

Advanced Bayesian Inference for Big Data of Lognormal Distribution Under Generalized, Logarithmic and Weighted Loss Function

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Abstract

In the era of Big Data, modeling asymmetric processes requires a departure from traditional symmetric loss functions to achieve reliable Bayesian inference. This paper addresses the challenge of estimating parameters for large-scale lognormal distributions by introducing a suite of advanced loss functions. We systematically analyze the impact of Generalized Entropy, Generalized Linex, Logarithmic Squared Error, and Weighted Linex loss functions on the efficiency of Bayesian estimators. Unlike standard approaches, these functions are specifically engineered to handle the skewedness and scale of massive datasets by providing a more nuanced penalty structure. Our comparative analysis reveals that the integration of generalized and weighted loss functions significantly enhances the stability and accuracy of posterior estimates in large-volume, asymmetric data environments. The findings provide critical insights for developing robust predictive models in fields where lognormal-distributed data are prevalent, such as finance, reliability engineering, and bioinformatics.

Keywords: (Bayes inference, lognormal distribution, big data, generalized linex loss function, logarithmic squared error loss function, generalized entropy loss function, weighted linex loss function)

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Introduction

The advancement of modern statistical inference is increasingly driven by the necessity to model complex, high-dimensional datasets that deviate significantly from classical assumptions. In particular, the lognormal distribution serves as a cornerstone in modeling phenomena characterized by inherent skewedness and heavy-tailed behavior, frequently encountered in fields such as financial econometrics, reliability engineering, and biological sciences. The novelty of this work stems from the systematic evaluation of these multi-faceted loss criteria within a unified Bayesian framework, specifically optimized for the complexities of massive, non-symmetric data regimes. Within the Bayesian paradigm, the quality of parameter estimation is fundamentally governed by the loss function, which dictates the decision-theoretic penalty for estimation errors.

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Traditionally, Bayesian estimation has been dominated by symmetric loss functions, most notably the Squared Error Loss (SEL) [3]. While these functions are mathematically tractable, they possess a critical limitation: they treat overestimations and underestimations with equal weight, failing to capture the asymmetric nature of lognormal-distributed data. Such a deficiency often leads to biased estimators and unreliable predictive intervals when applied to large-scale, skewed datasets. Consequently, there is a compelling theoretical need to move toward more sophisticated, non-linear, and asymmetric loss criteria that can adapt to the specific geometry of the posterior distribution.

This research addresses this gap by exploring a diverse class of advanced loss functions, categorized by their ability to provide nuanced penalty structures. We investigate the efficacy of Generalized Entropy and Generalized Linex loss functions, which offer enhanced flexibility through their generalization parameters. Furthermore, we incorporate Logarithmic Squared Error to better align with the scale of lognormal data, and Weighted Linex loss functions to allow for controlled asymmetry in the decision-making process. The complexity of these functions lies in their capacity to bridge the gap between pure mathematical abstraction and the practical requirements of asymmetric data modeling.

The methodological contribution of this study is presented in a structured sequence. First, we establish the rigorous mathematical and statistical foundations by deriving the optimal Bayesian estimators under each proposed loss function. Subsequently, we transition to the computational phase, where large-scale data simulations are performed using the R programming environment. This involves the implementation of custom algorithms within R to execute the estimation procedures and handle the computational demands of the generalized functions. Finally, the performance of these estimators is evaluated through a comprehensive presentation of numerical results, followed by a detailed comparative analysis and discussion of the findings.

