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Investigating fractional derivative coefficients in optimal control problems using Bell's polynomials to solve them numerically

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Abstract

In this research, we present a numerical method for solving fractional optimal control problems (FOCPs), and the working method is based on Bell's polynomial. Derivatives and fractional integrals of Caputo and Riemann type have been used. To obtain approximate solutions, we consider the fractional derivative of the state function and self-control as functions of Bell polynomials with unknown coefficients, and in the process of this work, we see the emergence of the derivative and integral operator matrices in terms of the fractional parameter. Finally, using a numerical method such as Lagrangian coefficients, the optimal control problem leads to a system of equations, from the solution of which all the unknowns will be found. All solutions are compared with each other on the basis of ordinary and fractional derivatives and the results are interesting.

Keywords: FOCPs, fractional integration and derivative, Bell Polynomials, Operational matrices.
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1 Introduction

Fractional optimal control problem (FOCP) is an optimal control problem (OCP) in which the differential equations governing the dynamics of the system contain at least one fractional derivative operator. The definition of fractional derivative is given by many authors. The commonly used definitions are the Riemann-Liouville (R-L) and Caputo [8, 11, 17, 24]. Jumari modified the left-hand fractional derivative to prevent the fractional derivative of constant functions from being non-zero, which in turn is valuable and useful [14]. Because the complexity or impossibility of analytically solving fractional optimal control problems (FOCPs) has become their absolute belief in most cases. Therefore, they use numerical methods to solve them. The numerical methods are based on the function approximation using operational matrix. So that, it is important to choose the method and accuracy of calculations. In recent years, different polynomials with various operation matrices have been presented by researchers. Including: The numerical solution for FOCPs, using Mott polynomials in the two papers presented by [1]. The numerical solution for FOCPs by Hermite polynomials provided by [31]. On the numerical solution for FOCPS, with Bell polynomials basis is

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discussed by [9]. Numerical evaluation of order six for fractional differential equations has been used by [3]. High order algorithms for numerical solution of fractional differential equations developed with [4]. New predictor-corrector approach for nonlinear fractional differential equations has been introduced by [5]. New views of the B-M fractional model based on the M-L law constructed with [26]. An efficient computational technique for fractal vehicular traffic flow has been designed by [20]. A fractional operator from the point of view of the M-L kernel in connection with the regular I-W equation is presented by [21]. The Ambartsumian equation with the fractional model has been investigated by [22]. A new approach for the NFOPCs with external persistent disturbances proposed by [16]. An efficient numerical solution of FOCPs by using the Ritz Method and Bernstein operational matrix has been provided by [25]. For the numerical solution of FOCP, a numerical approach based on Legendre canonical polynomial was introduced by [13]. Through the Bell polynomial, the solution of Fredholm-Volterra nonlinear integral equations is provided by [24]. Therefore, how to obtain the solution and also the interpretation of problems related to fractional differential equations is an active and developing field in applied mathematics. Our main focus in this research is on FOCPs with quadratic performance index and Caputo-type fractional dynamic system, and finally we convert them into a system of equations using Bell polynomials (BPs) as a basis. In this approximate method, the unknown functions $x(t)$ and $u(t)$ are expanded based on the unknown coefficients from the BPs base. The BPs were studied extensively by [6]. By using Bell base polynomials and finally fractional and ordinary integrated operational matrices, we solve the relevant problems without using Hamiltonian formulas. The formulation of the problem is given below.

$$Z = \int_a^b A_x(t)x^2(t) + B_u(t)u^2(t)dt, \tag{1.1}$$

$$s.t: \quad M_1\dot{x}(t) + M_2 \, {}_a^cD_t^\nu x(t) = h(t, x(t), u(t)), \tag{1.2}$$
$$x(a) = x_a,$$

where $A_x(t)$, $B_u(t)$ and $K(t)$ may be constants or functions of time, (M_1 and M_2) are not zero together. They are used in various branches of mathematics and are often used in solving differential equations, approximation theory and integral equations. To demonstrate the application and effectiveness of FOCPs in solving practical or physical problems where vibration and slip are important in terms of their controllability. Various examples are included to show the application and importance of the new approach. The work done in this paper is arranged as follows: In Section 2, we present an introduction to fractional calculus and required symbols. Important properties of BPs and necessary formulas in this paper are given in Section 3. Discussions of function approximation and operational matrix are reported in sections 4 and 5, respectively. Convergence is given in Section 6. Current technique for solving FOCPs is given in Section 7. Some examples with their solutions are given in Section 8 and the conclusion is written in section 9.

2 An introduction to fractional calculations and required symbols

Definitions of Riemann-Liouville fractional integrals (R-LFI), Riemann-Liouville fractional derivatives (R-LFD) and Caputo fraction derivative (CFD) used in this paper are mostly defined here, [8, 23, 26].

Definition 2.1. (R-LFI). The R-LFI of order $\nu > 0$ for arbitrary and definite f are defined as

$${}_0I_t^\nu f(t) = \begin{cases} \frac{1}{\Gamma(\nu)} \int_a^t \frac{f(s)}{(t-s)^{1-\nu}} ds & \nu > 0 \\ f(t) & \nu = 0, \end{cases} \tag{2.1}$$

and

$${}_tI_b^\nu f(t) = \begin{cases} \frac{1}{\Gamma(\nu)} \int_t^b \frac{f(s)}{(s-t)^{1-\nu}} ds & \nu > 0 \\ f(t) & \nu = 0, \end{cases} \tag{2.2}$$

where Γ is Euler's gamma function, that is

$$\Gamma(t) = \int_0^\infty t^{z-1}e^{-t}dt, \quad a < z < b.$$

Definition 2.2. (R-LFD). Let $\nu > 0$ for $q-1 < \nu \leq q$, $q \in N$. The R-LFD of order $\nu > 0$ for arbitrary and definite function f are defined as

$${}_aD_t^\nu f(t) := \frac{1}{\Gamma(q-\nu)} \frac{d^q}{dx^q} \int_a^t (t-s)^{q-1-\nu} f(s)ds, \tag{2.3}$$

and

$$ {}_tD_b^{\nu} f(t) := \frac{(-1)^q}{\Gamma(q-\nu)} \frac{d^q}{dx^q} \int_t^b (s-t)^{q-1-\nu} f(s)ds. \tag{2.4} $$

Definition 2.3. (CFD). Let $\nu > 0$ for $q-1 < \nu \leq q$, $q \in N$. The CFD of order $\nu > 0$ for arbitrary and definite function f are defined as

$$ {}_a^cD_t^{\nu} f(t) = \frac{1}{\Gamma(q-\nu)} \int_a^t (t-s)^{q-\nu-1} f^{(q)}(s)ds, \tag{2.5} $$

and

$$ {}_t^cD_b^{\nu} f(t) = \frac{1}{\Gamma(q-\nu)} \int_t^b (s-t)^{q-\nu-1} f^{(q)}(s)ds, \tag{2.6} $$

where for $\nu = q$, we have ${}_a^cD_t^q f(t) = {}_t^cD_b^q f(t) = f^{(q)}(t)$. Relation between R-LFI and CFD of f is as following:

$$ {}_aI_t^{\nu}({}_a^cD_t^{\nu} f(t)) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(a) \frac{t^k}{k!}, \tag{2.7} $$

and

$$ {}_a^cD_t^{\mu}({}_aI_t^{\nu} f(t)) = \begin{cases} {}_aI_t^{\nu-\mu} f(t) & \nu > \mu \\ f(t) & \nu = \mu \\ {}_aD_t^{\mu-\nu} f(t) & \nu < \mu. \end{cases} \tag{2.8} $$

