

Solutions of a parametric weighted double phase discrete problem with (p_1, p_2) -Laplacian equation

Mohamed Ousbika¹

Team of Modeling and Scientific Computing, Multidisciplinary Faculty of Nador, Oriental Applied Mathematics Laboratory, Mohamed First University, Oujda, Morocco

(Communicated by Abdolrahman Razani)

Abstract

We investigate a discrete nonlinear problem governed by a weighted double-phase (p_1, p_2) -Laplace operator subject to Dirichlet boundary conditions. By employing direct methods from the calculus of variations, we establish results on the existence and nonexistence of nontrivial solutions.

Keywords: Difference equations, Discrete boundary value problem, Variational methods
 2020 MSC: Primary 39A10, 39A12, 39A70; Secondary 34B15, 39F20

1 Introduction

Let $d \in \mathbb{N}$ and denote by $[1, d]_{\mathbb{N}} = \{1, \dots, d\}$ the finite set of integers. We consider the forward difference operator $\Delta x(t) = x(t+1) - x(t)$ and, for $s > 1$, Δ_s^a the weighted discrete s -Laplacian defined by

$$\Delta_s^a x(t-1) = \Delta \left(a(t) |\Delta x(t-1)|^{s-2} \Delta x(t-1) \right), \quad s > 1.$$

In this paper, we investigate the following nonlinear discrete Dirichlet boundary value problem governed by a weighted double-phase (p_1, p_2) -Laplace operator :

$$\begin{cases} \sum_{j=1}^2 \left(-\Delta_{p_j}^{a_j} x(t-1) \right) = \omega b(t) |x(t)|^{p_2-2} x(t) + \ell(t, x(t)), & \forall t \in [1, d]_{\mathbb{N}}, \\ x(0) = x(d+1) = 0, \end{cases} \quad (P_{\omega})$$

where $a_1, a_2 : [1, d+1]_{\mathbb{N}} \rightarrow [0, +\infty)$ and $b : [1, d]_{\mathbb{N}} \rightarrow [0, +\infty)$ are given functions, $1 < p_2 < p_1 < +\infty$, $\omega > 0$ is a real parameter, and $\ell(t, \cdot)$ is a continuous function for every $t \in [1, d]_{\mathbb{N}}$.

Difference equations and discrete boundary value problems have been intensively investigated over the last decades due to their broad applicability in numerical analysis, population dynamics, economics, control theory, and the modeling of processes evolving on discrete structures; see, for instance, [1, 13, 18, 30]. In many applications, discrete models arise either intrinsically, when the underlying phenomenon is governed by a finite lattice or network, or as

Email address: m.ousbika@ump.ac.ma (Mohamed Ousbika)

finite-dimensional counterparts of continuous differential equations obtained through discretization procedures. From a mathematical viewpoint, discrete boundary value problems provide a convenient framework in which variational and critical point methods can be applied in a finite-dimensional setting, leading to existence and multiplicity results for nonlinear systems [7, 16].

Among nonlinear operators, the discrete p -Laplacian has received particular attention, as it captures essential features of nonlinear diffusion and non-Newtonian phenomena while retaining a variational structure. Problems involving this operator have been investigated using critical point theory; see, for example, [8, 9, 11, 22, 23, 31].

In recent years, considerable interest has been devoted to problems governed by double-phase operators, which combine two different growth conditions characterized by distinct exponents. The prototype of such operators originates from the seminal work of Zhikov [32], motivated by variational problems arising in nonlinear elasticity and homogenization theory. In the continuous setting, double-phase energies of the form

$$\int_{\Omega} (|\nabla u|^{p_1} + a(x)|\nabla u|^{p_2}) \, dx, \quad 1 < p_2 < p_1,$$

have been extensively studied and have become increasingly relevant, as they naturally appear in the investigation of stationary solutions to reaction-diffusion systems [10].

The importance of the (p_1, p_2) -Laplace problem in the nonlinear setting bridges the gap between mathematically elegant toy models and the messy, non-homogeneous reality of modern materials science. It forces the development of cutting-edge nonlinear analysis while providing engineers with the mathematical teeth required to simulate everything from smart fluids to advanced geological anomalies, thereby demonstrating why these types of problems are worthy of study. Moreover, continuous problems driven by the (p, q) -Laplacian operator have been widely analyzed, yielding a rich variety of results concerning the existence, multiplicity, and qualitative properties of solutions (see, for instance, [3, 4, 5, 6, 12, 15, 17, 19, 20, 24, 25, 26, 27, 28]).