Step 1: Distribution of random variables

In the following formula derived from [9] we have:

$$\begin{aligned} X_i &\sim \text{LogNormal}(\mu, \sigma^2) \quad , \quad X_i = \exp(H_i), \quad X_i > 0 \\ H_i &= \ln(X_i) \sim \text{Normal}(\mu, \sigma^2) \\ f_{\mu, \sigma^2}(x) &= \frac{x^{-1}}{\sigma \cdot \sqrt{2\pi}} \cdot \exp\left(\frac{-(\ln(x) - \mu)^2}{2\sigma^2}\right), \quad x, \sigma > 0 \end{aligned}$$

Step 2: Prior distribution function

Non-informative priors are widely utilized when prior information is scarce, aiming to spread probability mass across the complete parameter space. Following the seminal model by Laplace (1812) for $S = 1$, and assuming the regularity conditions described by Lehmann (1996), the non-informative prior is constructed using the Fisher information from [4] as follows:

$$\begin{aligned} \Gamma(\psi) &= S, \quad S > 0 \\ \Gamma(\psi) &\propto \sqrt{\det(I(\psi))} \\ I(\psi) &= -E_{\gamma} \left[\frac{\partial^2 \log g(x, \gamma)}{\partial \gamma^2} \right] \end{aligned}$$

In scenarios where prior information regarding the parameters is minimal or highly diffuse, non-informative prior distributions are employed to ensure that the posterior inference is predominantly driven by the observed data. A prominent example is the Jeffreys' prior, which is derived from the Fisher information matrix. Often referred to as a standard or non-informative prior in continuous settings, the Jeffreys' prior provides a formal mechanism for constructing objective priors when parametric knowledge is insufficient. Specifically, the prior distribution is determined based on the Fisher information matrix, which is defined as follows:

$$\psi = (\mu, \sigma^2)$$

Under the condition of regularity, the Fisher information matrix is as follows:

$$\begin{aligned}
I_x(\psi) &= -E \left(\frac{\partial^2}{\partial \psi^2} \ln(g(x; \psi)) \right) = E \left(\left[\frac{\partial}{\partial \psi} \ln(g(x; \psi)) \right]^2 \right) \\
g_{\mu, \sigma^2}(x) &= \frac{x^{-1}}{\sigma \sqrt{2\pi}} \cdot \exp \left(-\frac{(\ln(x) - \mu)^2}{2\sigma^2} \right), \quad x; \sigma > 0 \\
\ln(g_{\mu, \sigma^2}(x)) &= -\ln(2\pi\sigma^2)^{\frac{1}{2}} - \ln(x) + \left(\frac{-(\ln(x) - \mu)^2}{2\sigma^2} \right) \\
\frac{\partial \ln(g_{\mu, \sigma^2}(x))}{\partial \sigma^2} &= \frac{1}{2\sigma^2} - \frac{(\ln(x) - \mu)^2}{2\sigma^4} \\
\frac{\partial \ln(g_{\mu, \sigma^2}(x))}{\partial \mu} &= \frac{\ln(x) - \mu}{\sigma^2} \\
\frac{\partial^2 \ln(g_{\mu, \sigma^2}(x))}{\partial \mu \partial \sigma^2} &= \frac{\ln(x) - \mu}{\sigma^4} \\
\frac{\partial^2 \ln(g_{\mu, \sigma^2}(x))}{\partial \mu^2} &= -\frac{1}{\sigma^2} \\
\frac{\partial^2 \ln(g_{\mu, \sigma^2}(x))}{\partial (\sigma^2)^2} &= -\frac{1}{2\sigma^4} + \frac{(\ln(x) - \mu)^2}{\sigma^6} \\
I(\mu, \sigma^2) &= E \left(\left[\frac{\partial}{\partial \mu} \ln(g_{\mu, \sigma^2}(x)) \right] \left[\frac{\partial}{\partial \mu} \ln(g_{\mu, \sigma^2}(x)) \quad \frac{\partial}{\partial \sigma^2} \ln(g_{\mu, \sigma^2}(x)) \right] \right) \\
I(\mu, \sigma^2) &= E \left(\begin{bmatrix} \frac{\partial^2 \ln(g_{\mu, \sigma^2}(x))}{\partial \mu^2} & \frac{\partial^2 \ln(g_{\mu, \sigma^2}(x))}{\partial \mu \partial \sigma^2} \\ \frac{\partial^2 \ln(g_{\mu, \sigma^2}(x))}{\partial \sigma^2 \partial \mu} & \frac{\partial^2 \ln(g_{\mu, \sigma^2}(x))}{\partial \sigma^4} \end{bmatrix} \right)
\end{aligned}$$