3 Content about Bell polynomials

3.1 BPs and their properties

BPs are sequences of classical polynomials that are mutually orthogonal and the partial exponential BPs of degree k are defined as [6]

$$ B_{n,k}(t_1, \ldots, t_{n-k+1}) = \sum \frac{n!}{d_1! \ldots d_{n-k+1}!} \left(\frac{t_1}{1!}\right)^{d_1} \cdots \left(\frac{t_{n-k+1}}{(n-k+1)!}\right)^{d_{n-k+1}}, \tag{3.1} $$

where $n, k \in Z^+$, $k \leq n$ and $d_i \in Z^+$, $i = 1, \ldots, n-k+1$, such that

$$ \begin{aligned} & d_1 + d_2 + \cdots + d_{n-k+1} = k, \\ & d_1 + 2d_2 + \cdots + (n+k-1)d_{n-k+1} = n. \end{aligned} $$

The complete exponential Bell polynomial of degree n is the summation of (3.1)

$$ B_n(t_1, t_2, \ldots, t_n) = \sum_{k=1}^{n} B_{n,k}(t_1, t_2, \ldots, t_{n-k+1}). \tag{3.2} $$

If all ideals are equal in (3.2) ($t_j = t$) for $j = 1, \ldots, n$ and $B_n(t)$ Bell polynomial with all variables equal to t, then

$$ B_n(t) = \sum_{k=0}^{n} S(n,k) t^k. \tag{3.3} $$

It is called complete or exponential Bell polynomial (CBP) or (EBP), of course, the other name of these polynomials is Touchard polynomials (TPs), which in this article we consider title of the Bell polynomial,

$$ S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n \tag{3.4} $$

are called Stirling numbers of the second kind (SNSK), but by changing the variable $i = k - j$, we can write (3.4) as

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} i^n, \tag{3.5}$$

which accuracy is computationally better than (3.4). And the recursive relation of (3.4) can be written as [6]

$$S(n,k) = kS(n-1,k) + S(n-1,k-1). \tag{3.6}$$

Some of the SNSK are as follow: $S(0,0) = S(n,1) = S(n,n) = 1, S(n,0) = 0, S(n,2) = 2^{n-1} - 1, S(n,3) = \frac{3^{n-1}-2^n+1}{2}$. The first few sentences of Bell polynomials are as following:

$$\begin{aligned}
B_0(t) &= 1,\\
B_1(t) &= t,\\
B_2(t) &= t^2 + t,\\
B_3(t) &= t^3 + 3t^2 + t,\\
B_4(t) &= t^4 + 6t^3 + 7t^2 + t,\\
B_5(t) &= t^5 + 10t^4 + 25t^3 + 15t^2 + t,\\
B_6(t) &= t^6 + 15t^5 + 65t^4 + 90t^3 + 31t^2 + t.
\end{aligned}$$

Some properties of Bell polynomials are as [6]

a: $B_n(1) = \sum_{k=0}^{n} S(n,k) = B_n$ (Bell number),
b: $B_n(-1) = \widetilde{B}_n$ (complement of Bell number),
c: $B_{0,0} = B_{n,n} = B_{n,1} = 1$ and $B_{n,0} = B_{0,k} = 0$, for $n, k \geq 1$,
d: $B_n(t) = e^{-t}\sum_{k=0}^{\infty} \frac{k^n}{k!} t^k$, $\forall n, k \in Z^+$ and $k \leq n$, the recursive relation of d can be written as
e: $B_n(t) = t\sum_{k=0}^{n}\binom{k-1}{n-1} B_{k-1}(t)$,
f: $B'_n(t) = \sum_{k=0}^{n-1}\binom{n}{k} B_k(t)$ (Derivative of $B_n(t)$ relative to t),
g: $B_{n+1}(t) = t(B_n(t) + B'_n(t)) = te^{-t}\frac{d}{dt}(e^t B_n(t))$,
h: $B_{n+1}(t+s) = \sum_{k=0}^{n}\binom{n}{k} B_k(t)B_{n-k}(s)$.

Definition 3.1. Semi-orthogonality of $B_n(t)$. $\forall n, m \in N$, we have

$$\int_0^{\infty} B_n(-t)B_m(-t)\frac{e^{-2t}}{t}dt = (-1)^{n-1}\frac{2^{m+n}-1}{m+n}\mathcal{B}_{m+n}, \tag{3.7}$$

where $\mathcal{B}_{m+n}$ are the Bernoulli numbers [7]. Note that $B_{m+n} = 0$ if $m + n = 2k + 1$, $k > 1$ and $k \in N$, N is Natural number. Using BPs, basis polynomials $\{1, t, t^2, \ldots, t^n\}$ can be written according to them as follow:

$$\begin{aligned}
1 &= B_0(t),\\
t &= B_1(t),\\
t^2 &= -B_1(t) + B_2(t),\\
t^3 &= B_1(t) - 3B_2(t) + B_3(t),\\
t^4 &= -6B_1(t) + 11B_2(t) - 6B_3(t) + B_4(t).
\end{aligned}$$

By mathematical induction the sentence t^n can be written in terms of BPs as follows:

$$t^n = \sum_{k=0}^{n}(-1)^{n-k}s(n,k)B_k(t), \tag{3.8}$$

where $s(n,k)$ are called Stirling numbers of the first kind (SNFK) and are given as [30]

$$\begin{aligned}
s(n,k) &= \sum_{0\leq i_1<\cdots<i_{n-k}<n} i_1 i_2 \ldots i_{n-k}\\
&= \sum_{i_1=0}^{n-1}\sum_{i_2=0}^{i_1-1}\cdots\sum_{i_{n-k}=0}^{i_{n-k-1}-1} i_1 i_2 \ldots i_{n-k}.
\end{aligned} \tag{3.9}$$

Repetition relation for (3.9) is given by [30]

$$s(n+1,k)=s(n,k-1)-ns(n,k), \tag{3.10}$$

where $s(0,0)=1, s(n,0)=s(0,n)=0$. We can write (3.8) for $j=0,1,2,\dots,n$, as in the following way:

$$t^j=\sum_{k=0}^{j} d_k B_k(t), \tag{3.11}$$

where $d_k, k=0,1,2,\dots,j$, are constant numbers and are given as

$$d_k=(-1)^{j-k}s(j,k). \tag{3.12}$$

4 Approximation of Function

According to chapter 6 (Approximation theory) of [18], it can be written that

$$h(t)\simeq h_0(t)=\sum_{i=0}^{n} c_i B_i(t)=C^T\Phi_n(t), \tag{4.1}$$

where $h(t)\in H=L^2([a,b])$ is arbitrary element and $h_0(t)$ is the best and unique approximation of $Y=Span\{\Phi_n(t)\}$, where Y is a complete subspace of H, where

$$\Phi_n(t)=[B_m(t)]^T, \quad m=0,\dots,n, \tag{4.2}$$

and T symbol means transposition. For computing the coefficients c_i, we consider for $i=0,1,2,\dots,n$,

$$k_i=\langle h(t),B_i(t)\rangle=\int_a^b h(t)B_i(t)dt, \tag{4.3}$$

where $\langle .,.\rangle$ denotes inner product. So, using (4.3), we have

$$\begin{aligned} k_i&=\int_a^b h(t)B_i(t)dt=\int_a^b \sum_{m=0}^{n} c_m B_m(t)B_i(t)\\ &=\sum_{m=0}^{n} c_m \int_a^b B_m(t)B_i(t)\\ &=\sum_{m=0}^{n} c_m\langle B_m(t),B_i(t)\rangle\\ &=\sum_{m=0}^{n} c_m R_{mi}, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} R_{mi}=&\langle B_m(t),B_i(t)\rangle=\int_a^b B_m(t)B_i(t)dt\\ =&\int_a^b \sum_{j=0}^{m} S(m,j)t^j\sum_{r=0}^{i} S(i,r)t^r dt\\ =&\sum_{j=0}^{m}\sum_{r=0}^{i} S(m,j)S(i,r)\frac{b^{j+r+1}-a^{j+r+1}}{j+r+1}. \end{aligned} \tag{4.5}$$