Motivated by these developments, it is natural to investigate discrete counterparts of double-phase problems. When continuous double-phase models are discretized, or when the underlying framework is intrinsically discrete, one is naturally led to difference operators with competing growth conditions. In the discrete Dirichlet setting, the authors of [21, 33], established the existence of at least two positive solutions and, moreover, infinitely many solutions for such discrete boundary value problems of the form

$$\begin{cases} \sum_{j=1}^2 \left(-\Delta_{p_j}^1 x(t-1) + \alpha_j(t) |x(t)|^{p_j-2} x(t) \right) = \lambda f(t, x(t)), & \forall t \in [1, d]_{\mathbb{N}}, \\ x(0) = x(d+1) = 0, \end{cases}$$

The presence of weights in the discrete operator makes it possible to encode heterogeneity effects analogous to those induced by spatially dependent coefficients in the continuous setting. Despite this strong conceptual connection, the analysis of discrete double-phase problems presents specific challenges, as the lack of homogeneity and the coexistence of different exponents prevent a direct extension of classical results for single-phase discrete p -Laplacian equations.

Inspired by the aforementioned studies, we initiate an investigation into the existence and non-existence of non-trivial solutions to problem (P_{ω}) , employing variational methods from critical point theory together with truncation techniques in an appropriate discrete functional space. In particular, we emphasize the interplay between the weighted double-phase structure, the parameter ω and nonlinear perturbation ℓ .

The structure of this paper is as follows. Section 2 provides some basic preliminaries and introduces the variational framework associated with problem (P_{ω}) . Section 3 is devoted to the statement and proofs of our main results.

2 Mathematical Background and Preliminaries

Problem (P_{ω}) is analysed in the finite-dimensional real Banach

$$X_d = \{x : [0, d+1]_{\mathbb{N}} \rightarrow \mathbb{R} \mid x(0) = x(d+1) = 0\},$$

endowed with the norm

$$\|x\| = \left(\sum_{t=0}^d |\Delta x(t)|^s \right)^{1/s}, \quad s > 1. \quad (2.1)$$

Moreover, for any $s > 1$, we introduce another standard norm on X_d given by

$$\|x\|_s = \left(\sum_{t=1}^d |x(t)|^s \right)^{1/s}, \quad (2.2)$$

Since X_d is finite-dimensional, these two norms are equivalent. More precisely, we have

Lemma 2.1. ([14]) For any $x \in X_d$ and $s \geq 1$, we have

$$\frac{1}{2}\|x\| \leq \|x\|_s \leq d^{\frac{1}{s}}(d+1)^{\frac{s-1}{s}}\|x\|. \quad (2.3)$$

We considered the functional $\mathcal{B} : X_d \rightarrow \mathbb{R}$ defined by

$$\mathcal{B}(x) = \sum_{t=1}^d \mathcal{F}(t, x(t)), \quad \text{for all } x \in X_d, \quad (2.4)$$

where

$$\mathcal{F}(t, s) = \int_0^s \ell(t, u) du, \quad \text{for all } s \in \mathbb{R}, \quad t \in [1, d]_{\mathbb{N}}. \quad (2.5)$$

Since ℓ is continuous, the functional $\mathcal{B}$ belongs to the class $C^1(X_d, \mathbb{R})$ and satisfies

$$\langle \mathcal{B}'(x), y \rangle = \sum_{t=1}^d \ell(t, x(t))y(t), \quad \text{for all } x, y \in X_d. \quad (2.6)$$

Let $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 : X_d \rightarrow \mathbb{R}$ be the functionals defined by

$$\mathcal{E}_1(x) = \frac{1}{p_1} \sum_{t=1}^{d+1} \left(a_1(t) |\Delta x(t-1)|^{p_1} \right), \quad \mathcal{E}_2(x) = \frac{1}{p_2} \sum_{t=1}^{d+1} \left(a_2(t) |\Delta x(t-1)|^{p_2} \right), \quad \text{and} \quad \mathcal{E}_3(x) = \frac{1}{p_2} \sum_{t=1}^d b(t) |x(t)|^{p_2} \quad (2.7)$$