So, as a result, the Fisher information matrix for the lognormal distribution is of the following form:

$$I(\mu, \sigma^2) = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{bmatrix}$$

As a result and from [7] we have:

$$\Gamma(\mu, \sigma^2) \propto |I(\mu, \sigma^2)|^{S_1}, \quad S_1 \in IR^+$$

From [8] we have:

$$\Rightarrow \Gamma(\mu, \sigma^2) \propto |I(\mu, \sigma^2)|^{\frac{1}{2}}$$

So the prior distribution obtained from the Fisher information matrix and used according to the conditions is in the following form.

$$\Gamma(\mu, \sigma^2) \propto \frac{1}{\sigma^2} \quad \Gamma(\mu, \sigma^2) \propto 1 \quad \text{uniform distribution} \quad \Gamma(\mu, \sigma^2) \propto \frac{1}{\sigma^3}$$

We also use prior distributions called (awareness) and non-informative prior distributions called (unawareness). So, as a result, we have:

$$\Gamma(\mu, \sigma^2) \propto \frac{1}{\sigma^2}$$

Step 3: Likelihood Function

In this study, due to the independence and distribution of random variables, the likelihood function is defined as the product of density functions [6], which is done as follows.

$$L(\mu, \sigma^2 | x) = \prod_{i=1}^n f_{\mu, \sigma^2}(x) = (2\pi\sigma^2)^{-\frac{n}{2}} \cdot \prod_{i=1}^n (x_i)^{-1} \cdot \exp \left(\sum_{i=1}^n \frac{-(\ln(x_i) - \mu)^2}{2\sigma^2} \right) \propto \sigma^{-n} \exp \left(\frac{-1}{2\sigma^2} \sum_{i=1}^n (H_i - \mu)^2 \right)$$

$$S_* = \sum_{i=1}^n (H_i - \bar{H})^2$$

$$\sum_{i=1}^n (H_i - \mu)^2 = n (\mu - \bar{H})^2 + S_*$$

$$\bar{H} = \frac{1}{n} \sum_{i=1}^n H_i = \frac{1}{n} \sum_{i=1}^n \ln(x_i)$$

Step 4: Posterior distribution function

$$\Gamma(\mu, \sigma^2 | x) \propto L(\mu, \sigma^2 | x) \cdot \Gamma(\mu, \sigma^2)$$

$$\Gamma(\mu, \sigma^2 | x) = \frac{1}{\sigma^2} \cdot \frac{1}{\sigma^n} \exp\left(\frac{-1}{2\sigma^2} \sum_{i=1}^n (H_i - \mu)^2\right) \propto \frac{1}{\sigma^{n+2}} \exp\left(\frac{-1}{2\sigma^2} [S_* + n(\mu - \bar{H})^2]\right)$$

Marginal distributions of lognormal distribution parameters from [11] and [5]

$$\Gamma(\mu | \sigma^2, x) = \int \Gamma(\mu, \sigma^2 | x) d\sigma^2$$

$$\Gamma(\mu | \sigma^2, x) \propto \exp\left(\frac{-1}{2\sigma^2} [n(\mu - \bar{H})^2]\right)$$

$$\mu | \sigma^2 \sim N(\bar{H}, \frac{\sigma^2}{n})$$

$$\Gamma(\sigma^2 | \mu, x) = \int \Gamma(\mu, \sigma^2 | x) d\mu = \int_{-\infty}^{+\infty} \frac{1}{\sigma^{n+2}} \cdot \exp\left(\frac{-1}{2\sigma^2} [S_* + n(\mu - \bar{H})^2]\right) d\mu$$

$$\propto \frac{1}{\sigma^{n+2}} \cdot \exp\left(\frac{-S_*}{2\sigma^2}\right) \times \int_{-\infty}^{+\infty} \exp\left(\frac{-1}{2\sigma^2} [n(\mu - \bar{H})^2]\right) d\mu$$

$$= \frac{1}{\sigma^{n+2}} \cdot \exp\left(\frac{-S_*}{2\sigma^2}\right) \cdot \sqrt{\frac{2\pi\sigma^2}{n}} = \frac{1}{\sigma^{n+1}} \cdot \exp\left(\frac{-S_*}{2\sigma^2}\right)$$

$$\sigma^2 | x \sim \text{Inverse} - \Gamma(\alpha = \frac{n-1}{2}, \beta = \frac{S_*}{2})$$