Let $R=[R_{ji}]_{(n+1)\times(n+1)}$, $K=[k_i]_{(n+1)\times 1}$, $i=0,\dots,n$ and $m=0,\dots,n$, where R is the dual matrix for basis Φ_n and it can be got as follows: From (5.5), we have

$$K^T=C^TR \;\Rightarrow C^T=K^TR^{-1}. \tag{4.6}$$

So that, any arbitrary element such as $g(t) \in H$ can be represented as a combination of BPs with certain coefficients

$$g(t) \simeq C^T \Phi_n(t), \tag{4.7}$$

where

$$C^T = \langle h(t), \Phi_n(t) \rangle R^{-1}. \tag{4.8}$$

On the other hand, the BPs can be written on a complete basis in an arbitrary and specific interval $[a, b]$. By (3.3), we write

$$\Phi_n(t) = ST(t), \tag{4.9}$$

where $T(t) = [t^i]_{(n+1)\times 1}$, $i = 0, \ldots, n$. And this matrix S has two important properties, firstly, it is a lower triangular matrix, secondly, the elements on the main diameter are nonzero [22], hence S^{-1} exists, if

$$\sum_{j=0}^{n} c_i B_i(t) = C^T \Phi_n(t) = 0, \tag{4.10}$$

where $C = [c_i]_{(n+1)\times 1}$. Then by (4.9) and (4.10), we get

$$C^T \Phi_n(t) = C^T ST(t) = 0. \tag{4.11}$$

Because $T(t)$, according to the Fundamental Theorem of Algebra, is a generating set [16], so that in (4.11), we have

$$C^T S = 0 \ \ \to C^T = [0, \ldots, 0].$$

5 The operational matrix of the BPs

5.1 Operational matrix of the derivative (OMD) on BPs

We define the OMD for the BPs. Then, we have

$$\Phi'_n(t) = D_{[1]} \Phi_n(t), \tag{5.1}$$

where $D_{[1]}$ is the OMD with dimension $(n + 1)$ for the BPs and it is given as

$$D_{[1]}(p, q) = \begin{cases} 0 & q \geq p \\ \binom{p-1}{q-1} & p > q, \end{cases} \tag{5.2}$$

where $p, q = 1, \ldots, n + 1$.

5.2 Operational matrix of fractional integral (OMFI) on BPs

Here we describe OMFI based on The Bell basis vector. The operational matrix can be approximated as

$$I^\nu \Phi_n(t) \simeq V \Phi_n(t), \tag{5.3}$$

where V is the $(n + 1) \times (n + 1)$ OMFI of type R-L. We achieve V as follows

$$\begin{aligned} I^\nu B_i(t) &= \sum_{m=0}^{i} I^\nu S(i, m) t^m = \sum_{m=0}^{i} S(i, m) I^\nu t^m \\ &= \sum_{m=0}^{i} S(i, m) \frac{\Gamma(m + 1)}{\Gamma(m + \nu + 1)} t^{m+\nu}, \end{aligned} \tag{5.4}$$

where $S(i, m)$ is defined in (3.4). Now, we approximate $t^{m+\nu}$ by $n + 1$ terms of the Bell basis

$$t^{m+\nu} \simeq \sum_{r=0}^{n} v_r B_r, \tag{5.5}$$

where

$$\begin{aligned} v_r =& R^{-1} \int_a^b t^{m+\nu} B_r(t) dt \\ =& R^{-1} \int_a^b t^{m+\nu} \sum_{q=0}^{r} S(r,q) t^q dt \\ =& \sum_{q=0}^{r} S(r,q) R^{-1} \int_a^b t^{m+q+\nu} dt \\ =& \sum_{q=0}^{r} S(r,q) \times s_{mq}^{\nu}, \end{aligned} \tag{5.6}$$

and $s_{mq}^{\nu} = \frac{b^{m+q+\nu+1} - a^{m+q+\nu+1}}{m+q+\nu+1} R^{-1}$. Therefore, we have

$$I^{\nu} B_i(t) \simeq \sum_{r=0}^{n} V_{ir} B_r(t), \tag{5.7}$$

where

$$V_{ir} = \sum_{m=0}^{i} \sum_{q=0}^{r} S(i,m) \times S(r,q) \times \frac{\Gamma(m+1)}{\Gamma(m+\nu+1)} \times s_{mq}^{\nu}. \tag{5.8}$$

Finally, we obtain operational matrix defined in (5.3) as

$$V = \begin{bmatrix} V_{00} & \dots & V_{0n} \\ \vdots & \ddots & \vdots \\ V_{n0} & \dots & V_{nn} \end{bmatrix}, \tag{5.9}$$

where V is called OMFI corresponding to BPs. Equation (5.9) can be considered as the product of matrices

$$V = S_{\Gamma} S S_{\nu}, \tag{5.10}$$

where

$$S_{\Gamma} = \begin{bmatrix} \frac{\Gamma(1)}{\Gamma(1+\nu)} & 0 & \dots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2+\nu)} & 0 & \dots \\ \vdots & \ddots & \vdots & \\ 0 & S(n,1)\frac{\Gamma(2)}{\Gamma(2+\nu)} & \dots & \frac{\Gamma(n+1)}{\Gamma(n+1+\nu)} \end{bmatrix},$$

$$S = \begin{bmatrix} S(0,0) & 0 & \dots & 0 \\ 0 & S(1,1) & 0 & \dots \\ \vdots & \ddots & \vdots & \\ 0 & S(n,1) & \dots & S(n,n) \end{bmatrix}, \quad S(i,i) = 1$$

and $S_{\nu} = (s_{ir}^{\nu}) = \frac{b^{i+r+1+\nu} - a^{i+r+1+\nu}}{i+r+1+\nu} R^{-1}$, $i, r = 0, \dots, n$.

6 Review and analysis of Convergence

In this part, we check the convergence and finally find the upper limit of the created error (eI^{ν}), which tends to zero.

Theorem 6.1. Let $g_n(t)$ be the factor approximating the function $g(t) \in L^2[0,1]$ as following:

$$g_n(t) = \sum_{m=0}^{n} c_m B_m(t) = C^T \Phi_n(t), \tag{6.1}$$

where C and $\Phi_n(t)$ are given in (4.1) and (4.5). In this case, we have

$$\lim_{n\to\infty} \|g(t) - g_n(t)\|_{L^2[0,1]} = 0. \tag{6.2}$$

By variable variation $u = \frac{t-a}{b-a}$ any interval as $[a, b]$ can be transformed to interval as $[0, 1]$. The error vector eI^ν of the operational matrix is given by

$$eI^\nu = [eI_0^\nu, eI_1^\nu, \dots, eI_n^\nu]^T = V\Phi_n(t) - I^\nu \Phi_n(t). \tag{6.3}$$

From Eq. (5.5) and Theorem 4.4 of [31], we have

$$\begin{aligned}&\left\| t^{k+\nu} - \sum_{j=0}^{n} v_j B_j \right\|_2 \\ &= \left(\frac{G(t^{k+\nu}, B_0(t), B_1(t), \dots, B_n(t))}{G(B_0(t), B_1(t), \dots, B_n(t))} \right).\end{aligned} \tag{6.4}$$

