int_a^b e^{-\Phi(\varsigma)} \delta(\varsigma) |z(\varsigma)|^2 \mathrm{d}\varsigma \right)^{\frac{1}{2}},$$

where $\Phi(\varsigma) = \displaystyle\int_a^{\varsigma} \gamma(\xi) d\xi$, for all $\varsigma \in [a,b]$.

Proposition 2.4. [4, Proposition 2.3] Assume (1.1). Then $\|\cdot\|_{\mathrm{E}}$ is a norm on the space E and it is equivalent to $\|\cdot\|_0$. In particular, one has

$$m\|z\|_0 \leq \|z\|_{\mathrm{E}} \leq M\|z\|_0, \tag{2.1}$$

for all $z \in \mathrm{E}$, where m, M with $M \geq m > 0$, are given by

$$m = \begin{cases} \left(\min_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)}\right)^{\frac{1}{2}}, & \text{if} \quad \operatorname{ess\,inf}_{\varsigma\in[a,b]} \delta(\varsigma) \geq 0, \\ \left(\min_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \left(1 + \operatorname{ess\,inf}_{\varsigma\in[a,b]} \delta(\varsigma) \left(\dfrac{2(b-a)}{\pi}\right)^2\right)\right)^{\frac{1}{2}}, & \text{if} \quad \operatorname{ess\,inf}_{\varsigma\in[a,b]} \delta(\varsigma) < 0 \end{cases}$$

and

$$M = \begin{cases} \left(\max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \left(1 + \operatorname{ess\,sup}_{\varsigma\in[a,b]} \delta(\varsigma) \left(\dfrac{2(b-a)}{\pi}\right)^2\right)\right)^{\frac{1}{2}}, & \text{if} \quad \operatorname{ess\,inf}_{\varsigma\in[a,b]} \delta(\varsigma) \geq 0, \\ \left(\max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)}\right)^{\frac{1}{2}}, & \text{if} \quad \operatorname{ess\,inf}_{\varsigma\in[a,b]} \delta(\varsigma) < 0. \end{cases}$$

Remark 2.5. We observe that, since $\|z\|_0$ is equivalent to $\|z\|$, as proved in Proposition 2.1, thanks to the transitivity property, we obtain the equivalence between $\|z\|_{\mathrm{E}}$ and $\|z\|$.

Remark 2.6. The space E is a Hilbert space with the dot product

$$< z, v >= \int_a^b e^{-\Phi(\varsigma)} z'(\varsigma)v'(\varsigma)\mathrm{d}\varsigma + \int_a^b e^{-\Phi(\varsigma)}\delta(\varsigma)z(\varsigma)v(\varsigma)\mathrm{d}\varsigma$$

that clearly induces the norm $\|z\|_{\mathrm{E}}$.

Remark 2.7. Taking into account (1) in Proposition 2.4 and (2.1), the following inequality holds:

$$\max_{\varsigma\in[a,b]} |z(\varsigma)| \leq \frac{(b-a)^{\frac{1}{2}}}{m}\|z\|_{\mathrm{E}}, \quad \forall z \in \mathrm{E}. \tag{2.2}$$

Definition 2.8. A function $z \in \mathrm{E}$ is called a weak solution of problem (P^h), if

$$\int_a^b e^{-\Phi(\varsigma)} z'(\varsigma)v'(\varsigma)\mathrm{d}\varsigma + \int_a^b e^{-\Phi(\varsigma)}\delta(\varsigma)(z(\varsigma))v(\varsigma)\mathrm{d}\varsigma - \lambda \int_a^b e^{-\Phi(\varsigma)}h(\varsigma, z(\varsigma))v(\varsigma)\mathrm{d}\varsigma$$

holds for any $v \in \mathrm{E}$.

