

Mountain pass solutions for nonlocal fractional elliptic operators

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Abstract

The purpose of this paper is to check the existence of solutions to nonlocal integrodifferential operators. These equations have a nontrivial variational structure solution using the Mountain Pass Theorem. To effectively employ the nonlinear methodologies, a comprehensive and rigorous analysis of the fractional spaces involved is imperative. In addition, we require rather general assumptions on the local operator.

Keywords: Variational techniques, Mountain Pass Theorem, Integrodifferential operators, Fractional Laplacian, Nonlocal problems

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1 Introduction

In this work, we study the following nonlocal fractional elliptic operators

$$\begin{cases} -M\left(\int_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x - y) dx dy\right) \zeta_K u(x) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded domain in $(\mathbb{R}^n, |\cdot|)$ with smooth boundary $\partial\Omega$, $s \in (0, 1)$ that fixes s as $s < \frac{n}{2}$, $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function, $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function, and the nonlocal Lebesgue integrodifferential operator ζ_K is defined by

$$\zeta_K u(x) := \frac{1}{2} \int_{\mathbb{R}^n} (u(x + y) + u(x - y) - 2u(x)) K(y) dy, \quad x \in \mathbb{R}^n.$$

The kernel $K : \mathbb{R}^n \setminus \{0\} \rightarrow (0, +\infty)$ is a measurable function with the following properties:

$$\gamma K \in L^1(\mathbb{R}^n), \quad \text{where } \gamma(x) = \min\{|x|^2, 1\}, \quad (1.2)$$

$$\exists \lambda > 0, \text{ s.t. } K(x) \geq \lambda |x|^{-(n+2s)} \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \quad (1.3)$$

$$K(x) = K(-x), \quad \forall x \in \mathbb{R}^n \setminus \{0\}. \quad (1.4)$$

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A typical example for K is given by $K(x) = |x|^{-(n+2s)}$. In this case the nonlocal operator ζ_K becomes the fractional Laplacian operator, i.e.

$$-(-\Delta)^s u(x) := P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \quad x \in \mathbb{R}^n, \quad (1.5)$$

along any $u \in C_0^\infty(\Omega)$, where P.V. denotes the Cauchy principal value.

In our context, to circumvent potential technical challenges arising from the inclusion of the following term

$$M \left(\int_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x - y) dx dy \right),$$

we impose some restrictions on the continuous function M :

$$(M_1) \quad 0 < m_0 < M(t), \quad \text{for all } t \in [0, +\infty),$$

$$(M_2) \quad \text{There exists } t_0 \geq 0 \text{ so that } \widehat{M}(t) \geq tM(t), \text{ for all } t \in [t_0, \infty), \text{ with } \widehat{M}(t) = \int_0^t M(s) ds.$$

A typical example for M is given by $M(t) = 2 + \frac{\sin t}{1+t^2}$ for all $t \geq 0$. We remark that as a consequence of conditions (M_1) and (M_2) , there exist two positive constants m_1 and m_2 such that

$$m_0 \frac{\|u\|_{X_0}^2}{2} \leq \frac{1}{2} \widehat{M} \left(\int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \right) \leq m_1 \frac{\|u\|_{X_0}^2}{2} + \frac{m_2}{2}, \quad (1.6)$$

see [7, Lemma 1], so that the space X_0 and its norm is defined in Section 2.

Also, we assume that $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function that satisfies the following hypotheses:

$$(f_1) \quad \text{There exist } a_1, a_2 > 0 \text{ and } q \in (2, 2^*) \text{ with } 2^* = \frac{2n}{n-2s} \text{ such that } |f(x, t)| \leq a_1 + a_2 |t|^{q-1}, \text{ a.e. } x \in \Omega, t \in \mathbb{R},$$

$$(f_2) \quad \lim_{|t| \rightarrow 0} \frac{f(x, t)}{|t|} = 0, \quad \text{uniformly in } x \in \Omega,$$

$$(f_3) \quad \text{there exist } \mu > 2 \text{ and } r > 0 \text{ so that } 0 \leq \mu F(x, t) \leq t f(x, t), \text{ for a.e. } x \in \Omega, t \in \mathbb{R}, |t| \geq r, \text{ where the function } F \text{ is the primitive of } f \text{ with respect to the second variable, that is, } F(x, t) = \int_0^t f(x, \tau) d\tau.$$

A simple computation shows that the function $f(x, u) = |u|^{p-2}u$ where $2 < p < 2_s^*$, satisfies conditions (f_1) - (f_3) .