Hence, according to Eqs. (5.6) and (6.4), we have

$$\begin{aligned}\|eI_i^\nu\|_2 =& \|I^\nu B_i(t) - \sum_{r=0}^{n} v_{ir} B_r(t)\|_2 \\ =& \left\| \sum_{m=0}^{i} S(i,m) \times \frac{\Gamma(m+1)}{\Gamma(m+\nu+1)} \left(t^{m+\nu} - \sum_{r=0}^{n}\sum_{q=0}^{r} S(r,q) s_{mq}^\nu B_r \right) \right\|_2 \\ \leq& \sum_{m=0}^{i} S(i,m) \times \frac{\Gamma(m+1)}{\Gamma(m+\nu+1)} \times \left\| t^{m+\nu} - \sum_{r=0}^{n} v_r B_r \right\|_2 \\ \leq& \sum_{m=0}^{i} S(i,m) \times \frac{\Gamma(m+1)}{\Gamma(m+\nu+1)} \times \left(\frac{G(t^{m+\nu}, B_0(t), B_1(t), \dots, B_n(t))}{G(B_0(t), B_1(t), \dots, B_n(t))} \right)^{\frac{1}{2}},\end{aligned} \tag{6.5}$$

where $S(.,.)$ is defined in (4.4). Using Theorem 4.5 of [31] and Eq.(6.5), it can concluded that with the increase of Bell's base degree, the error eI^ν tends to zero.

7 Investigation of the proposed Method

Considering the following FOCP

$$\mathrm{Min} Z = \int_a^b A_x(t)x^2(t) + B_u(t)u^2(t) + K(t)x(t)u(t)dt, \tag{7.1}$$

$$M_1 \dot{x}(t) + M_2\, {}_0^cD_t^\nu x(t) = cx(t) + du(t) + f_0(t), \tag{7.2}$$

$$x(a) = x_a, \quad x(b) = x_b, \tag{7.3}$$

in this article, we consider the right side of (1.2) i.e. $h(t, x(t), u(t)) = cx(t) + du(t) + f_0(t)$, where $f_0(t)$ can be any function, $c, d \in R$ and $0 < \nu \leq 1$. We expand the fractional derivative of the state, control variables and also $f_0(t)$ by the Bell basis $\Phi_n(t)$

$${}_0^cD_t^\nu x(t) \simeq X\Phi_n(t), \quad X = [x_i]_{(n+1)\times 1}, \tag{7.4}$$

$$u(t) \simeq U\Phi_n(t), \quad U = [u_i]_{(n+1)\times 1}, \tag{7.5}$$

$$f_0(t) \simeq F\Phi_n(t), \quad F = [f_i]_{(n+1)\times 1}, \tag{7.6}$$

where $i = 0, \dots, n$. Using (3.11), (7.3) and (7.4), $x(t)$ can be written as

$$x(t) = {}_0I_t^\nu\, {}_0^cD_t^\nu x(t) + x(a) \simeq (XV + d_a)\Phi_n(t), \tag{7.7}$$

where V is the OMFI of order ν and

$$d_a = [x_a, 0, \ldots, 0]. \tag{7.8}$$

By derivating of (7.7), we get

$$\dot{x}(t) = (XV + d_a)\dot{\Phi}_n(t) = (XV + d_a)D_\phi \Phi_n(t), \tag{7.9}$$

where D_ϕ is defined in (6.4). Also using (7.4), (7.5), (7.6), (7.7) with (7.9), (7.2) be as

$$[M_1(XV + d_0)D_\phi + M_2X - c(XV + d_0) - dU - F] \times \Phi_n(t) = 0. \tag{7.10}$$

Since Eq. (7.10) satisfy for any $t \in [a, b]$, then

$$M_1(XV + d_0)D_\phi + M_2X - c(XV + d_0) - dU - F = 0. \tag{7.11}$$

The boundary conditions $x(a) = x_a$, $x(b) = x_b$ can be expressed as

$$(XV + d_a)\Phi_n(a) - x_a = 0, \quad (XV + d_a)\Phi_n(b) - x_b = 0. \tag{7.12}$$

By Eqs.(7.5) and (7.7), the approximate objective function Z,

$$\begin{aligned} Z \simeq& Z(X, U) \\ =& \int_a^b A[(XV + d_a)\Phi_n(t)]^T[(XV + d_a)\Phi_n(t)] + B[U\Phi_n(t)]^T[U\Phi_n(t)] + K(XV + d_a)\Phi_n(t)U\Phi_n(t)dt. \end{aligned} \tag{7.13}$$

Using the Lagrange multiplier technique (LMT), we find the extremum of Eq.(7.13) considering (7.11) and (7.12). Let

$$\begin{aligned} Z^*[X, U, \lambda] =& Z[X, U] + (M_1(XV + d_a)D_\phi + M_2X - c(XV + d_a) - dU - F)\lambda_1 \\ &+ ((XV + d_a)\Phi_n(a) - x_a)\lambda_2 + ((XV + d_a)\Phi_n(b) - x_b)\lambda_3, \end{aligned} \tag{7.14}$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)^T$, is the Lagrange coefficient. By zeroing the derivatives of (7.14) with respect to the variables, we have

$$\frac{\partial Z^*}{\partial X} = 0, \quad \frac{\partial Z^*}{\partial U} = 0, \quad \frac{\partial Z^*}{\partial \lambda} = 0. \tag{7.15}$$

By solving the Eq.(7.15), all the unknowns are found and as a result the variables of state, control and objective function are obtained approximately.

8 Some numerical examples

In this part, we have given some examples and solved them with the proposed method, and by evaluating and comparing the numerical results with other methods its nature and performance can be seen which has met our expectations.

Example 8.1. Select the FOCP as follow:

$$\begin{aligned} \text{Min} \quad & Z(x(t), u(t)) = \frac{1}{2}\int_0^1 x^2(t) + u^2(t)dt, \\ \text{s.t.} \quad & M_1\dot{x}(t) + M_2D^\nu x(t) = -x(t) + u(t), \\ & x(0) = 1, x(1) = \cosh(\sqrt{2}) + \beta\sinh(\sqrt{2}), \end{aligned}$$

where $A_x(t) = B_u(t) = 1$, $K(t) = 0$, $c = -1$, $d = 1$, $f(t) = 0$ and $M_1 + M_2 = 1$. We examine the numerical results in the following two case $(a) : M_1 = \frac{4}{5}, M_2 = \frac{1}{5}$ and case $(b) : M_1 = \frac{1}{5}, M_2 = \frac{4}{5}$.

Solution. The analytical solutions of this problem for $\nu = 1$ and $M_1 = M_2 = \frac{1}{2}$ are as follow:

$$x_{exa}(t) = \cosh(\sqrt{2}t) + \beta \sinh(\sqrt{2}t),$$
$$u_{exa}(t) = (1+\sqrt{2}\beta)\cosh(\sqrt{2}t) + (\sqrt{2}+\beta)\sinh(\sqrt{2}t),$$
$$\beta = -\frac{\cosh(\sqrt{2}) + \sqrt{2}\sinh(\sqrt{2})}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2})}$$

and $Z_{exa} = 0.19290929809316939551$. To obtain the approximation solution, let $n = 5$,

$$ {}_0^cD_t^\alpha x(t) \simeq X\Phi(t),\ u(t) \simeq U\Phi(t),$$

where

$$X = [x_0, x_1, x_2, x_3, x_4, x_5],\ U = [u_0, u_1, u_2, u_3, u_4, u_5],\ \Phi(t) = [B_0, B_1, B_2, B_3, B_4, B_5]^T.$$