Put

$$H(\varsigma, \zeta) = \int_0^\zeta h(\varsigma, x)\mathrm{d}x \quad \text{for any } (\varsigma, \zeta) \in (a, b) \times \mathbb{R}.$$

For each $z \in \mathrm{E}$, consider the functional φ defined on E by

$$\varphi(z) = \frac{1}{2}\|z\|_{\mathrm{E}}^2 + \sum_{j=1}^{d} \int_0^{z(\varsigma_j)} I_j(s)\mathrm{d}s - \lambda \int_a^b e^{-\Phi(\varsigma)} H(\varsigma, z(\varsigma))\mathrm{d}\varsigma. \tag{2.3}$$

It is clear that φ is differentiable at any $z \in \mathrm{E}$ and

$$\varphi'(z)(v) = \int_a^b e^{-\Phi(\varsigma)} z'(\varsigma)v'(\varsigma)\mathrm{d}\varsigma + \int_a^b e^{-\Phi(\varsigma)}\delta(\varsigma)z(\varsigma)v(\varsigma)\mathrm{d}\varsigma + \sum_{j=1}^{d} I_j(z(\varsigma_j))v(\varsigma_j) - \lambda \int_a^b e^{-\Phi(\varsigma)}h(\varsigma, z(\varsigma))v(\varsigma)\mathrm{d}\varsigma, \tag{2.4}$$

for any $v \in \mathrm{E}$. Obviously, φ' is continuous. We formulate our main results on the existence infinitely many weak solutions for the problem (P^h). Our main tool to ensure the results is Theorem 9.12 of [14] that we now recall here.

Theorem 2.9. [14, Theorem 9.12] Let X be an infinite dimensional real Banach space. Let $\varphi \in C^1(X,\mathbb{R})$ be an even functional which satisfies the Palais-Smale condition, and $\varphi(0)=0$. Suppose that $X = V \oplus \mathrm{E}$, where V is infinite dimensional, and φ satisfies that

(i) there exist $\alpha > 0$ and $\rho > 0$ such that $\varphi(z) \geq \alpha$ for every $z \in \mathrm{E}$ with $\|z\| = \rho$,

(ii) for any finite dimensional subspace $W \subset X$, there is $R = R(W)$ such that $\varphi(z) \leq 0$ on $W \setminus B_{R(W)}$.

Then φ possesses an unbounded sequence of critical values.

Theorem 2.9 has been successfully used to ensure the existence of infinitely many solutions for boundary value problems in the papers [1, 20].

3 Main result

In this section, we present our main results regarding the existence of infinitely many weak solutions for the problem described in (P^h).

Theorem 3.1. Assume that the following conditions hold true:

(A_1) there exists a constant $\nu > 2$ such that

$$0 < \nu H(\varsigma, z) \leq z h(\varsigma, z) \quad \text{for every} \quad (\varsigma, z) \in (a,b) \times \mathbb{R}\backslash\{0\},$$

(A_2)

$$0 < z I_j(z) \leq \nu \int_0^z I_j(s)\mathrm{d}s \text{ for every } z \in \mathbb{R}\backslash\{0\},\ j = 1, \ldots, d.$$

Moreover, if $h(\varsigma, z)$ and I_j $(j = 1, \ldots, d)$ are odd about z, then the impulsive problem (P^h) has infinitely many weak solutions for $\lambda > 0$.

Lemma 3.2. Assume that $(\mathrm{A}_1) - (\mathrm{A}_2)$ hold and $\lambda > 0$. Then $\varphi(z)$ satisfies the Palais-Smale condition.

Proof . Let $\{z_n\}$ be a sequence in E such that $\{\varphi(z_n)\}$ is bounded and $\varphi'(z_n) \to 0$ as $n \to +\infty$. First, we prove that $\{z_n\}$ is bounded. By (2.3) and (2.4), one has

$$\begin{aligned}\nu\varphi(z_n) - \varphi'(z_n)(z_n) = \left(\frac{\nu}{2} - 1\right)\|z_n\|_{\mathrm{E}}^2 + \nu \sum_{j=1}^{d} \int_0^{z_n(\varsigma_j)} I_j(s)\mathrm{d}s - \sum_{j=1}^{d} I_j(z_n(\varsigma_j))z_n(\varsigma_j) \\ - \lambda \int_a^b e^{-\Phi(\varsigma)} \left(\nu H(\varsigma, z_n(\varsigma)) - h(\varsigma, z_n(\varsigma))z_n(\varsigma)\right) \mathrm{d}\varsigma.\end{aligned}$$