Elliptic fractional operators are a class of non-local operators that generalize classical elliptic differential operators, such as the Laplacian to fractional orders. These operators are characterized by their ability to model phenomena with long-range interactions, memory-driven dynamics, and irregular or anomalous characteristics, which cannot be adequately described by classical local operators.

In recent years, fractional differential operators and equations have garnered significant attention due to their exceptional ability to model complex systems [11, 24]. So far, these operators have found extensive applications across diverse scientific and engineering disciplines, including optimization theory [18], fractional quantum mechanics [13], financial modeling [10], image processing [15], and biomedical engineering [19]. For a comprehensive overview of related studies, readers are directed to [4, 5, 6, 12, 16, 20, 25, 27]. In [2], the authors investigated a singular fractional $p(x, \cdot)$ -Kirchhoff type problem. Using the Nehari manifold approach and variational techniques within a weighted fractional Sobolev space with variable exponents, they established the existence of at least two distinct nontrivial weak solutions for sufficiently small values of the parameter λ . Also, in the work of Zuo et al. [26], a bi-nonlocal problem driven by a variable-order fractional $p(\cdot), q(\cdot)$ -Kirchhoff operator has been studied. They showed the existence of at least one nontrivial positive energy solution. Furthermore, under a symmetry assumption on the nonlinearity, they employ the fountain theorem to establish the existence of infinitely many positive energy solutions. The study highlights the challenges and novel techniques required to handle the double nonlocality from both the Kirchhoff term and the variable-order fractional operator. In [3], Azroul et al. considered a weighted Kirchhoff-type problem involving

a variable-order fractional $p(x, \cdot)$ -Laplacian operator. By introducing a suitable functional setting and proving a key continuous and compact embedding theorem, the authors employ Ricceri's three critical point approach to derive existence results under appropriate assumptions on the parameters and nonlinear terms.

A particularly interesting class of problems in this context is the fractional Kirchhoff-type equations, which generalize the well-known Kirchhoff model for vibrating strings [17]. The classical Kirchhoff equation accounts for the dependence of the tension on the deformation of the string, leading to a non-local term in the partial differential equation. In the fractional setting, this non-locality is further enriched by the presence of fractional Laplacian operators, introducing long-range interactions and singular behaviors.

Cheng [9] investigated the existence and multiplicity of nontrivial weak solutions to the nonlocal elliptic Kirchhoff type problem

$$\begin{cases} -\left(a + b \int_{\Omega} |\nabla u|^2 dx\right) \Delta u = f(x, u), & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N = 1, 2, 3$) is a bounded domain with smooth boundary, $a, b > 0$ are constants, and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function satisfying some restrictions. The author proves the existence of at least one nontrivial solution using the Mountain Pass Theorem and further obtains multiplicity results via a cut-off technique, which truncates the nonlinearity to ensure geometric conditions for multiple critical points.

Our assumption $(\mathbf{f}_1)$ imposes a direct subcritical growth bound, which aligns with Cheng's (1.2), but is more explicit in the fractional critical exponent. This differs from Cheng's reliance on asymptotic limits of F (e.g., $(f_1) - (f_2)$ tied to eigenvalues λ_1, μ_1 , as our fractional spaces lack a direct spectral decomposition; instead, we ensure coercivity via the properties (1.2)–(1.4) of the kernel K . Moreover, conditions $(\mathbf{M}_1) - (\mathbf{M}_2)$ generalize Cheng's linear Kirchhoff term $(a + b \int_{\Omega} |\nabla u|^2 dx)$ to a continuous function M with strict positivity and an integral growth condition, ensuring boundedness below and above by quadratic terms in the X_0 -norm (cf. (1.6)). This allows handling nonlinear nonlocal effects absent in Cheng, while implying coercivity for the energy functional in fractional spaces. In fact, our assumptions extend Cheng's by decoupling the nonlinearity from eigenvalue asymptotics (suited to fractional operators without classical spectra) and introducing kernel-dependent bounds, while differing in their focus: Cheng's emphasize multiplicity via cut-off in low dimensions, whereas ours prioritize variational structure in higher-dimensional fractional settings.