1- Bayesian estimation under the generalized linex loss function

$$L(\psi, B) = b \cdot (\exp\left\{c \left(\frac{B}{\psi} - 1\right)\right\} - c \cdot \left(\frac{B}{\psi} - 1\right) - 1)$$

$$\arg \min E(L(\psi, B | X)) = \frac{\partial}{\partial B} E\left[b \cdot (\exp\left\{c \left(\frac{B}{\psi} - 1\right)\right\} - c \left(\frac{B}{\psi} - 1\right) - 1) | X\right] = 0$$

$$E(L(\psi, B | X)) = b \cdot [E(\exp\left\{c \left(\frac{B}{\psi} - 1\right)\right\}) - c \cdot E\left(\left(\frac{B}{\psi} - 1\right)\right) - 1]$$

$$\frac{\partial}{\partial B} E(L(\psi, B | X)) = b \cdot \left[c \cdot E\left(\frac{1}{\psi} \cdot \exp\left\{\left(\frac{B}{\psi} - 1\right)\right\}\right) - c \cdot E\left(\frac{1}{\psi}\right)\right] = 0$$

$$\rightarrow E\left(\frac{1}{\psi} \cdot \exp\left\{\left(\frac{B}{\psi} - 1\right)\right\}\right) = E\left(\frac{1}{\psi}\right)$$

Now for the parameter σ^2 we have

$$\begin{aligned}
E\left(\frac{1}{\sigma^2} \middle| X\right) &= \frac{n-1}{S_0} \\
E\left(\frac{1}{\sigma^2} \cdot \exp\left\{\left(\frac{B}{\sigma^2} - 1\right)\right\} \middle| X\right) &= \int \frac{1}{\sigma^2} \cdot \exp\left\{\left(\frac{B}{\sigma^2} - 1\right)\right\} \cdot \Gamma(\sigma^2 | X) d\sigma^2 \\
&= \int \frac{1}{\sigma^2} \cdot \exp\left\{\left(\frac{B}{\sigma^2} - 1\right)\right\} \cdot \frac{\left(\frac{S_0}{2}\right)^{\left(\frac{n-1}{2}\right)}}{\Gamma\left(\frac{n-1}{2}\right)} \cdot (\sigma^2)^{-\left(\frac{n-1}{2}\right)} \cdot \exp\left(-\frac{S_0}{2\sigma^2}\right) d\sigma^2
\end{aligned}$$

By placing the following values

$$d\sigma^2 = -\frac{1}{u^2}, \quad u = \frac{1}{\sigma^2}$$

$$E\left(\frac{1}{\sigma^2} \cdot \exp\left\{\left(\frac{B}{\sigma^2} - 1\right)\right\} \middle| X\right) = \int u \cdot \exp\{(Tu - 1)\} \cdot \frac{\left(\frac{S_0}{2}\right)^{\left(\frac{n-1}{2}\right)}}{\Gamma\left(\frac{n-1}{2}\right)} \cdot u^{-\left(\frac{n-1}{2}\right)} \cdot \exp\left(-\frac{S_0 \cdot u}{2}\right) \cdot \frac{1}{u^2} du$$

Given the complexity of this integral, to solve using the delta approximation:

$$\begin{aligned}
E\left(\frac{1}{\sigma^2} \cdot \exp\left\{\left(\frac{B}{\sigma^2} - 1\right)\right\} \middle| X\right) &\approx \frac{n}{s_0} \cdot \exp\left\{\left(\frac{B}{s_0} - 1\right)\right\} \\
\frac{n}{s_0} \cdot \exp\left\{\left(\frac{B}{s_0} - 1\right)\right\} &= \frac{n-1}{s_0} \approx \frac{n}{s_0} \\
\exp\left\{\left(\frac{B}{s_0} - 1\right)\right\} &\approx 1, \quad \frac{B}{s_0} - 1 \approx 0 \quad \rightarrow B = \frac{s_0}{n} \\
B_2^{Bayes} &= \frac{s_0}{n}
\end{aligned}$$

This loss function is only dependent on the scale parameter, so for the location parameter we use its maximum likelihood estimate in the numerical comparison table.