According to conditions (A_1) and (A_2), it follows that

$$\nu \sum_{j=1}^{d} \int_0^{z_n(\varsigma_j)} I_j(s)\mathrm{d}s - \sum_{j=1}^{d} I_j(z_n(\varsigma_j))z_n(\varsigma_j) \geq 0 \tag{3.1}$$

and

$$\lambda \left(\int_a^b e^{-\Phi(\varsigma)} h(\varsigma, z_n(\varsigma))z_n(\varsigma)\mathrm{d}\varsigma - \nu \int_a^b e^{-\Phi(\varsigma)} H(\varsigma, z_n(\varsigma))\mathrm{d}\varsigma \right) \geq 0. \tag{3.2}$$

By examining equations (3.1) through (3.2), we can deduce that

$$\nu\varphi(z_n) - \varphi'(z_n)(z_n) \geq \left(\frac{\nu}{2} - 1\right)\|z_n\|_{\mathrm{E}}^2.$$

Given that $\nu > 2$, it follows that the sequence $\{z_n\}$ is bounded. Therefore, since E is a reflexive Banach space, we can assert, up to a subsequence, that

$$z_n \rightharpoonup z \quad \text{in E},$$
$$z_n \to z \quad \text{in} \quad C([a,b])$$

and

$$z_n \to z \quad \text{a.e. on} \quad [a,b].$$

Since $\varphi'(z_n) \to 0$ and $z_n \to z$, we obtain that

$$(\varphi'(z_n) - \varphi'(z))\,(z_n - z) \to 0.$$

Due to the continuity of h and I_j $(j = 1, \ldots, d)$, we can conclude that

$$\int_a^b e^{-\Phi(\varsigma)}\,(h(\varsigma, z_n(\varsigma)) - h(\varsigma, z(\varsigma)))\,(z_n(\varsigma) - z(\varsigma))\mathrm{d}\varsigma \to 0 \quad \text{as } n \to \infty$$

and

$$\sum_{j=1}^{d} (I_j(z_n(\varsigma_j)) - I_j(z(\varsigma_j)))\,(z_n(\varsigma_j) - z(\varsigma_j)) \to 0 \quad \text{as } n \to \infty.$$

Furthermore, a straightforward calculation demonstrates that

$$\begin{aligned}
(\varphi'(z_n) - \varphi'(z))\,(z_n - z) &= \int_a^b e^{-\Phi(\varsigma)}|z_n'(\varsigma) - z'(\varsigma)|^2\mathrm{d}\varsigma + \int_a^b e^{-\Phi(\varsigma)}\delta(\varsigma)|z_n(\varsigma) - z(\varsigma)|^2\mathrm{d}\varsigma \\
&+ \sum_{j=1}^{d} (I_j(z_n(\varsigma_j) - I_j(z(\varsigma_j))\,(z_n(\varsigma_j) - z(\varsigma_j)) \\
&- \lambda \int_a^b e^{-\Phi(\varsigma)}(h(\varsigma, z_n(\varsigma) - h(\varsigma, z(\varsigma))(z_n(\varsigma) - z(\varsigma))\mathrm{d}\varsigma \\
&\geq \|z_n - z\|_{\mathrm{E}}^2 \qquad \text{as } n \to \infty.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} (\varphi'(z_n) - \varphi'(z))\,(z_n - z) \geq \lim_{n\to\infty} \|z_n - z\|_{\mathrm{E}}^2.$$

So $\|z_n - z\|_{\mathrm{E}} \to 0$ as $n \to \infty$, which implies that $\{z_n\}$ converges strongly to z in E. Therefore, φ satisfies the Palais-Smale condition. □