In [14], the authors considered the following fractional elliptic problem involving Kirchhoff type function with critical exponent

$$\begin{cases} -M\left(\int_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x - y) dx dy\right) \zeta_K u(x) = \lambda f(x, u) + |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $n > 2s$ with $s \in (0, 1)$, $2^* = \frac{2n}{n-2s}$ and λ is a positive parameter. Applying the mountain pass lemma, the authors proved that there is at least one nontrivial weak solution and showed the asymptotic behavior of this solution. [14] tackled the critical case via the Hardy-Littlewood-Sobolev inequality to handle the term $|u|^{2^*-2}u$, but this introduces concentration-compactness challenges absent in subcritical settings. Moreover, solutions depended on a parameter λ . In the present work, we removed this term entirely (focusing on $f(x, u)$ alone), leveraging $(\mathbf{f}_1)$'s explicit subcritical growth to simplify Palais-Smale verification without advanced inequalities. Considering the nonlinearity assumptions on f in [14], the conditions $(\mathbf{f}_1) - (\mathbf{f}_3)$ in our work inherit the subcritical bound, the zero limit and the AR-condition, but extend to all $t \in \mathbb{R}$, allowing sign-changing solutions. Also, we added $(\mathbf{M}_2)$, which is crucial for the Mountain pass geometry in the subcritical setting.

Motivated by the above papers, we consider the fractional Kirchhoff-type problem (1.1). More precisely, our results extend the previous work [22]. Now, we state the main results of the present paper.

Theorem 1.1. Suppose that $s \in (0, 1)$, $n > 2s$ and Ω be a bounded open subset of $\mathbb{R}^n$. Assume that the functions K , M and f satisfy conditions (1.2)–(1.4), $(\mathbf{M}_1) - (\mathbf{M}_2)$ and $(\mathbf{f}_1) - (\mathbf{f}_3)$. Then problem (1.1) has a nontrivial Mountain Pass type solution.

The fractional Laplacian represents the most elementary example within the non-local framework, a relationship established by the subsequent result:

Theorem 1.2. If $s \in (0, 1)$, $n > 2s$ and Ω be a bounded open subset of $\mathbb{R}^n$ and the functions M and f satisfy $(\mathbf{M}_1)$ $-(\mathbf{M}_2)$ and $(\mathbf{f}_1)$ $-(\mathbf{f}_3)$. Then the following problem

$$M\left(\int_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} dx dy\right) = \int_{\Omega} f(x, u(x))\varphi(x) dx \quad (1.7)$$

for every $\varphi \in H^s(\mathbb{R}^n)$ with $\varphi = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, has a Mountain Pass type solution.

This paper has the following structure. In Section 2, we state some preliminary results needed for the framework of the paper. In Section 3, we prove Theorem 1.1 performing the classical Mountain Pass Theorem [1]. Moreover, as an application, we discuss the case of an equation driven by the fractional Laplacian operator and we prove Theorem 1.2.

2 Some preliminary results

In this section, we recall some preliminary and useful results which will be useful in the sequel.

The functional space X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and the map $(x, y) \mapsto (g(x) - g(y))\sqrt{K(x - y)}$ is in $L^2((\mathbb{R}^n \times \mathbb{R}^n) \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega), dxdy)$ (here $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$). Furthermore, we denote by X_0 the following linear subspace of X

$$X_0 := \{g \in X : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}.$$

Since $C_0^\infty(\Omega) \subseteq X_0$, then X and X_0 are non-empty. Moreover, the space X is endowed with the norm

$$\|g\|_X := \|g\|_{L^2(\Omega)} + \left(\int_Q |g(x) - g(y)|^2 K(x - y) dx dy\right)^{\frac{1}{2}}, \quad (2.1)$$

where $Q = \mathbb{R}^{2n} \setminus \mathcal{O}$ and $\mathcal{O} = (\mathcal{C}\Omega) \times (\mathcal{C}\Omega) \subset \mathbb{R}^{2n}$. Also, X_0 is endowed with the norm defined as

$$X_0 \ni g \mapsto \|g\|_{X_0} := \left(\int_Q |g(x) - g(y)|^2 K(x - y) dx dy\right)^{\frac{1}{2}}$$

and $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space (see [22, Lemma 7]), with the scalar product

$$\langle u, v \rangle_{X_0} := \int_Q (u(x) - u(y))(v(x) - v(y))K(x - y) dx dy.$$

Now, we define by $H^s(\Omega)$ the usual fractional Sobolev space endowed with the norm (the so-called Gagliardo norm)

$$\|g\|_{H^s(\Omega)} := \|g\|_{L^2(\Omega)} + \left(\int_{\Omega \times \Omega} \frac{|g(x) - g(y)|^2}{|x - y|^{n+2s}} dx dy\right)^{\frac{1}{2}}. \quad (2.2)$$

We note that, even in the model case in which $K(x) = |x|^{-(n+2s)}$, the norms in (2.1) and (2.2) are not the same, since $\Omega \times \Omega$ is strictly contained in Q ; this makes the fractional Sobolev space $H^s(\Omega)$ not sufficient for studying our problem from a variational point of view (see [22]).