We obtain the approximate solutions for $\nu = 1$ and case (a) as follow:

$$\begin{aligned}
X =& [1, -\frac{7693}{1641}, \frac{1057}{209}, -\frac{1303}{620}, \frac{885}{1856}, -\frac{368}{16315}],\\
U =& [-\frac{393}{1019}, \frac{352}{725}, \frac{1563}{2108}, -\frac{5084}{5975}, \frac{845}{3297}, -\frac{368}{16315}],\\
x_{app}(t) =& -\frac{368}{16315}t^5 + \frac{437}{3045}t^4 - \frac{1074}{2381}t^3 + \frac{1333}{1336}t^2 - \frac{1412}{1019}t + 1,\\
u_{app}(t) =& -\frac{368}{16315}t^5 + \frac{149}{4848}t^4 + \frac{267}{2171}t^3 - \frac{407}{1145}t^2 + \frac{1449}{2704}t - \frac{393}{1019},\\
Z_{app} =& 0.1929092989576520.
\end{aligned}$$

In Table 1, we obtain the optimal values of objective function Z for different values of ν and n. We give different values of α for different values of ν at $n = 7$. In Table 2, by comparing these tables, it can be seen that for different ν for case (a), the difference in the degree of deviation of the $f(x, u)$ (Figure 3) curve from the origin is a maximum of 58.5 degree, while in case (b), this difference is almost 31.74 degree, in other words, the dispersion range of angles in mode (a) is much smaller than mode (b). That is for case (a), the density of the value of Z for different ν is close to the real state, while in case (b), the dispersion is increased, in other words, the demand space for case (a) and case (b) is $\alpha \in [56.76, 62.21]$ and $\alpha \in [46.77, 78.51]$ respectively.

By comparing graphs in Figure 1 and Figure 2 it can be seen that the dispersion of the left graphs is less than the right graphs for different ν which confirms the contents of Table 2 and also Figure 3. We compare optimal values of objective function J with different values of ν for case (a) and case (b) at $n = 7$ in Table 3. We obtained the objective, state and control function errors with different choices of n at $\nu = 1$ and $M_1 = M_2 = \frac{1}{2}$ using with two methods (a: the method of [11] and b: our method) in Table 4. We indicate the angle or the amount of deviation of the object from the horizontal state with the sign α, where $\alpha = atan(\frac{df}{dt})|_{t=0}$.

Table 1: case (a): optimal values of objective function Z for various values of ν and n for Example 1

$\nu \setminus n$	5	7	9	11
0.1	0.151767945494	0.17351626096115035	0.17351626167574861	0.17315905522288654
0.3	0.159416621137	0.17685835761895106	0.17685829651840556	0.17626551370113597
0.5	0.167929233457	0.18078661356758523	0.18078587645735884	0.18010459562606711
0.7	0.177261526483	0.18524961475448683	0.18524694995377512	0.18478557932528402
0.9	0.187406928026	0.19022151143960065	0.19021966534469185	0.19034717188032983
1	0.192909298958	0.19290929809317703	0.19290929809316939	0.19290929809316939

Table 2: Different values of α for various values of ν at $n = 7$ for Example 1

	ν	1	0.9	0.7	0.5	0.3
	exa	*app*				
case(a):	58.3575	58.3574	56.76	58.62	61.21	62.61
case(b):	58.3575	58.3574	46.77	58.75	72.64	78.51

Table 3: Optimal values of objective function Z for $n = 7$ and various values of M_1, M_2 for Example 1

$\nu \setminus M_1, M_2$	$\frac{4}{5}, \frac{1}{5}$	$\frac{1}{5}, \frac{4}{5}$
0.1	0.1735162609611504	0.1209227933645721
0.3	0.1768583576189511	0.1335720678019314
0.5	0.1807866135675852	0.1480391491012388
0.7	0.1852496147544868	0.1641667844896235
0.9	0.1902215114360065	0.1824090568860903
1	0.1929092980931770	0.1929092980931770

Table 4: Errors of objective, state and control function with $\nu = 1$ and various values of n (table a and table b) for Example 1

	n	$\mid Z_e - Z_a \mid$	$\mid x_e - x_a \mid$	$\mid u_e - u_a \mid$
a:Method [11]	4	1.024×10^{-6}	1.782×10^{-4}	5.325×10^{-3}
	6	6.795×10^{-11}	1.015×10^{-6}	4.923×10^{-5}
	8	1.554×10^{-15}	3.433×10^{-9}	2.405×10^{-7}
	10	6.671×10^{-16}	7.636×10^{-12}	7.215×10^{-10}

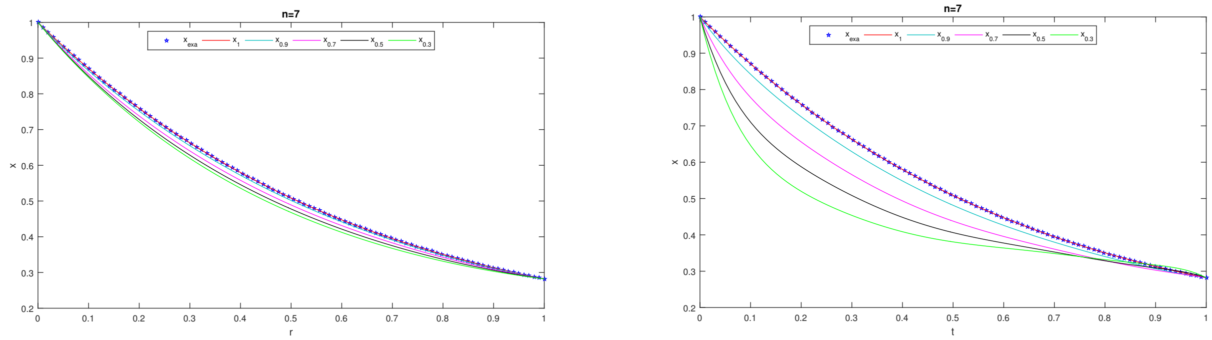

Figure 1: Behavior of $x(t)$ for different values of ν for Example 1(left figure for case (a) and right figure for case (b))

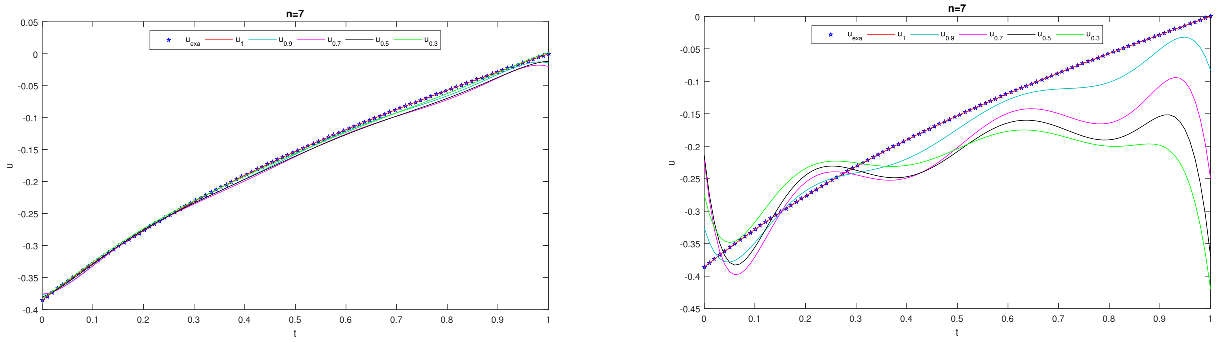

Figure 2: Behavior of $u(t)$ for different values of ν for Example 1(left figure for case (a) and right figure for case (b))