Lemma 3.3. [1] Assume that (A_1) holds, it follows that for each $\varsigma \in (a,b)$, the following inequalities are satisfied:

$$H(\varsigma, z) \leq H\left(\varsigma, \frac{z}{|z|}\right)|z|^\nu \;\; \text{if} \;\; 0 < |z| \leq 1$$

and

$$H(\varsigma, z) \geq H\left(\varsigma, \frac{z}{|z|}\right)|z|^\nu \;\; \text{if} \;\; |z| \geq 1.$$

In light of Lemma 3.3, (A_1) implies that for each $\varsigma \in (a,b)$

$$H(\varsigma, z) \leq \mathbf{M}|z|^\nu \;\; \text{if} \;\; 0 < |z| \leq 1 \tag{3.3}$$

and

$$H(\varsigma, z) \geq \mathbf{m}|z|^\nu \;\; \text{if} \;\; |z| \geq 1, \tag{3.4}$$

where $\mathbf{M} = \max_{\varsigma\in[a,b],|z|=1} H(\varsigma, z)$, $\mathbf{m} = \min_{\varsigma\in[a,b],|z|=1} H(\varsigma, z)$. Thanks to (A_1), one has $\mathbf{M} > 0$ and $\mathbf{m} > 0$. Since $H(\varsigma, z) - \mathbf{m}|z|^\beta$ is continuous on $(a,b) \times [-1,1]$, there exists a constant C_2 such that

$$H(\varsigma, z) \geq \mathbf{m}|z|^\nu - C_2 \quad \text{for each } (\varsigma, z) \in (a,b) \times [-1,1]. \tag{3.5}$$

So it follows from (3.4) and (3.5) that

$$H(\varsigma, z) \geq \mathbf{m}|z|^\nu - C_2 \quad \text{for each } (\varsigma, z) \in (a,b) \times \mathbb{R}. \tag{3.6}$$

Remark 3.4. From (A_2), it follows that there are constants $C_3, C_4 > 0$ such that

$$\int_0^z I_j(s)\mathrm{d}s < C_3|z|^\nu + C_4, \tag{3.7}$$

for every $z \in \mathbb{R}$ and $j = 1, 2, \dots, d$.

Proof .[**Proof of Theorem 3.1**] Using the continuity of h and I_j $(j = 1, \dots, d)$, we obtain that $\varphi(z)$ is continuously differentiable. In view of (2.3), it is obvious that $\varphi(z)$ is even and $\varphi(0) = 0$. By Lemma 3.2, $\varphi(z)$ satisfies the Palais-Smale condition. First, we show that φ satisfies condition (i) in Theorem 2.9. For any $z \in \mathrm{E}$, by (2.3) and (A_2), one has

$$\begin{aligned}\varphi(z) =&\frac{1}{2}\|z\|_{\mathrm{E}}^2 + \sum_{j=1}^{d}\int_0^{z(\varsigma_j)} I_j(s)\mathrm{d}s - \lambda\int_a^b e^{-\Phi(\varsigma)}H(\varsigma, z(\varsigma))\mathrm{d}\varsigma\\ \geq&\frac{1}{2}\|z\|_{\mathrm{E}}^2 - \lambda\int_a^b e^{-\Phi(\varsigma)}H(\varsigma, z(\varsigma))\mathrm{d}\varsigma.\end{aligned}$$

It is clear that $\|z\|_{\mathrm{E}} \leq \dfrac{m}{(b-a)^{\frac{1}{2}}}$ implies $\|z\|_\infty \leq 1$. Thanks to (3.3), one has

$$\begin{aligned}\int_a^b e^{-\Phi(\varsigma)}H(\varsigma, z(\varsigma))\mathrm{d}\varsigma \leq&\mathbf{M}\int_a^b e^{-\Phi(\varsigma)}|z|^\nu \mathrm{d}\varsigma\\ \leq&\mathbf{M}\max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)}\frac{(b-a)^{\frac{\nu+2}{2}}}{m^\nu}\|z\|_{\mathrm{E}}^\nu,\ \|z\|_{\mathrm{E}} \leq \frac{m}{(b-a)^{\frac{1}{2}}}.\end{aligned} \tag{3.8}$$