Lemma 2.1. [22, Lemma 6] If $K : \mathbb{R}^n \setminus \{0\} \rightarrow (0, +\infty)$ be satisfied in conditions (1.2)-(1.4), then

a) there exists a positive constant C depend on s and n such that for every $v \in X_0$

$$\|v\|_{L^{2^*}(\Omega)}^2 = \|v\|_{L^{2^*}(\mathbb{R}^n)}^2 \leq C \int_{\mathbb{R}^{2n}} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} dx dy,$$

b) there is constant number $C > 1$, that is depend on λ, n, s and Ω such that for every $v \in X_0$,

$$\int_Q |v(x) - v(y)|^2 K(x - y) dx dy \leq \|v\|_X^2 \leq C \int_Q |v(x) - v(y)|^2 K(x - y) dx dy,$$

i.e.

$$\|v\|_{X_0} = \left(\int_Q |v(x) - v(y)|^2 K(x - y) dx dy \right)^{\frac{1}{2}} \quad (2.3)$$

is a norm on X_0 equivalent to the usual one defined in (2.1).

Lemma 2.2. [22, Lemma 8] Let $K : \mathbb{R}^n \setminus \{0\} \rightarrow (0, +\infty)$ be satisfied in conditions (1.2)-(1.4) and $\{v_j\}$ be a bounded sequence in X_0 . Then there exists $v_\infty \in L^\nu(\mathbb{R}^n)$ such that up to a subsequence $\{v_j\} \rightarrow v_\infty$ in $L^\nu(\mathbb{R}^n)$ as $j \rightarrow +\infty$, for any $\nu \in [1, 2^*)$.

Lemma 2.3. [22, Lemma 3] Suppose $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function which is satisfied in conditions $(\mathbf{f}_1)$ and $(\mathbf{f}_2)$. Then for every $\epsilon > 0$ exists $\delta = \delta(\epsilon)$ that a.e. $x \in \Omega$ and for every $t \in \mathbb{R}$,

$$|f(x, t)| \leq 2\epsilon|t| + q\delta(\epsilon)|t|^{q-1}, \quad (2.4)$$

so as a consequence:

$$|F(x, t)| \leq \epsilon|t|^2 + \delta(\epsilon)|t|^q, \quad (2.5)$$

where F is defined as in $(\mathbf{f}_3)$.

Lemma 2.4. [22, Lemma 4] Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function and F be a primitive of f with respect to second variable. Suppose $(\mathbf{f}_3)$ holds, then there exist two positive solutions of measurable function $m' = m'(x)$ and $M' = M'(x)$ such that a.e. $x \in \Omega$ and for every $t \in \mathbb{R}$,

$$F(x, t) \geq m'(x)|t|^\mu - M'(x), \quad (2.6)$$

moreover, if f be satisfied in conditions $(\mathbf{f}_1)$ and $(\mathbf{f}_2)$, then $m', M' \in L^\infty(\Omega)$.

3 Mountain Pass Solutions in a Non-Local Framework

It is obvious that problem (1.1) has a variational structure and the associated Euler-Lagrange functional to the problem is defined as follow:

$$J(u) = \frac{\widehat{M}}{2} \left(\int_Q |u(x) - u(y)|^2 K(x - y) dx dy \right) - \int_\Omega F(x, u(x)) dx.$$

Note that the function J is Fréchet differentiable in $u \in X_0$, and for every $\varphi \in X_0$ we have

$$\begin{aligned} \langle J'(u), \varphi \rangle &= M \left(\int_Q |u(x) - u(y)|^2 K(x - y) dx dy \right) \int_Q (u(x) - u(y))(\varphi(x) - \varphi(y)) K(x - y) dx dy \\ &\quad - \int_\Omega f(x, u(x)) \varphi(x) dx. \end{aligned}$$

Thus, the critical point of J is a solution of problem (1.1). For getting critical point we can use Mountain Pass Theorem (see [1, 21, 23]), then it must be checked whether the functional J is satisfied in a special geometric structure and also in Palais-Smale compactness condition. Now, we check the special geometric structure of functional J .

Proposition 3.1. If the Carathéodory function f satisfies conditions $(\mathbf{f}_1)$ and $(\mathbf{f}_2)$, then there exist $\rho > 0$ and $\beta > 0$ so that for every $u \in X_0$ with $\|u\|_{X_0} = \rho$, we have $J(u) \geq \beta$.