2- Bayesian estimation under the generalized entropy loss function

In the following formula derived from [10] we have:

$$\begin{aligned}
L(\psi, B) &= \left(\frac{B}{\psi}\right)^c - c \cdot \ln\left(\frac{B}{\psi}\right) - 1 \\
\arg \min E(L(\psi, B | X)) &= \frac{\partial}{\partial B} E\left[\left(\frac{B}{\psi}\right)^c - c \cdot \ln\left(\frac{B}{\psi}\right) - 1\right] = 0 \\
E\left[\left(\frac{B}{\psi}\right)^c - c \cdot \ln\left(\frac{B}{\psi}\right) - 1\right] &= B^c \cdot E\left(\left(\frac{1}{\psi}\right)^c \middle| X\right) - c \cdot \ln(B) + c \cdot E(\psi | X) - 1 \\
\frac{\partial}{\partial B} \left(B^c \cdot E\left(\left(\frac{1}{\psi}\right)^c \middle| X\right) - c \cdot \ln(B) + c \cdot E(\psi | X) - 1\right) &= 0 \\
\rightarrow c \cdot B^{c-1} \cdot E\left(\left(\frac{1}{\psi}\right)^c \middle| X\right) - \frac{c}{B} &= 0 \\
\rightarrow B^c &= \frac{1}{E\left(\left(\frac{1}{\psi}\right)^c \middle| X\right)} \\
\rightarrow B &= \left(\frac{1}{E\left(\left(\frac{1}{\psi}\right)^c \middle| X\right)}\right)^{\frac{-1}{c}} \\
B_1^{bayes} &= \left(\frac{1}{E\left(\left(\frac{1}{\mu}\right)^c \middle| X\right)}\right)^{\frac{-1}{c}} = \left(\frac{1}{\int_{-\infty}^{\infty} \left(\frac{1}{\mu}\right)^c \cdot \Gamma(\mu | \sigma^2, X) d\mu}\right)^{\frac{-1}{c}}
\end{aligned}$$

The value of the above integral diverges. The Bayesian estimate for $\hat{\mu}_{Bayes}$ does not exist under the generalized entropy loss function, and this loss function depends on the scale parameter, and we use Maximum likelihood Estimation (mle) [1] for comparisons in the numerical results table.

$$\begin{aligned}
 B_2^{bayes} &= \left(\frac{1}{E\left(\left(\frac{1}{\sigma^2}\right)^c | X\right)} \right)^{-\frac{1}{c}} \\
 &\rightarrow E\left(\left(\frac{1}{\sigma^2}\right)^c | X\right) = \frac{\Gamma(\alpha + c)}{\Gamma(\alpha)} \cdot \beta^{-c} = \frac{\Gamma(\frac{n-1}{2} + c)}{\Gamma(\frac{n-1}{2})} \cdot \left(\frac{S_0}{2}\right)^{-c} \\
 B_2^{bayes} &= \left(\frac{1}{\frac{\Gamma(\frac{n-1}{2} + c)}{\Gamma(\frac{n-1}{2})} \cdot \left(\frac{S_0}{2}\right)^{-c}} \right)^{-\frac{1}{c}} = \frac{S_0}{2} \cdot \left(\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-1}{2} + c)} \right)^{\frac{1}{c}}
 \end{aligned}$$