Combining (3.8) and (A_1), one has

$$\varphi(z) \geq \frac{1}{2}\|z\|_{\mathrm{E}}^2 - \lambda\mathbf{M}\max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)}\frac{(b-a)^{\frac{\nu+2}{2}}}{m^\nu}\|z\|_{\mathrm{E}}^\nu,$$

which implies that we can choose $\rho > 0$ small enough such that $\varphi(z) \geq \alpha > 0$ with $\|z\|_{\mathrm{E}} = \rho$. Second, we show that φ satisfies condition (ii) in Theorem 2.9. Let $W \subset \mathrm{E}$ is a finite dimensional subspace. For every $r \in \mathbb{R}\setminus\{0\}$ and $z \in W\setminus\{0\}$ with $\|z\|_{\mathrm{E}} = 1$, by (3.6)-(3.7), we can imply that

$$\begin{aligned}\varphi(rz) =& \frac{1}{2}\|rz\|_{\mathrm{E}}^2 + \sum_{j=1}^{d}\int_0^{rz(\varsigma_j)} I_j(s)\mathrm{d}s - \lambda\int_a^b e^{-\Phi(\varsigma)}H(\varsigma, rz(\varsigma))\mathrm{d}\varsigma\\ \leq& \left(\frac{r^2}{2}\right)\|z\|_{\mathrm{E}}^2 + \sum_{j=1}^{d}(C_3|r|^\nu|z(\varsigma)|^\nu + C_4) - \lambda|r|^\nu\mathbf{m}\min_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)}\frac{(b-a)^{\frac{\nu+2}{2}}}{m^\nu}\|z\|_{\mathrm{E}}^\nu + \lambda C_2\max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)}(b-a).\end{aligned}$$

Noting that $\nu > 2$, the above inequality implies that there exists r_0 such that $\|rz\|_{\mathrm{E}} > \rho$ and $\varphi(rz) < 0$ for each $r \geq r_0 > 0$. Since W is a finite dimensional subspace, there exists $R(W) > 0$ such that $\varphi(z) \leq 0$ on $W \setminus B_{R(W)}$. According to Theorem 2.9, the functional $\varphi(z)$ possesses infinitely many critical points, i.e. the impulsive problem (P^h) has infinitely many weak solutions.□ Now we illustrate Theorem 3.1 by presenting the following example.

Example 3.5. Consider the problem

$$\begin{cases} -z'' + z' - z = \lambda h(z(\varsigma)), \quad \varsigma \neq \dfrac{1}{3},\ \varsigma \in (0,1),\\ \Delta(z'(\dfrac{1}{3}) = I_1(z(\dfrac{1}{3})),\\ z(0) = z'(1) = 0,\end{cases} \tag{3.9}$$

where $I_1(\zeta) = \dfrac{1}{4}\zeta$ for every $\zeta \in \mathbb{R}$ and

$$h(\zeta) = \zeta^3 + \zeta^3 e^{\zeta^4} + \zeta^7 e^{\zeta^4}$$

for each $\zeta \in \mathbb{R}$. By the expression of h, we have

$$H(\zeta) = \frac{\zeta^4}{4} + \frac{\zeta^4 e^{\zeta^4}}{4}$$

for each $\zeta \in \mathbb{R}$. By choosing $\nu = 3 > 2$, the assumptions (A_1) and (A_2) are fulfilled. Since $h(z)$ and I_1 are odd about z, we clearly see that all assumptions of Theorem 3.1 are fulfilled. Therefore, the impulsive problem (3.9) has infinitely many weak solutions.