Example 8.2. Consider the following (FOCP) [2]:

$$\text{Min} \quad Z = \int_0^1 (tu(t) - (\nu + 2)x(t))^2 dt,$$

subject to the dynamical fractional control system

$$M_1 \dot{x}(t) + M_2 {}_0^c D_t^\nu x(t) = M_3 u(t) + t^2,$$

and the boundary conditions

$$x(0) = 0, \;\; x(1) = \frac{2}{M_2 \Gamma(3 + \nu)}.$$

	n	$\mid Z_e - Z_a \mid$	$\mid x_e - x_a \mid$	$\mid u_e - u_a \mid$
b:Our method	4	1.469×10^{-7}	4.295×10^{-5}	5.404×10^{-4}
	6	2.524×10^{-12}	1.231×10^{-7}	2.250×10^{-6}
	8	4.670×10^{-18}	2.097×10^{-10}	5.021×10^{-9}
	10	7.951×10^{-18}	2.354×10^{-13}	6.973×10^{-12}

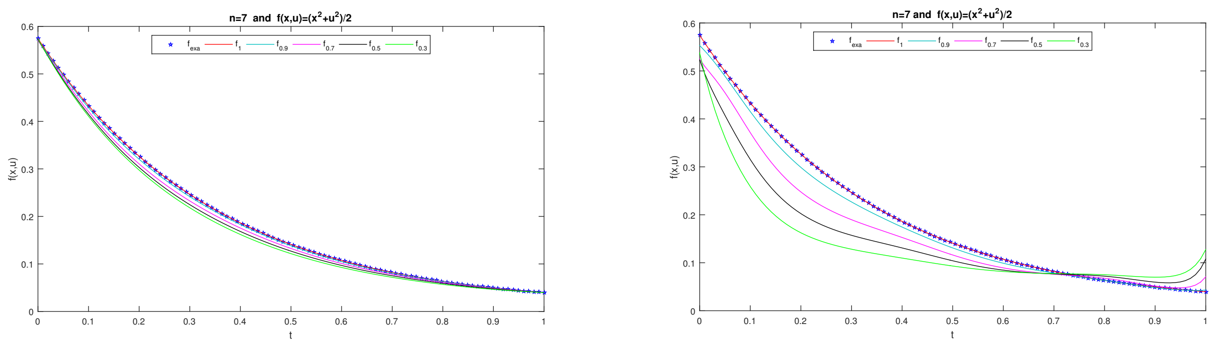

Figure 3: Behavior of f_{exa} and f_{app} for different values of ν for Example 1(left figure for case (a) and right figure for case (b))

In this example, $c = 0$ and $f_0(t) = t^2$. So that to obtain real solutions, by using Eqs (7.4) and (7.6), we have ${}_0^cD_t^\nu x(t) = f_0(t) = F_0\Phi_n(t)$, $F_0 = [0, -1, 1, 0, \dots, 0]$. By fractional integrating of it we obtain the real solutions as

$$(x^*(t), u^*(t)) = \left(\frac{2t^{\nu+2}}{M_2\Gamma(\nu+3)}, \frac{2M_1t^{\nu+1}}{M_2M_3\Gamma(\nu+2)}\right)$$

and

$$Z^* = \frac{1}{2\nu+5}\left(\frac{2}{M_2\Gamma(\nu+2)}\right)^2\left[\frac{M_1}{M_3} - 1\right]^2.$$

According to the real value of Z it can be concluded that if $\frac{M_1}{M_3} \approx 1$, then for each couple (x, u), where $M_2 \neq 0, x = x(t, \nu, M_2), u = u(t, \nu, M_1, M_2, M_3)$, the value of $Z = 0$. The result can be seen in Tables 5 and 6 for different values of ν and M_2. By deriving from x we see that the deviation value of the control(u) from the speed($\dot{x}$) is 45 degrees. This degree is different for approximate solutions, see Table 7, in this Table for $M_2 = 1$, $M_2 = 100$ and $M_2 = 1000$, the values of α are: $\alpha \in [44.715, 45]$, $\alpha \in [26.565, 45]$ and $\alpha \in [5.1944, 45]$, respectively. It means that with the increase of M_2 the dispersion of the deviations is away from 45 degrees. And this scattering for $\nu = 1$ is done from the bottom and in this case the dynamic system of example 2 is so

$$(M_1 + M_2)\dot{x}(t) = M_3u(t) + t^2 \to \alpha = a\tan\left(\frac{M_3}{M_1 + M_2}\right).$$

It is clear that the deviation of α from 45 degrees depends on M_2. If M_2 is too weak compared to M_1, α will approach 45 degrees with the mentioned condition ($\frac{M_1}{M_3} \approx 1$). For $\nu = 0$ the dynamic system of example 2 is so

$$M_1\dot{x}(t) = M_3u(t) + t^2 - M_2 \to \alpha = atan\left(\frac{M_3}{M_1}\right).$$

It can be seen that the value of α with condition $\frac{M_3}{M_1} \approx 1$ and for each value of M_2 is equal to $\frac{\pi}{4}$(see Table 7). For $M_i = 1 (i = 1, 2, 3)$ and $\nu = 1$ the real answers are so

$$x_{ex}^1(t) = \frac{1}{3}t^3, u_{ex}^1(t) = t^2, Z_{ex}^1 = 0.$$

Let $n = 7$ and $\nu = 1$, we obtain approximation solutions as

$$X_{ap}^1 = \left[0, \frac{2}{3}, -1, \frac{1}{3}, \frac{933}{578} \times 10^{-27}, -\frac{4285}{1252} \times 10^{-28}, \frac{1679}{471} \times 10^{-29}, -\frac{1887}{1315} \times 10^{-30}\right]$$

$$U^1_{ap}=\left[\frac{2503}{336}\times 10^{-32},-1,1,\frac{7039}{952}\times 10^{-27},-\frac{2007}{818}\times 10^{-27},\frac{5722}{1557}\times 10^{-28},-\frac{1567}{780}\times 10^{-29},\frac{2620}{859}\times 10^{-56}\right]$$

$$\begin{aligned}x^1_{ap}(t)=&-\frac{287}{200}\times 10^{-30}t^7+\frac{3181}{577}\times 10^{-30}t^6-\frac{2843}{337}\times 10^{-30}t^5+\frac{2113}{324}\times 10^{-30}t^4+\frac{1}{3}t^3\\&+\frac{3103}{599}\times 10^{-32}t^2-\frac{920}{247}\times 10^{-32}t,\end{aligned}$$

$$\begin{aligned}u^1_{ap}(t)=&-\frac{1522}{499}\times 10^{-56}t^7-\frac{2009}{1000}\times 10^{-29}t^6+\frac{2375}{359}\times 10^{-29}t^5-\frac{2843}{337}\times 10^{-29}t^4\\&+\frac{2579}{495}\times 10^{-29}t^3+t^2+\frac{661}{319}\times 10^{-30}t-\frac{2503}{336}\times 10^{-32},\end{aligned}$$

$$Z^1_{ap}=1.0102\times 10^{-39}.$$

By ignoring the insignificant coefficients in the approximate answers it is observed that the approximate answers are equal to the real answers.