3- Bayesian estimation under the Squared Error in logarithm loss function

$$\begin{aligned}
 L(\psi, B) &= (\log(B) - \log(\psi))^2 \\
 \arg \min E(L(\psi, B)) &= \frac{\partial}{\partial \log(B)} E((\log(B) - \log(\psi))^2) = 0 \\
 \rightarrow \frac{\partial}{\partial \log(B)} [(\log(B))^2 + E((\log(\psi))^2) - 2\log(B)E(\log(\psi))] &= 0 \\
 \rightarrow 2\log(B) - 2E(\log(\psi)) &= 0 \\
 B &= \exp(E(\log(\psi|X))) \\
 B_1^{bayes} &= \arg \min E(L(\mu, B)) = \exp(E(\log(\mu|X))) \\
 B_2^{bayes} &= \arg \min E(L(\sigma^2, B)) = \exp(E(\log(\sigma^2|X))) \\
 E(\log(\sigma^2|X)) &= \log(\beta) - \varphi(\alpha) \\
 \rightarrow \exp(E(\log(\sigma^2|X))) &= \exp(\log(\beta) - \varphi(\alpha))
 \end{aligned}$$

where $\varphi(\alpha)$ is the digamma function.

$$\begin{aligned}
 \varphi(\alpha) &= \log(\alpha) - \frac{1}{2\alpha} - \frac{1}{12\alpha^2} + o(\alpha^{-4}) \rightarrow \exp(\log(\beta) - \varphi(\alpha)) \\
 &= \exp\left(\log\left(\frac{S_0}{2}\right) - \log(\alpha) + \frac{1}{2\alpha} + \frac{1}{12\alpha^2} - o(\alpha^{-4})\right) \\
 &= \exp\left(\log\left(\frac{S_0}{2}\right) - \log\left(\frac{n-1}{2}\right) + \frac{1}{2 \times \frac{n-1}{2}} + \frac{1}{12\left(\frac{n-1}{2}\right)^2} - o\left(\left(\frac{n-1}{2}\right)^{-4}\right)\right) \\
 &\rightarrow \exp(\log(\beta) - \varphi(\alpha)) \sim \log\left(\frac{S_0}{n-1}\right) + \frac{1}{n-1} + \frac{1}{3(n-1)^2} \\
 \rightarrow B_2^{bayes} &= \log\left(\frac{S_0}{n-1}\right) + \frac{1}{n-1} + \frac{1}{3(n-1)^2}
 \end{aligned}$$

$$\begin{aligned}
 B_1^{bayes} &= \exp(E(\log(\mu|X))) = \int \log(\mu) \cdot \Gamma(\mu|\sigma^2, x) d\mu \\
 &= \int \log(\mu) \cdot \exp\left(\frac{-1}{2\sigma^2} [n(\mu - \bar{H})^2]\right) d\mu = \exp\left(\frac{-n}{2\sigma^2} \bar{H}^2\right) \int \log(\mu) \cdot \exp\left(\frac{-n}{2\sigma^2} [\mu^2 - 2\mu\bar{H}]\right) d\mu
 \end{aligned}$$

With using Taylor's explanation approximation, we have:

$$\begin{aligned}
 \exp(E(\log(\mu|X))) &= \exp\left(\ln(\bar{H}) + \left(\frac{-var(H)}{2n\bar{H}^2}\right)\right) = \bar{H} \cdot \exp\left(\left(\frac{-var(H)}{2n\bar{H}^2}\right)\right) \\
 \rightarrow B_1^{bayes} &= \bar{H} \cdot \exp\left(\left(\frac{-S_*}{2n\bar{H}^2}\right)\right)
 \end{aligned}$$

4- Bayesian estimation under the weighted linex loss function

This function, represent an advanced extension of the linex loss, which not only preserves inherent asymmetry of the original formulation but also introduce a tunable weight parameter to precisely modulate the intensity of penalization.