Proof . Suppose that u be a function in X_0 . By $(\mathbf{M}_1)$ and (2.5), for every $\epsilon > 0$ we have

$$\begin{aligned} J(u) &= \frac{\widehat{M}}{2} \left(\int_Q |u(x) - u(y)|^2 K(x-y) dx dy \right) - \int_{\Omega} F(x, u(x)) dx \\ &\geq \frac{m_0}{2} \int_Q |u(x) - u(y)|^2 K(x-y) dx dy - \epsilon \int_{\Omega} |u(x)|^2 dx - \delta(\epsilon) \int_{\Omega} |u(x)|^q dx \\ &\geq \frac{m_0}{2} \int_Q |u(x) - u(y)|^2 K(x-y) dx dy - \epsilon \|u\|_{L^2(\Omega)}^2 - \delta(\epsilon) \|u\|_{L^q(\Omega)}^q. \end{aligned} \quad (3.1)$$

We know that the embeddings $L^{2^*}(\Omega) \hookrightarrow L^2(\Omega)$ and $L^{2^*}(\Omega) \hookrightarrow L^q(\Omega)$ are continuous, thus according to these facts, Lemma 2.1-a), and (3.1), we get for every $\epsilon > 0$,

$$\begin{aligned} J(u) &\geq \frac{m_0}{2} \int_Q |u(x) - u(y)|^2 K(x-y) dx dy - \epsilon |\Omega|^{\frac{2^*-2}{2}} \|u\|_{L^{2^*}(\Omega)}^2 - |\Omega|^{\frac{(2^*-q)}{2}} \delta(\epsilon) \|u\|_{L^{2^*}(\Omega)}^q \\ &\geq \frac{m_0}{2} \int_Q |u(x) - u(y)|^2 K(x-y) dx dy - \epsilon C |\Omega|^{\frac{2^*-2}{2^*}} \int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dx dy \\ &\quad - \delta(\epsilon) C^{\frac{q}{2}} |\Omega|^{\frac{2^*-q}{2^*}} \left(\int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dx dy \right)^{\frac{q}{2}}. \end{aligned}$$

Now, considering (3.1), we have

$$\begin{aligned} J(u) &\geq \frac{m_0}{2} \int_Q |u(x) - u(y)|^2 K(x-y) dx dy - \frac{\epsilon C |\Omega|^{\frac{(2^*-2)}{2^*}}}{\lambda} \int_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y) dx dy \\ &\quad - \frac{\delta(\epsilon) C^{\frac{q}{2}} |\Omega|^{\frac{(2^*-q)}{2^*}}}{\lambda} \left(\int_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y) dx dy \right)^{\frac{q}{2}} \\ &= \left(\frac{m_0}{2} - \frac{\epsilon C |\Omega|^{\frac{(2^*-2)}{2^*}}}{\lambda} \right) \int_Q |u(x) - u(y)|^2 K(x-y) dx dy \\ &\quad - \frac{\delta(\epsilon) C^{\frac{q}{2}} |\Omega|^{\frac{(2^*-q)}{2^*}}}{\lambda} \left(\int_Q |u(x) - u(y)|^2 K(x-y) dx dy \right)^{\frac{q}{2}}. \end{aligned} \quad (3.2)$$

Considering $\epsilon > 0$ so that $\frac{2}{m_0} \epsilon C |\Omega|^{\frac{(2^*-2)}{2^*}} < \lambda$, and using (3.2) and Lemma 2.1-b), we conclude that

$$\begin{aligned} J(u) &\geq \alpha \int_Q |u(x) - u(y)|^2 K(x-y) dx dy \left[1 - \kappa \left(\int_Q |u(x) - u(y)|^2 K(x-y) dx dy \right)^{\frac{q}{2}-1} \right] \\ &\geq \alpha \|u\|_{X_0}^2 (1 - \kappa \|u\|_{X_0}^{q-2}) \end{aligned}$$

for α and κ that are suitable positive constants. Now, suppose $u \in X_0$ that $\|u\|_{X_0} = \rho > 0$. Due to $q > 2$, we can select ρ sufficiently small so that $1 - \kappa \rho^{q-2} > 0$, then

$$\inf_{u \in X_0, \|u\|_{X_0} = \rho} J(u) \geq \alpha \rho^2 (1 - \kappa \rho^{q-2}) =: \beta > 0,$$

thus, the assertion is proved. $\square$

Proposition 3.2. Assume f be a Carathéodory function which is satisfied in conditions $(\mathbf{f}_1)$ -($\mathbf{f}_3$). Then, there exists $e \in X_0$ that $e \geq 0$ a.e. in $\mathbb{R}^n$, $\|e\|_{X_0} > \rho$ and $J(e) < \beta$, where ρ and β are the same as in Proposition 3.1.