Table 5: Errors of objective, state and control function with $n=7$, $M_1=M_3=M$ and various values of ($M_2\neq 0$) (M is Free and arbitrary number) with $M=100$ for Example 2

	M_2	$\mid Z_e-Z_a\mid$	$\mid x_e-x_a\mid$	$\mid u_e-u_a\mid$
case (a):$\nu=1$	1	9.0181e-59	5.6604e-55	2.9174e-53
	100	6.923e-66	3.7329e-62	3.7009e-60
	1000	4.2278e-66	5.247e-63	2.6686e-61
	-1	8.3233e-60	1.2941e-55	5.5383e-54
	-100	1.4873e-58	8.2809e-56	1.8992e-54
	-1000	1.0417e-65	5.6869e-63	2.069e-61

	M_2	$\mid Z_e-Z_a\mid$	$\mid x_e-x_a\mid$	$\mid u_e-u_a\mid$
case (b):$\nu=0.9$	1	1.1286e-11	5.7687e-13	6.342e-10
	100	5.0954e-15	2.149e-16	4.1289e-13
	1000	1.9616e-15	1.8503e-18	7.6918e-14
	-1	1.0812e-11	5.5492e-13	6.0161e-10
	-100	9.2836e-11	5.5492e-13	6.0161e-10
	-1000	6.8247e-12	3.4281e-15	1.4961e-11

	M_2	$\mid Z_e-Z_a\mid$	$\mid x_e-x_a\mid$	$\mid u_e-u_a\mid$
case (c):$\nu=0.7$	1	2.2318e-10	1.26e-11	1.4486e-08
	100	3.2254e-14	1.893e-15	3.0728e-12
	1000	3.3245e-15	2.042e-17	2.7021e-13
	-1	2.2137e-10	1.2468e-11	1.4226e-08
	-100	1.5451e-14	7.8275e-16	5.4041e-13
	-1000	2.1605e-15	8.7233e-17	1.5625 e-13

For $\nu=0.3$ the exact solutions are as

$$x^{0.3}_{ex}(t)=\frac{1311}{1759}\sqrt[10]{t^{23}},\quad u^{0.3}_{ex}(t)=\frac{2861}{1669}\sqrt[10]{t^{13}}.$$

For $n=7$ the approximate solutions are as

$$X^{0.3}_{ap}=\left[0,\frac{629}{4},-\frac{18179}{50},\frac{15363}{50},-\frac{12411}{100},\frac{4702}{183},\frac{1438}{551},\frac{271}{2647}\right],$$

$$U^{0.3}_{ap}=\left[-\frac{55}{10786},-\frac{20899}{100},\frac{44767}{100},-\frac{33453}{100},\frac{568}{5},-\frac{4571}{250},\frac{324}{257},-\frac{96}{4001}\right],$$

Table 6: Values of objective function with $n = 7$, $M_1 = M_3 = M$ and various values of ($M_2 \neq 0$) (M is free and arbitrary number) with $M = 100$ for Example 2

M_2	$\nu = 1$	$\nu = 0.9$	$\nu = 0.7$	$\nu = 0.5$	$\nu = 0.3$	$\nu = 0.1$	$\nu = 0$
1	2.9999e-60	1.1286e-11	2.2318e-10	1.0332e-09	2.138e-09	1.0295e-09	1.8988e-60
100	2.6257e-65	5.0954e-15	3.2254e-14	9.9209e-14	1.8978e-13	9.3558e-14	2.5385e-60
1000	3.3817e-69	1.9616e-15	3.3245e-15	1.8249e-15	1.1241e-15	5.0156e-16	1.5976e-53
-1	2.1311e-60	1.0812e-11	2.2137e-10	1.0344e-09	2.1434e-09	2.5421e-15	1.8892e-60
-100	8.2085e-64	9.2836e-15	1.5451e-14	1.1161e-13	2.4428e-13	1.1362e-13	1.4916e-60
-1000	1.5055e-66	6.8247e-12	2.1605e-15	4.3515e-15	7.8841e-15	2.5421e-15	9.7111e-58

Table 7: Values of α for various values of ν at $n = 7$ with $M_1 = M_3 = M$ and various values of ($M_2 \neq 0$) (M is free and arbitrary number) with $M = 100$

	M_2	$\nu = 1$	$\nu = 0.9$	$\nu = 0.7$	$\nu = 0.5$	$\nu = 0.3$	$\nu = 0.1$	$\nu = 0$
real solution	1	45	45	45	45	45	45	45
	100	45	45	45	45	45	45	45
	1000	45	45	45	45	45	45	45
app solution	1	44.715	44.854	44.938	44.973	44.989	44.996	45
	100	26.565	32.823	39.249	42.472	43.955	44.597	45
	1000	5.1944	8.8186	17.242	27.599	36.161	40.94	45

$$x_{ap}^{0.3}(t) = \frac{5119}{50000}t^7 - \frac{45979}{10000}t^6 + \frac{88059}{10000}t^5 - \frac{48843}{50000}t^4 + \frac{85799}{10000}t^3 + \frac{3459}{10000}t^2 - \frac{384}{78125}t,$$

$$\begin{aligned} u_{ap}^{0.3}(t) = &- \frac{1728949911346043}{72057594037927936}t^7 + \frac{37843}{50000}t^6 - \frac{27323}{10000}t^5 + \frac{4303}{1000}t^4 - \frac{9563}{2500}t^3 \\ &+ \frac{509}{200}t^2 + \frac{69611}{100000}t - \frac{2939488668145617}{576407523034234 88}, \end{aligned}$$

$$Z_{ap}^{0.3} = 1.8834 \times 10^{-9}.$$

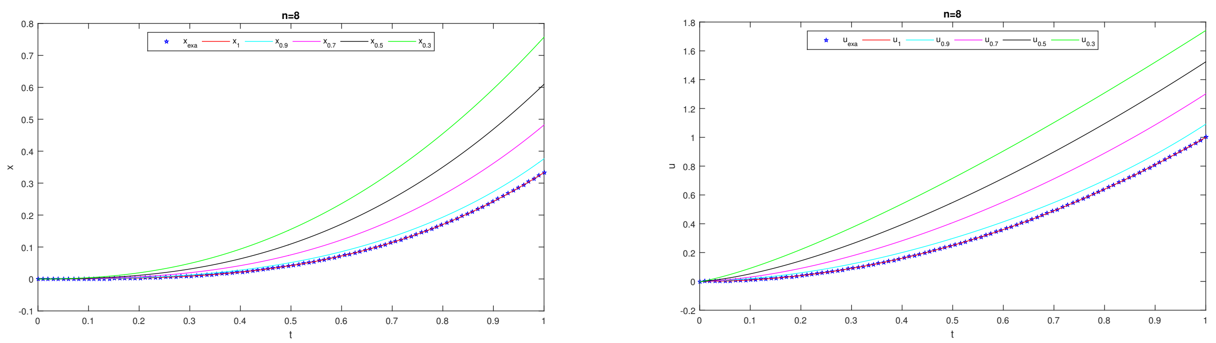

Figure 4: Behavior of $x(t)$ (left) and $u(t)$ (right) for different values of ν for Example 2.

We obtained the errors of objective, state and control functions for $n = 7$, three cases of ν (case (a):$\nu = 1$, case (b):$\nu = 0.9$, case (c):$\nu = 0.7$) and coefficients different from M_2 in Table 5. We also obtained the value of Z for different ν and distinct values of M_2 in Table 6. The value of α for different values of ν and M_2 is obtained in Table 7 where $\alpha = atan(\dot{x}, u)$. The movement of the state and control functions can be seen in Figure 4 for various values of ν at $n = 8$. And also in Figure 5 the results of the errors of state and control functions are shown.