In the following formula derived from [2] we have:

$$\begin{aligned}
 L(\psi, B) &= w(\psi).(\exp \{c(B - \psi)\} - c(B - \psi) - 1) \\
 \arg \min E(L(\psi, B)) &= \frac{\partial}{\partial B} E(w(\psi).(\exp \{c(B - \psi)\} - c(B - \psi) - 1)|X) = 0 \\
 &\rightarrow c.e^{cB}.E(w(\psi).e^{-c\psi}|X) - c.E(w(\psi)|X) = 0 \\
 &\rightarrow e^{cB} = \frac{E(w(\psi)|X)}{E(w(\psi).e^{-c\psi}|X)} \\
 &\rightarrow B = -\frac{1}{c} \ln\left(\frac{E(w(\psi).e^{-c\psi}|X)}{E(w(\psi)|X)}\right) \\
 B_1^{bayes} &= -\frac{1}{c} \ln\left(\frac{E(w(\mu).e^{-c\mu}|X)}{E(w(\mu)|X)}\right) \\
 B_2^{bayes} &= -\frac{1}{c} \ln\left(\frac{E(w(\sigma^2).e^{-c\sigma^2}|X)}{E(w(\sigma^2)|X)}\right)
 \end{aligned}$$

For the weight $w(\mu) = e^{b\mu}$ we have:

$$\begin{aligned}
 B_1^{bayes} &= -\frac{1}{c} \ln\left(\frac{E(e^{b\mu}.e^{-c\mu}|X)}{E(e^{b\mu}|X)}\right) \\
 \rightarrow E(e^{b\mu}.e^{-c\mu}|X) &= \int e^{b\mu}.e^{-c\mu}.\exp\left(\frac{-1}{2\sigma^2} [n(\mu - \bar{H})^2]\right) d\mu \\
 \rightarrow E(e^{b\mu}|X) &= \int e^{b\mu}.\exp\left(\frac{-1}{2\sigma^2} [n(\mu - \bar{H})^2]\right) d\mu
 \end{aligned}$$

The above integral does not have a closed form and the final solution is reached using numerical methods in the software.

For the weight $w(\sigma^2) = \frac{1}{\sigma^2}$ we have:

$$\begin{aligned}
 B_2^{bayes} &= -\frac{1}{c} \ln\left(\frac{E\left(\frac{1}{\sigma^2}.e^{-c\sigma^2}|X\right)}{E\left(\frac{1}{\sigma^2}|X\right)}\right) \\
 E\left(\frac{1}{\sigma^2}.e^{-c\sigma^2}|X\right) &= \int \frac{1}{\sigma^2}.e^{-c\sigma^2}.\sigma^{-(n+1)} \exp\left(\frac{-S_0}{2\sigma^2}\right) d\sigma^2
 \end{aligned}$$

The above integral is solved using the modified Bessel function, just like the weightless case, with the difference that $\lambda = \alpha + 1$. So, as a result, we have:

$$\begin{aligned}
 \rightarrow E\left(\frac{1}{\sigma^2}.e^{-c\sigma^2}|X\right) &= 2.\left(\frac{S_0}{2}\right)^{\frac{n-1}{2}}.\left(\frac{S_0}{2} - c\right).K_{\frac{n+1}{2}}.(2\sqrt{-c.\frac{S_0}{2}}) \\
 \rightarrow E\left(\frac{1}{\sigma^2}|X\right) &= \frac{\alpha}{\beta} = \frac{n-1}{S_0} \\
 \rightarrow B_2^{bayes} &= -\frac{1}{b} \ln\left(\frac{2.\left(\frac{S_0}{2}\right)^{\frac{n-1}{2}}.\left(\frac{S_0}{2} - c\right).K_{\frac{n+1}{2}}.(2\sqrt{-c.\frac{S_0}{2}})}{\frac{n-1}{S_0}}\right)
 \end{aligned}$$

Simulation studies

To obtain numerical estimates, a large-scale stochastic simulation was conducted, involving the generation of a synthetic dataset exceeding one million observations. Specifically, a big data sample was initially drawn from a lognormal distribution with predefined parameters, $\mu = 1.5$, $\sigma^2 = 0.5$. Subsequently, the simulation procedures and numerical computations were implemented independently for each loss function using the R programming environment. This approach allowed for the isolated calculation of Bayesian parameter estimates under each specific loss criterion. The resulting numerical outputs are synthesized and presented in the following comparative table for rigorous analysis.