Proof . Suppose $u \in X_0$ is fixed so that $\|u\|_{X_0} = 1$ and assume $t > 0$. According to Lemma 2.4 and (1.6), we obtain

$$\begin{aligned} J(tu) &= \frac{\widehat{M}}{2} \left(\int_Q |tu(x) - tu(y)|^2 K(x-y) dx dy \right) - \int_{\Omega} F(x, tu(x)) dx \\ &\leq \frac{m_1 t^2}{2} + \frac{m_2}{2} - t^\mu \int_{\Omega} m'(x) |u(x)|^\mu dx + \int_{\Omega} M'(x) dx. \end{aligned}$$

Since $\mu > 2$, we conclude that $J(tu) \rightarrow -\infty$, as $t \rightarrow +\infty$. Then the claim follows by considering $e = Tu$, with T sufficiently large. $\square$

The geometry of the Mountain Pass Theorem is fulfilled by Propositions 3.1 and 3.2. Therefore, in order to apply the Mountain Pass Theorem, we have to check the validity of the Palais-Smale condition.

Proposition 3.3. Assume f is a Carathéodory function which is satisfied in conditions $(\mathbf{f}_1)$ – $(\mathbf{f}_3)$. If $C \in \mathbb{R}$, $\{u_j\}$ be a sequence in X_0 such that

$$J(u_j) \rightarrow C, \quad (3.3)$$

and

$$\sup \{ |\langle J'(u_j), \varphi \rangle| : \varphi \in X_0, \|\varphi\|_{X_0} = 1 \} \rightarrow 0, \quad \text{as } j \rightarrow \infty, \quad (3.4)$$

then $\{u_j\}$ is bounded in X_0 .

Proof . For every $j \in N$, using (3.3) and (3.4), it is obviously concluded that exists $\kappa > 0$ such that

$$|\langle J'(u_j), \frac{u_j}{\|u_j\|_{X_0}} \rangle| \leq \kappa, \quad (3.5)$$

and

$$|J(u_j)| \leq \kappa. \quad (3.6)$$

Also, according to Lemma 2.3, by substituting $\epsilon = 1$, we can write

$$\left| \int_{\Omega \cap \{|u_j| \leq r\}} (F(x, u_j(x)) - \frac{1}{\mu} f(x, u_j(x)u_j(x)) dx \right| \leq (r^2 + \delta(1)r^q + \frac{2}{\mu}r + \frac{q}{\mu}\delta(1)r^{q-1})|\Omega| =: \tilde{\kappa}. \quad (3.7)$$

By $(\mathbf{f}_3)$, we have

$$\begin{aligned} J(u_j) - \frac{1}{\mu} \langle J'(u_j), u_j \rangle &= \frac{\widehat{M}}{2} \left(\int_Q |u(x) - u(y)|^2 K(x - y) dx dy \right) - \int_{\Omega} F(x, u_j) dx \\ &\quad - \frac{1}{\mu} M \left(\int_Q |u(x) - u(y)|^2 k(x - y) dx dy \right) \left(\int_Q |u(x) - u(y)|^2 k(x - y) dx dy \right) \\ &\quad - \frac{1}{\mu} \int_{\Omega} f(x, u_j) u_j dx \\ &= (M \|u_j\|_{X_0}^2) \left[\frac{\widehat{M}(\|u_j\|_{X_0}^2)}{2M \|u_j\|_{X_0}^2} - \frac{\|u_j\|_{X_0}}{\mu} \right] - \frac{1}{\mu} \int_{\Omega} (\mu F(x, u_j) - f(x, u_j)u_j) dx. \end{aligned}$$

Hence, using $(\mathbf{M}_1)$ and (3.7) we get

$$\begin{aligned} J(u_j) - \frac{1}{\mu} \langle J'(u_j), u_j \rangle &\geq m_0 \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u_j\|_{X_0}^2 - \frac{1}{\mu} \int_{\Omega} (\mu F(x, u_j) - f(x, u_j)u_j) dx \\ &\geq m_0 \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u_j\|_{X_0}^2 - \int_{\Omega \cap \{|u_j| \leq r\}} \left(F(x, u_j) - \frac{1}{\mu} f(x, u_j)u_j \right) dx \\ &\geq m_0 \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u_j\|_{X_0}^2 - \tilde{\kappa}. \end{aligned} \quad (3.8)$$

As a result of (3.5) and (3.6), we have

$$J(u_j) - \frac{1}{\mu} \langle J'(u_j), u_j \rangle \leq \kappa(1 + \|u_j\|_{X_0}),$$

and using (3.8), for every $j \in N$

$$\|u_j\|_{X_0}^2 \leq \kappa_*(1 + \|u_j\|_{X_0}),$$

for a proper positive constant κ_* . Then the claim of Proposition 3.3 is completed. $\square$

Now, we state the following Proposition that its proof follows the general approach outlined in Proposition 12 of [22].