A straightforward computation shows that $\mathcal{E}_1, \mathcal{E}_2$, and $\mathcal{E}_3$ belong to $C^1(X_d, \mathbb{R})$, and their Gâteaux derivatives at $x \in X_d$ are given by

$$\langle \mathcal{E}'_1(x), y \rangle = \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{E}_1(x + \varepsilon y) - \mathcal{E}_1(x)}{\varepsilon} = \sum_{t=1}^{d+1} a_1(t) |\Delta x(t-1)|^{p_1-2} \Delta x(t-1) \Delta y(t-1), \quad (2.8)$$

$$\langle \mathcal{E}'_2(x), y \rangle = \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{E}_2(x + \varepsilon y) - \mathcal{E}_2(x)}{\varepsilon} = \sum_{t=1}^{d+1} a_2(t) |\Delta x(t-1)|^{p_2-2} \Delta x(t-1) \Delta y(t-1), \quad (2.9)$$

$$\langle \mathcal{E}'_3(x), y \rangle = \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{E}_3(x + \varepsilon y) - \mathcal{E}_3(x)}{\varepsilon} = \sum_{t=1}^d b(t) |x(t)|^{p_2-2} x(t) y(t), \quad (2.10)$$

for all $y \in X_d$. For each $\omega > 0$, we define the functional $\mathcal{E}_\omega : X_d \rightarrow \mathbb{R}$ associated with problem (P_ω) by

$$\mathcal{E}_\omega(x) = \mathcal{E}_1(x) + \mathcal{E}_2(x) - \omega \mathcal{E}_3(x) - \mathcal{B}(x). \quad (2.11)$$

Clearly, $\mathcal{E}_\omega \in C^1(X_d, \mathbb{R})$ and its derivative at $x \in X_d$ satisfies

$$\langle \mathcal{E}'_\omega(x), y \rangle = \langle \mathcal{E}'_1(x), y \rangle + \langle \mathcal{E}'_2(x), y \rangle - \omega \langle \mathcal{E}'_3(x), y \rangle - \langle \mathcal{B}'(x), y \rangle, \quad \text{for all } y \in X_d. \quad (2.12)$$

Using summation by parts, for all $x, y \in X_d$, we deduce

$$\begin{aligned} \langle \mathcal{E}'_1(x), y \rangle &= \sum_{t=1}^d a_1(t) |\Delta x(t-1)|^{p_1-2} \Delta x(t-1) \Delta y(t-1) + a_1(d+1) |\Delta x(d)|^{p_1-2} \Delta x(d) \Delta y(d) \\ &= \sum_{t=1}^d a_1(t) |\Delta x(t-1)|^{p_1-2} \Delta x(t-1) y(t) - \sum_{t=1}^d a_1(t+1) |\Delta x(t)|^{p_1-2} \Delta x(t) y(t) \\ &\quad + a_1(1) |\Delta x(0)|^{p_1-2} \Delta x(0) y(1) - a_1(d+1) |\Delta x(d)|^{p_1-2} \Delta x(d) y(d) \\ &= - \sum_{t=1}^d \Delta [a_1(t) |\Delta x(t-1)|^{p_1-2} \Delta x(t-1)] y(t). \end{aligned} \quad (2.13)$$

Similarly, we obtain

$$\langle \mathcal{E}'_2(x), y \rangle = - \sum_{t=1}^d \Delta [a_2(t) |\Delta x(t-1)|^{p_2-2} \Delta x(t-1)] y(t). \quad (2.14)$$

Consequently, for all $x, y \in X_d$, we have

$$\begin{aligned} \langle \mathcal{E}'_\omega(x), y \rangle &= - \sum_{t=1}^d [\Delta (a_1(t) |\Delta x(t-1)|^{p_1-2} \Delta x(t-1)) + \Delta (a_2(t) |\Delta x(t-1)|^{p_2-2} \Delta x(t-1))] y(t) \\ &\quad - \omega \sum_{t=1}^d b(t) |x(t)|^{p_2-2} x(t) y(t) - \sum_{t=1}^d \ell(t, x(t)) y(t). \end{aligned} \quad (2.15)$$

Since y is arbitrary, we obtain the following result:

Lemma 2.2. The vector $x \in X_d$, is a solution to problem (P_ω) if and only if x is a critical point of the functional $\mathcal{E}_\omega$.

Let $\hat{\lambda}_b(s, a)$, [9], be the first eigenvalue of the problem

$$\begin{cases} -\Delta_s^a x(t-1) = \omega b(t) |x(t)|^{s-2} x(t), & \forall t \in [1, d]_{\mathbb{N}}, \\ x(0) = x(d+1) = 0, \end{cases} \quad (2.16)$$

and $\hat{x}_b(s, a)$ the positive eigenfunction of problem (2.16) with eigenvalue $\hat{\lambda}_b(s, a)$. Recall that $\hat{\lambda}_b(s, a)$, is characterised by

$$\hat{\lambda}_b(s, a) = \min_{x \in X_d \setminus \{0\}} \left(\frac{\sum_{t=1}^{d+1} a(t) |\Delta x(t-1)|^s}{\sum_{t=1}^d b(t) |x(t)|^s} \right). \quad (2.17)$$