Table of numerical results

$MSE(\sigma^2)$	$MSE(\mu)$	$\hat{\sigma}_{\text{Bayes}}^2$	$\hat{\mu}_{\text{Bayes}}$	Loss function
0.000203	0.000381	0.497961	1.499619	Generalized Linex
0.000004	0.000003	0.499996	1.501000	Generalized entropy
0.000744	0.000003	0.499256	1.499997	Squared error in logarithm
0.000004	0.000001	0.498	1.500011	Weighted Linex

Comprehensive explanations for each table column

1. Loss function: This Loss Functions: Specifies the criterion employed for Bayesian estimation (logarithmic squared error, weighted linex, generalized linex, generalized entropy). Each loss function has a different effect on Bayesian estimates.
2. $\hat{\mu}_{\text{Bayes}}$: The Bayesian point estimate of the lognormal mean parameter, which is different under each loss function and can vary slightly from the true value of the parameter μ . In fact, each loss function causes a small change in the estimated value of μ .
3. $\hat{\sigma}_{\text{Bayes}}^2$: The Bayesian estimator for the lognormal variance. The value of this estimate can be different under each loss function and may increase or decrease depending on the chosen loss function.
4. $MSE(\mu)$: The Mean Squared Error of the μ estimator, utilized to quantify estimation precision and assess the impact of diverse loss functions on predictive accuracy. Lower MSE values indicate superior estimator performance. The variation in MSE across different loss functions serves as a primary metric for evaluating the relative efficacy of each criterion in minimizing estimation error, calculated as follows:

$$MSE(\mu) = \frac{1}{n} \sum_{i=1}^n (\mu_{\text{true}} - \hat{\mu}_i)^2$$

μ_{true} : The actual value of the parameter used in the data simulation.

$\hat{\mu}_i$: Estimated value of the parameter.

5. $MSE(\sigma^2)$: The Mean Squared Error of the σ^2 estimator, utilized to quantify estimation precision and assess the impact of diverse loss functions on predictive accuracy. , and like a μ lower MSE values indicate superior estimator performance. The variation in MSE across different loss functions serves as a primary metric for evaluating the relative efficacy of each criterion in minimizing estimation error, calculated as follows:

$$MSE(\sigma^2) = \frac{1}{n} \sum_{i=1}^n (\sigma_{\text{true}}^2 - \hat{\sigma}_i^2)^2$$

σ_{true}^2 : The actual value of the parameter used in the data simulation.

$\hat{\sigma}_i^2$: Estimated value of the parameter.

Conclusion

This research has provided a comprehensive investigation into the Bayesian inference of lognormal-distributed Big Data, with a particular emphasis on the influence of diverse and generalized loss functions on parameter estimation. By evaluating a wide spectrum of loss structures ranging from advanced generalized forms, including generalized

entropy, generalizes linex, logarithmic square error and weighted linex. we have elucidated the critical role of loss-function selection in the presence of large-scale, asymmetric datasets. The empirical findings derived from our extensive stochastic simulations yield several significant insights. Most notably, the results demonstrate that the choice of loss function profoundly impacts both the point estimates and the estimation precision.

The Weighted LINEX loss function emerged as the most robust estimator under the examined scenarios. This superiority was evidenced by its ability to consistently yield the minimum Mean Squared Error (MSE) and provide point estimates μ and σ^2 that exhibited the highest degree of proximity to the true parameter values used in the simulation.

Furthermore the Generalized Entropy loss function demonstrated a superior capacity to mitigate the risks associated with asymmetric estimation errors, making them more suitable for high-stakes decision-making in Big Data analytics. In conclusion, this study underscores that for lognormal-distributed data, a “one-size-fits-all” approach to loss functions is suboptimal. The transition toward generalized, asymmetric loss structures is essential for achieving high-fidelity Bayesian inference. Future research may extend these findings by exploring the performance of these estimators in more complex, multi-dimensional hierarchical models or by integrating machine learning-based prior specifications to further enhance predictive accuracy.

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