Proposition 3.4. Assume that f is a Carathéodory function which is satisfied in conditions $(\mathbf{f}_1)$ – $(\mathbf{f}_3)$. If $\{u_j\}$ be a bounded sequence in X_0 and (3.4) is satisfied, then there exists $u_\infty \in X_0$ so that, up to a subsequence, $\|u_j - u_\infty\|_{X_0} \rightarrow 0$, when $j \rightarrow +\infty$.

Proof . As we know, every bounded sequence such as $\{u_j\}$ in a reflexive space, like X_0 , has a weakly convergent subsequence. Therefore, there exists a subsequence, still denote by $\{u_j\}$ and a function $u_\infty \in X_0$, so that

$$\int_Q (u_j(x) - u_j(y))(\varphi(x) - \varphi(y))K(x - y) dx dy \longrightarrow \int_Q (u_\infty(x) - u_\infty(y))(\varphi(x) - \varphi(y))K(x - y) dx dy \quad (3.9)$$

for every $\varphi \in X_0$, as $j \rightarrow +\infty$. Moreover by Lemma 2.2, up to a sub-sequence

$$\begin{aligned} u_j &\rightarrow u_\infty && \text{in } L^q(\mathbb{R}^n), \\ u_j &\rightarrow u_\infty && \text{a.e. in } \mathbb{R}^n, \end{aligned} \quad (3.10)$$

as $j \rightarrow +\infty$ and there exists $l \in L^q(\mathbb{R}^n)$ such that

$$|u_j(x)| \leq l(x) \quad \text{a.e. in } \mathbb{R}^n \quad \text{for all } j \in N, \quad (3.11)$$

(see [8, Theorem IV.9]). Using $(\mathbf{f}_1)$, (3.9)–(3.11) and the fact that the map $t \rightarrow f(\cdot, t)$ is continuous in $t \in \mathbb{R}$, and by Dominated Convergence Theorem, we arrive at

$$\int_\Omega f(x, u_j(x))u_j(x) dx \rightarrow \int_\Omega f(x, u_\infty(x))u_\infty(x) dx, \quad (3.12)$$

and

$$\int_\Omega f(x, u_j(x))u_\infty(x) dx \rightarrow \int_\Omega f(x, u_\infty(x))u_\infty(x) dx, \quad (3.13)$$

as $j \rightarrow +\infty$. Moreover, by (3.4), we get

$$0 \leftarrow \langle J'(u_j), u_j \rangle = M \left(\int_Q |u_j(x) - u_j(y)|^2 K(x - y) dx dy \right) \left(\int_Q |u(x) - u(y)|^2 K(x - y) dx dy \right) - \int_\Omega f(x, u_j(x))u_j(x) dx.$$

Also, considering (3.12), we conclude that

$$M \left(\int_Q |u_j(x) - u_j(y)|^2 K(x - y) dx dy \right) \left(\int_Q |u(x) - u(y)|^2 K(x - y) dx dy \right) \rightarrow \int_\Omega f(x, u_\infty)u_\infty(x) dx, \quad (3.14)$$

as $j \rightarrow +\infty$. So

$$\begin{aligned} 0 \leftarrow \langle J'(u_j), u_\infty \rangle &= M \left(\int_Q |u_j(x) - u_j(y)|^2 K(x - y) dx dy \right) \left(\int_Q (u_j(x) - u_j(y))(u_\infty(x) - u_\infty(y))K(x - y) dx dy \right) \\ &\quad - \int_\Omega f(x, u_j(x))u_\infty(x) dx, \end{aligned} \quad (3.15)$$

as $j \rightarrow \infty$. By (3.9), (3.13) and (3.15) we obtain

$$\int_Q |u_\infty(x) - u_\infty(y)|^2 K(x - y) dx dy = \int_\Omega f(x, u_\infty(x))u_\infty(x) dx, \quad (3.16)$$

Thus, considering relations (3.14) and (3.16), we have

$$\int_Q |u_j(x) - u_j(y)|^2 K(x-y) dx dy \rightarrow \int_Q |u_\infty(x) - u_\infty(y)|^2 K(x-y) dx dy, \quad (3.17)$$

then $\|u_j\|_{X_0} \rightarrow \|u_\infty\|_{X_0}$ as $j \rightarrow \infty$. Finally

$$\begin{aligned} \|u_j - u_\infty\|_{X_0}^2 &= \|u_j\|_{X_0}^2 + \|u_\infty\|_{X_0}^2 - 2 \int_Q (u_j(x) - u_j(y))(u_\infty(x) - u_\infty(y)) K(x-y) dx dy \\ &\rightarrow 2\|u_\infty\|_{X_0}^2 - 2 \int_Q |u_\infty(x) - u_\infty(y)|^2 K(x-y) dx dy = 0, \end{aligned}$$

as $j \rightarrow \infty$, by considering (3.9) and (3.17). Then, we get the assertion of Proposition 3.4. $\square$

3.1 Proof of Theorem 1.1

As Propositions 3.1–3.4 are satisfied, the Mountain Pass Theorem [23] leads to a critical point $u \in X_0$ of J . Moreover

$$J(u) \geq \beta > 0 = J(0),$$

and the claim is completed.