Theorem 3.6. Suppose that the following conditions are satisfied:

(A_3) there exist constants $R > 0$ and $0 < \lambda L_1 < \dfrac{1}{\max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \frac{(b-a)^2}{m^2}}$ such that

$$H(\varsigma, z) \le L_1|z|^2 \text{ for every } (\varsigma, z) \in (a, b) \times \mathbb{R},\ |z| \le R,$$

(A_4) $H(\varsigma, z) \ge 0$ for every $(\varsigma, z) \in (a, b) \times \mathbb{R}$ and there exist constants $R_1 > 0$, $\delta_1 > 0$ and $\alpha_1 > \nu$ such that

$$H(\varsigma, z) \ge \delta_1|z|^{\alpha_1} \quad \text{for every} \quad (\varsigma, z) \in (a, b) \times \mathbb{R},\ |z| \ge R_1,$$

(A_5) there exists a constant $\nu > 2$, $\delta_1 \ge 0$ and $0 < \alpha_2 < 2$ such that

$$\nu H(\varsigma, z) - zh(\varsigma, z) \le \delta_2|z|^{\alpha_2} \quad \text{for every} \quad (\varsigma, z) \in (a, b) \times \mathbb{R},$$

(A_6)

$$0 < zI_j(z) \le \nu \int_0^z I_j(s)\mathrm{d}s \text{ for every } z \in \mathbb{R}\backslash\{0\}, \quad j = 1, \ldots, d.$$

Moreover, if $h(\varsigma, z)$ and I_j $(j = 1, \ldots, d)$ are odd about z, then the impulsive problem (P^h) has infinitely many weak solutions for $\lambda > 0$.

Lemma 3.7. Under the conditions outlined in Theorem 3.6, the function $\varphi(z)$ meets the Palais-Smale condition.

Proof . Let $\{z_n\}$ be a sequence in E such that $\{\varphi(z_n)\}$ is bounded and $\varphi'(z_n) \to 0$ as $n \to +\infty$. By (2.3), (2.4) and (A_5), one has

$$\begin{aligned}\nu\varphi(z_n) - \varphi'(z_n)(z_n) = &\left(\frac{\nu}{2} - 1\right) \|z_n\|^2_{\mathrm{E}} + \nu \sum_{j=1}^{d} \int_0^{z_n(\varsigma_j)} I_j(s)\mathrm{d}s - \sum_{j=1}^{d} I_j(z_n(\varsigma_j))z_n(\varsigma_j)\\
&- \lambda \int_a^b e^{-\Phi(\varsigma)} \left(\nu H(\varsigma, z_n(\varsigma)) - h(\varsigma, z_n(\varsigma))z_n(\varsigma)\right) \mathrm{d}\varsigma\\
\ge &\left(\frac{\nu}{2} - 1\right) \|z_n\|^2_{\mathrm{E}} - \lambda\delta_2 \max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \frac{(b-a)^{\frac{\alpha_2+2}{2}}}{m^{\alpha_2}} \|z\|^{\alpha_2}_{\mathrm{E}}.\end{aligned}$$

Since $0 < \alpha_2 < 2$ it follows that $\{z_n\}$ is bounded on E. The proof of the Palais-Smale condition is similar to that in Lemma 3.2. We omit it here. □

Proof .[**Proof of Theorem 3.6**] Using the continuity of h and I_j $(j = 1, \ldots, d)$, we obtain that $\varphi(z)$ is continuously differentiable. In view of (2.3), it is obvious that $\varphi(z)$ is even and $\varphi(0) = 0$. By Lemma 3.7, $\varphi(z)$ satisfies the Palais-Smale condition. First, we show that φ satisfies condition (i) in Theorem 2.9. From (2.2), we have

$$\|z\|_{\mathrm{E}} \le \frac{m}{(b-a)^{\frac{1}{2}}} R$$

implies $\|z\|_\infty \le R$. Thanks to (A_3), one has

$$\begin{aligned}\lambda \int_a^b e^{-\Phi(\varsigma)} H(\varsigma, z(\varsigma))\mathrm{d}\varsigma \le &\lambda L_1 \int_a^b e^{-\Phi(\varsigma)}|z(\varsigma)|^2\mathrm{d}\varsigma\\
\le &\lambda L_1 \max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \frac{(b-a)^2}{m^2}\|z\|^2_{\mathrm{E}},\ \|z\|_{\mathrm{E}} \le \frac{m}{(b-a)^{\frac{1}{2}}} R.\end{aligned} \tag{3.10}$$