For $b : [1, d]_{\mathbb{N}} \rightarrow [0, +\infty)$, we set

$$b_0 = \min_{1 \leq t \leq d} b(t) \quad \text{and} \quad b^+ = \max_{1 \leq t \leq d} b(t).$$

In studying problem (P_ω) , we make the following assumptions:

- $(H_1) :$ $\limsup_{|x| \rightarrow +\infty} \frac{\ell(t, x)}{|x|^{p_1-2} x} \leq \hat{\lambda}_1(p_1, a_1)$ uniformly for all $t \in [1, d]_{\mathbb{N}}$;
- $(H_2) :$ there exists $s \in (p_2, p_1)$ with $\lim_{|x| \rightarrow +\infty} \frac{\ell(t, x)x - p_1 \mathcal{F}(t, x)}{|x|^s} = +\infty$ uniformly for all $t \in [1, d]_{\mathbb{N}}$;
- $(H_3) :$ $\lim_{x \rightarrow 0} \frac{\ell(t, x)}{|x|^{p_2-2} x} = 0$ uniformly for all $t \in [1, d]_{\mathbb{N}}$;
- $(H_4) :$ $x\ell(t, x) \leq \hat{\lambda}_1(p_1, a_1)|x|^{p_1}$ for all $t \in [1, d]_{\mathbb{N}}$ and $x \in \mathbb{R}$.

3 Main results

Theorem 3.1. Assume that the hypotheses $(H_1) - (H_3)$ are satisfied. Then for each $\omega > \hat{\lambda}_b(p_2, a_2)$, problem (P_ω) admits at least one nontrivial solution in X_d .

Proof . We investigate the properties of the associated energy functional $\mathcal{E}_\omega$ in order to obtain a solution of (P_ω) . The coercivity of $\mathcal{E}_\omega$ is a crucial ingredient for applying the direct methods of calculus of variations; see [29]. To prove

that $\mathcal{E}_\omega$ is coercive, we argue by contradiction. Assume that there exist a sequence $(x_n)_n \subset X_d$ and a constant $M > 0$ such that

$$\|x_n\| \longrightarrow +\infty, \quad \text{as } n \rightarrow +\infty \quad (3.1)$$

and

$$\mathcal{E}_\omega(x_n) \leq M, \quad \text{for all } n \in \mathbb{N}. \quad (3.2)$$

Set $y_n = \frac{x_n}{\|x_n\|}$. Then $\|y_n\| = 1$. Since X_d is finite-dimensional, up to a subsequence, $y_n \longrightarrow y$ in X_d as $n \rightarrow +\infty$. From (3.2), we have

$$\mathcal{E}_1(x_n) + \mathcal{E}_2(x_n) \leq M + \omega \mathcal{E}_3(x_n) + \mathcal{B}(x_n), \quad \text{for all } n \in \mathbb{N}.$$

Using (2.7) and the identity $x_n = \|x_n\| y_n$, we obtain

$$\|x_n\|^{p_1} \mathcal{E}_1(y_n) + \|x_n\|^{p_2} \mathcal{E}_2(y_n) \leq M + \|x_n\|^{p_2} \omega \mathcal{E}_3(y_n) + \sum_{t=1}^d \mathcal{F}(t, \|x_n\| y_n(t)). \quad (3.3)$$

By hypothesis (H_1) , there exists $R > 0$ such that

$$\mathcal{F}(t, x) \leq \frac{1}{p_1} \hat{\lambda}_1(p_1, a_1) |x|^{p_1}, \quad \text{for all } t \in [1, d]_{\mathbb{N}} \text{ and } |x| > R. \quad (3.4)$$

Combining (3.3) and (3.4) we obtain

$$\mathcal{E}_1(y_n) + \frac{1}{\|x_n\|^{p_1-p_2}} \mathcal{E}_2(y_n) \leq \frac{M}{\|x_n\|^{p_1}} + \omega \frac{b^+}{p_2} \frac{\|y_n\|_{p_2}^{p_2}}{\|x_n\|^{p_1-p_2}} + \frac{1}{p_1} \hat{\lambda}_1(p_1, a_1) \|y_n\|_{p_1}^{p_1}. \quad (3.5)$$

Letting $n \rightarrow +\infty$ and using (3.1), we deduce that

$$\sum_{t=1}^{d+1} a_1(t) |\Delta y(t-1)|^{p_1} \leq \hat{\lambda}_1(p_1, a_1) \|y\|_{p_1}^{p_1}.$$