As a classic model of Laplacian (see [23, pages 103,104]), a sign changing Mountain Pass solution could be defined. Actually, the following conclusion is correct:

Corollary 3.5. Assume all assumptions of Theorem 1.1 are satisfied, then problem (1.1) refer to a non-negative solution $u_+ \in X_0$ and a non-positive solution $u_0 \in X_0$ that are Mountain Pass solution and are not equivalent to zero.

Proof . For showing the existence of a non-negative (non-positive) solution of problem (1.1), it is needed to define the following function:

$$F_\pm(x, t) = \int_0^t f_\pm(x, \tau) d\tau$$

with

$$f_+(x, t) = \begin{cases} f(x, t), & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases} \quad \text{and} \quad f_-(x, t) = \begin{cases} 0, & \text{if } t < 0 \\ f(x, t), & \text{if } t \geq 0 \end{cases}$$

Note that $f_\pm$ is satisfied in $(\mathbf{f}_1)$ and $(\mathbf{f}_2)$, As assumption $(\mathbf{f}_3)$ by f_+ and F_+ a.e. is satisfied on Ω and for every $t > r$ by f_- and F_- a.e. in Ω and for every $t < -r$.

Suppose $J_\pm : X_0 \rightarrow \mathbb{R}$ be a function by following definition:

$$J_\pm(u) = \frac{\widehat{M}}{2} \left(\int_Q |u(x) - u(y)|^2 K(x-y) dx dy \right) - \int_\Omega F_\pm(x, u(x)) dx.$$

Obviously, we can see that function J is well-defined and is Fréchet differentiable in $u \in X_0$ and for every $\varphi \in X_0$, so

$$\begin{aligned} \langle J'_\pm(u), \varphi \rangle &= M \left(\int_Q |u(x) - u(y)|^2 K(x-y) dx dy \right) \left(\int_Q (u(x) - u(y))(\varphi(x) - \varphi(y)) \right. \\ &\quad \left. K(x-y) dx dy \right) - \int_\Omega f_\pm(x, u(x)) \varphi(x) dx. \end{aligned} \quad (3.18)$$

Moreover, Propositions 3.1–3.4 hold for $J_\pm$. (As it was mentioned in proposition 3.2 that $e \geq 0$) and $J_\pm(0) = 0$. Then according to the Mountain Pass theorem, there is a non-trivial critical point $u_\pm \in X_0$ of $J_\pm$. We claim that $u_+ \in \mathbb{R}^n$ is non-negative. In fact, we define $\varphi := (u_+)^-$, so that v^- is negative part of v , i.e. $v^- = \max\{-v, 0\}$. We

know $u_+ \in X_0$ leads to $u_+^- \in X_0$ and we can use φ in (3.18). Also, using (M_1) and the definition of f_+ and of negative part, we have

$$\begin{aligned} 0 &= \langle J'_\pm(u_+), (u_+)^- \rangle \\ &= M \left(\int_Q (u_+(x) - u_+(y)) ((u_+)^-(x) - (u_+)^-(y)) K(x-y) dx dy \right) \\ &\quad \left(\int_Q (u_+(x) - u_+(y)) ((u_+)^-(x) - (u_+)^-(y)) K(x-y) dx dy \right) - \int_\Omega f_+(x, u_+(x)) (u_+)^- dx \\ &> m_0 \int_Q (u_+(x) - u_+(y)) ((u_+)^-(x) - (u_+)^-(y)) K(x-y) dx dy \\ &= m_0 \|(u_+)^-\|_{X_0}^2. \end{aligned}$$

Therefore, $\|(u_+)^-\|_{X_0} = 0$, then $u_+ \geq 0$ a.e. in $\mathbb{R}^n$, that completes the claim. In a similar process it can be shown that $u_- \in \mathbb{R}^n$ is non-positive. $\square$

3.2 Proof of Theorem 1.2

If in Theorem 1.1, we choose $K(x) = |x|^{-(n+2s)}$ the claim of Theorem 1.2 is obtained.

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