Combining (3.10) and (A_6), one has

$$\begin{aligned}\varphi(z) \geq& \frac{1}{2}\|z\|_{\mathrm{E}}^2 - \lambda L_1 \max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \frac{(b-a)^2}{m^2}\|z\|_{\mathrm{E}}^2\\ =& \left(\frac{1}{2} - \lambda L_1 \max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \frac{(b-a)^2}{m^2}\right)\|z\|_{\mathrm{E}}^2,\end{aligned}$$

which implies that we can choose $\rho > 0$ small enough such that $\varphi(z) \geq \alpha > 0$ with $\|z\|_{\mathrm{E}} = \rho$. Second, we show that φ satisfies condition (ii) in Theorem 2.9. Let $W \subset \mathrm{E}$ is a finite dimensional subspace. According to Remark 3.4, it follows from (A_6) that there exist constants $C_3, C_4 > 0$ such that

$$\int_0^z I_j(s)\mathrm{d}s < C_3|z|^{\nu} + C_4 \tag{3.11}$$

for every $z \in \mathbb{R}$. By (A_4), we can imply that there exists a constant $C_5 > 0$ such that

$$H(\varsigma, z) \geq \delta_1|z|^{\alpha_1} - C_5 \quad \text{for each} \quad (\varsigma, z) \in (a,b)\times\mathbb{R}. \tag{3.12}$$

For every $r \in \mathbb{R}\setminus\{0\}$ and $z \in W\setminus\{0\}$ with $\|z\|_{\mathrm{E}} = 1$, by (3.11)-(3.12), we can imply that

$$\begin{aligned}\varphi(rz) =& \frac{1}{2}\|rz\|_{\mathrm{E}}^2 - \sum_{j=1}^{d}\int_0^{rz(\varsigma_j)} I_j(s)\mathrm{d}s - \lambda\sum_{j=0}^{d}\int_{s_j}^{\varsigma_{j+1}} H(\varsigma, rz(\varsigma))\mathrm{d}\varsigma\\ \leq& \left(\frac{r^2}{2}\right)\|z\|_{\mathrm{E}}^2 + \sum_{j=1}^{d} C_3|r|^{\nu}|z(\varsigma)|^{\nu} + C_4 - \lambda|r|^{\alpha_1}\delta_1\int_a^b e^{-\Phi(\varsigma)}|z(\varsigma)|^{\alpha_1}\mathrm{d}\varsigma + \lambda C_5.\end{aligned}$$

Noting that $\alpha_1 > \nu$, the above inequality implies that there exists r_0 such that $\|ru\| > \rho$ and $\varphi(rz) < 0$ for each $r \geq r_0 > 0$. Since W is a finite dimensional subspace, there exists $R(W) > 0$ such that $\varphi(z) \leq 0$ on $W \setminus B_{R(W)}$. □
Finally, we present a straightforward implication of Theorem 3.1 for the case where h is independent of ς.

Theorem 3.8. Suppose that the following conditions are satisfied:

(A_7) there exists a constant $\nu > 2$ such that

$$0 < \nu H(z) \leq zh(z) \quad \text{for every} \quad z \in \mathbb{R}\backslash\{0\},$$

(A_8)

$$0 < zI_j(z) \leq \nu\int_0^z I_j(s)\mathrm{d}s \quad \text{for every} \quad z \in \mathbb{R}\backslash\{0\},\ j = 1,\ldots,d.$$

Moreover, if $h(z)$ and I_j $(j = 1,\ldots,d)$ are odd about z, then the impulsive problem

$$\begin{cases} -z'' + \gamma(\varsigma)z' + \delta(\varsigma)z = \lambda h(z(\varsigma)), & \varsigma \neq \varsigma_j,\ \varsigma \in (a,b),\\ \Delta(z'(\varsigma_j) = I_j(z(\varsigma_j)), & j = 1,2,\ldots,d,\\ z(a) = z'(b) = 0 \end{cases}$$

has infinitely many weak solutions for $\lambda > 0$.

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The authors declare that there are no conflicts of interest.