9 Conclusion

In this paper, we presented a numerical scheme for solving quadratic optimal control problems with nonlinear constrained. The Bell polynomials was employed. In two examples, we show as the M_2 coefficient increases, the dispersion of angles α increases too. For example 1 the maximum value of deviation by reducing the order of the

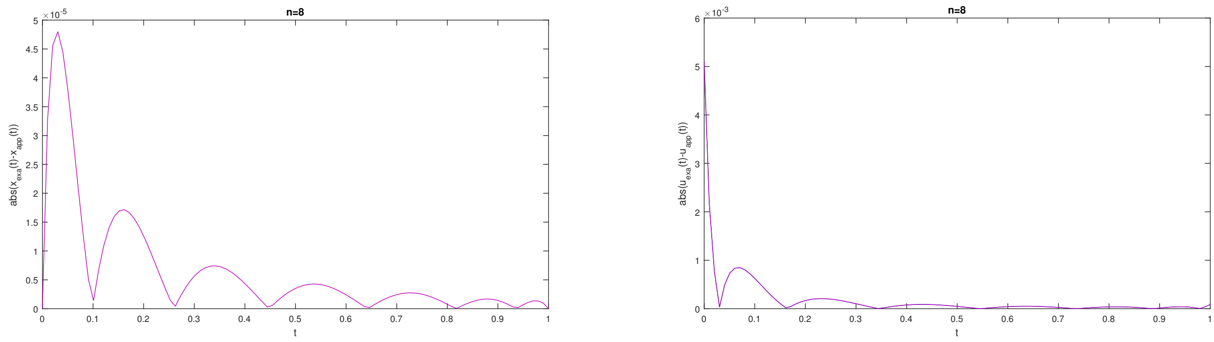

Figure 5: Absolute errors of $x(t)$(left) and $u(t)$(right) for $\nu = 0.3$, for Example 2.

fraction ν for $M_2 < M_1$ is 4.2525, and for case $M_1 < M_2$ is 20.1525. In other words by decreasing the fractional order of the derivative the required space for the $f(x,u) = \frac{x^2+u^2}{2}$ function becomes larger. Therefore the value of the objective function (Z) has a direct relationship with the fractional order of the derivative(ν) that is as it decreases its value decreases too. In example 2 for $M_1 = M_3$ as M_2 increases, the deviation value increases as the fractional order of ν decreases. It means that is the coefficient of the fractional term and the fractional order in the angle dispersion value have an inverse relationship with each other. The obtained results showed that the new approach can solve the problem effectively.

References

[1] S.A. Alavi, A.R. Haghighi, A. Yari, and F. Soltanian, *Using Mott polynomials operational matrices to optimize multi-dimensional fractional optimal control problems*, Iran. J. Numer. Anal. Optim. **12** (2022), no. 1, 201–227.

[2] S.A. Alavi, A.R. Haghighi, A. Yari, and F. Soltanian, *A numerical method for solving fractional optimal control problems using the operational matrix of Mott polynomials*, Computational Meth. Differ. Equ. **10** (2022), no. 3, 755–773.

[3] M.S. Asl and M. Javidi, *Numerical evaluation of order six for fractional differential equations: stability and convergency*, Bull. Belg. Mathe. Soc.-Simon Stevin **26** (2019), no. 2, 203–221.

[4] M.S. Asl, M. Javidi, and Y. Yan, *High order algorithms for numerical solution of fractional differential equations*, Adv. Differ. Equ. **2021** (2021), no. 1, 1–23.

[5] M.S. Asl, M. Javidi, and B. Ahmad, *New predictor-corrector approach for nonlinear fractional differential equations: error analysis and stability*, J. Appl. Anal. Comput. **9** (2019), no. 4, 1527–1557.

[6] E.T. Bell, *Exponential polynomials*, Ann. Math. **35** (1934), no. 2, 258–277.

[7] K.N. Boyadzhiev, *Exponential polynomials, Stirling numbers and evaluation of some gamma integrals*, Abstr. Appl. Anal. **2009** (2009), 1–18.

[8] M. Caputo, *Linear models of dissipation whose q is almost frequency independent*, Geophys. J. Royal Astron. Soc. **13** (1967), no. 5, 529–539.

[9] M.R. Dadashi, A.R. Haghighi, F. Soltanian, and A. Yari, *On the numerical solution of optimal control problems via Bell polynomials basis*, Iran. J. Numer. Anal. Optim. **10** (2020), no. 2, 197-221.

[10] D.J. Inman, *Vibration with Control*, John Wiley & Sons, Ltd. ISBN: 0-470-01051-7, 2006.

[11] K. Diethelm, *The analysis of Fractional Differential Equations*, Springer-Verlag, 2010.

[12] M. Dehghan, E.A. Hamedi, and A.H. Khosravian, *A numerical scheme for the solution of a class of fractional variational and optimal control problems using the modified Jacobi polynomials*, J. Vib. Control. **22** (2016), no. 6, 1547–1559.

[13] S.S. Ezz-Eldien, E.H. Doha, D. Baleanu, and A.H. Bhrawy, *A numerical approach based on Legendre orthonormal*

polynomials for numerical solutions of fractional optimal control problems, J. Vib. Control. **23** (2017), no. 1, 16-30.

[14] G. Jumarie, *Modified Riemann-Liouville derivative and fractional Taylor series of non-differentiable functions Further results*, Comput. Math. Appl. **51** (2006), no. 9-10, 1367–1376.

[15] S. Jahanshahi and D.F.M. Torres, *A simple accurate method for solving fractional variational and optimal control problems*, J. Optim. Theory. Appl. **174** (2016), no. 1, 156–175.

[16] A. Jajarmi, H. Mojtaba, E. Mohammadzadeh, and D. Baleanu, *A new approach for the nonlinear fractional optimal control problems with external persistent disturbances*, J. Franklin Inst. **355** (2018), no. 9, 3938–3967.

[17] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier Science, Amsterdam, the Netherlands, 2006.

[18] E. Kreyszig, *Introductory Functional Analysis with Applications*, New York, NY: John Wiley and Sons, 1978.

[19] E. Keshavarz, Y. Ordokhani, and M. Razzaghi, *A numerical solution for fractional optimal control problems via Bernoulli polynomials*, J. Vib. Control **22** (2016), no. 18, 3889–3903.

[20] D. Kumar, F. Tchier, J. Singh, and D. Baleanu, *An Efficient Computational Technique for Fractal Vehicular Traffic Flow*, Entropy **20** (2018), no. 4, 259.

[21] D. Kumar, J. Singh, D. Baleanu, and Sushila, *Analysis of regularized long-wave equation associated with a new fractional operator with Mittag-Leffler type kernel Authors*, Phys. A: Statist. Mech. Appl. **492** (2018), 155–167.

[22] D. Kumar, J. Singh, D. Baleanu, and S. Rathore, *Analysis of a fractional model of the Ambartsumian equation*, Eur. J. Phys. Plus. **133** (2018), 259.

[23] K.S. Miller and B. Ross, *An Introduction to the Fractional Differential Equations*, John Wiley and Sons, New York, NY, USA, 1993.

[24] F. Mirzaee, *Numerical solution of nonlinear Fredholm-Volterra integral equations via Bell polynomials*, Comput. Meth. Differ. Equ. **5** (2017), no. 2, 88–102.

[25] A. Nemati, S.A. Yousefi, F. Soltanian, and A.J. Saffar, *An efficient numerical solution of FOCPs by using the Ritz Method and Bernstein operational matrix*, Asian J. Control **1** (2017), no. 19, 1-11.

[26] I. Podlubny, *Fractional Differential Equations, Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[27] T.J. Rivlin, *An Introduction to the Approximation of Functions*, Dover Publications, 1981.

[28] J. Singh, D. Kumar, and D. Baleanu, *New aspects of fractional Biswas-Milovic model with Mittag-Leffler law*, Math. Model. Nat. Phenom. **14** (2019), no. 3, 303.

[29] J. Singh, D. Kumar, D. Baleanu, and S. Rathore, *An efficient numerical algorithm for the fractional Drinfeld–Sokolov–Wilson equation*, Appl. Math. Comput. **335** (2018), 12–24.

[30] R.P. Stanley, *Enumerative Combinatorics*, Cambridge University Press, 2011.

[31] A. Yari, *Numerical solution for FOCPs by Hermite polynomials*, J. Vib. Control **27** (2021), no. 5-6, 698–716.