References

[1] G.A. Afrouzi, S. Moradi, and G. Caristi, *Infinitely many solutions for impulsive nonlocal elastic beam equations,* Differ. Equ. Dyn. Syst. **30** (2022), 287–300.

[2] E. Amoroso, G. Bonanno, G. Molica Bisci, G. D'Aguì, S. De Caro, S. Foti, D. ÓRegan, and A. Testa, *Second order differential equations for the power converters dynamical performance analysis,* Math. Methods Appl. Sci. **45** (2022), 5573–5591.

[3] D. Averna, N. Giovannelli, and E. Tornatore, *Existence of three solutions for a mixed boundary value problem with the Sturm-Liouville equation,* Bull. Korean Math. Soc. **49** (2012) 1213-1222.

[4] G. Bonanno, G. D'Aguì, and V. Morabito, *Mixed boundary value problems involving Sturm-Liouville differential equations with possibly negative coefficients,* Bound. Value Probl. **2024** (2009), 1-16.

[5] G. Bonanno and E. Tornatore, *Infinitely many solutions for a mixed boundary value problem,* Ann. Pol. Math. **99** (2010), 285–293.

[6] L. Chen and J. Sun, *Nonlinear boundary value problem for first order impulsive functional differential equations,* J. Math. Anal. Appl. **318** (2006), 726–741.

[7] J. Chu and J.J. Nieto, *Impulsive periodic solutions of first-order singular differential equations,* Bull. Lond. Math. Soc. **40** (2008), 143–150.

[8] G. D'Aguì, *Existence results for a mixed boundary value problem with Sturm-Liouville equation,* Adv. Pure Appl. Math. **2** (2011), 237–248.

[9] G. D'Aguì, *Multiplicity results for nonlinear mixed boundary value problem,* Bound. Value Prob. **2012** (2012), 1–12.

[10] G. D'Aguì, A. Sciammetta, and E. Tornatore, *Two non-zero solutions for Sturm-Liouville equations with mixed boundary conditions,* Nonlinear Anal. RWA **47** (2019), 324–331.

[11] D. Ehsani, M.R. Mokhtarzadeh, and A. Razani, *Expansions of Dirichlet to Neumann operators under perturbations,* Appl. Anal. **89** (2010), no. 5, 789–799.

[12] H. El-Houari, L.S. Chadli, and H. Moussa, *Multiple solutions in fractional Orlicz-Sobolev Spaces for a class of nonlocal Kirchhoff systems,* Filomat **38** (2024), no. 8, 2857–2875.

[13] M. Makvand Chaharlang and A. Razani, *Two weak solutions for some Kirchhoff-type problem with Neumann boundary condition,* Georgian Math. J. **28** (2021), no. 3, 429–438.

[14] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations,* in: CBMS Reg. Conf. Ser. Math., vol. 65, Amer. Math. Soc., Providence, RI, 1986.

[15] M.A. Ragusa, A. Razani, andF. Safari, *Existence of radial solutions for a $p(x)$-Laplacian Dirichlet problem,* Adv. Difference Equ. **2021** (2021), 1–14.

[16] F. Safari and A. Razani, *Nonlinear nonhomogeneous Neumann problem on the Heisenberg group,* Appl. Anal. **101** (2022), no. 7, 2387–2400.

[17] F. Safari and A. Razani, *Positive weak solutions of a generalized supercritical Neumann problem,* Iran. J. Sci. Technol. Trans. A: Sci. **44** (2020), no. 6, 1891–1898.

[18] R. Salvati, *Multiple solutions for a mixed boundary value problem,* Math. Sci. Res. J. **7** (2003), 275–283.

[19] J. Sun, H. Chen, J.J. Nieto, and M. Otero-Novoa, *Multiplicity of solutions for perturbed second-order Hamiltonian systems with impulsive effects,* Nonlinear Anal. TMA **72** (2010), 4575–4586.

[20] D. Zhang, B. Dai, *Infinitely many solutions for a class of nonlinear impulsive differential equations with periodic boundary conditions,* Compu. Math. Appl. **61** (2011), 3153–3160.