On the other hand, by the variational characterization (2.17),

$$\sum_{t=1}^{d+1} a_1(t) |\Delta y(t-1)|^{p_1} \geq \hat{\lambda}_1(p_1, a_1) \|y\|_{p_1}^{p_1}, \quad (3.6)$$

therefore,

$$\sum_{t=1}^{d+1} a_1(t) |\Delta y(t-1)|^{p_1} = \hat{\lambda}_1(p_1, a_1) \|y\|_{p_1}^{p_1}.$$

Hence equality holds and y is an eigenfunction associated with $\hat{\lambda}_1(p_1, a_1)$. Since this eigenvalue is simple [9], there exists $\nu \geq 0$ such that $y = \nu \varphi_1$ where φ_1 is the eigenfunction associated with $\hat{\lambda}_1(p_1, a_1)$. If $\nu = 0$ then $y = 0$, contradicting $\|y_n\| = 1$, since $\nu \neq 0$. Assume that $\nu > 0$ and so we have $x_n(t) \rightarrow \infty$ for all $t \in [1, d]_{\mathbb{N}}$. By hypothesis (H_2) given $\xi > 0$ there exists $M_\xi > 0$ such that

$$\ell(t, x)x - p_1 \mathcal{F}(t, x) \geq \xi |x|^s, \quad \text{for all } t \in [1, d]_{\mathbb{N}} \text{ and for all } |x| > M_\xi. \quad (3.7)$$

Additionally, we have

$$\partial_x \left(\frac{\mathcal{F}(t, x)}{|x|^{p_1}} \right) = \frac{\ell(t, x)|x|^{p_1} - p_1 |x|^{p_1-2} x \mathcal{F}(t, x)}{|x|^{2p_1}} = \frac{\ell(t, x)x - p_1 \mathcal{F}(t, x)}{|x|^{p_1} x}.$$

From (3.7), for all $t \in [1, d]_{\mathbb{N}}$ we obtain that

$$\partial_x \left(\frac{\mathcal{F}(t, x)}{|x|^{p_1}} \right) \geq \frac{\xi}{|x|^{p_1-s+1}} \quad \text{for all } x > M_\xi, \quad (3.8)$$

and

$$\partial_x \left(\frac{\mathcal{F}(t, x)}{|x|^{p_1}} \right) \leq -\frac{\xi}{|x|^{p_1-s+1}} \quad \text{for all } x < -M_\xi. \quad (3.9)$$

Moreover, let $z \in \mathbb{R}$ such that $|z| \geq |x| > M_\xi$. by integrating (3.8) between x and z such that $z \geq x > M_\xi$, we find that

$$\frac{\mathcal{F}(t, z)}{|z|^{p_1}} - \frac{\mathcal{F}(t, x)}{|x|^{p_1}} \geq -\frac{\xi}{p_1 - s} \left(\frac{1}{|z|^{p_1-s}} - \frac{1}{|x|^{p_1-s}} \right), \quad \forall z \geq x > M_\xi; \quad (3.10)$$

else, we integrate (3.9) with respect to z and x such that $z \leq x < -M_\xi$, we have

$$\frac{\mathcal{F}(t, x)}{|x|^{p_1}} - \frac{\mathcal{F}(t, z)}{|z|^{p_1}} \leq -\int_z^x \frac{\xi}{|u|^{p_1-s+1}} du, \quad \forall z \leq x < -M_\xi;$$

so, by making a change of variables $v = -u > 0$, we get

$$\int_z^x \frac{\xi}{|u|^{p_1-s+1}} du = \int_{-z}^{-x} \frac{\xi}{v^{p_1-s+1}} dv = \frac{-\xi}{p_1 - s} \left(\frac{1}{(-x)^{p_1-s}} - \frac{1}{(-z)^{p_1-s}} \right);$$

therefore,

$$\frac{\mathcal{F}(t, x)}{|x|^{p_1}} - \frac{\mathcal{F}(t, z)}{|z|^{p_1}} \leq \frac{-\xi}{p_1 - s} \left(\frac{1}{|x|^{p_1-s}} - \frac{1}{|z|^{p_1-s}} \right), \quad \forall z \leq x < -M_\xi;$$

so

$$\frac{\mathcal{F}(t, z)}{|z|^{p_1}} - \frac{\mathcal{F}(t, x)}{|x|^{p_1}} \geq \frac{-\xi}{p_1 - s} \left(\frac{1}{|z|^{p_1-s}} - \frac{1}{|x|^{p_1-s}} \right), \quad \forall z \leq x < -M_\xi. \quad (3.11)$$

combining (3.10) and (3.11), we deduce that

$$\frac{\mathcal{F}(t, z)}{|z|^{p_1}} - \frac{\mathcal{F}(t, x)}{|x|^{p_1}} \geq \frac{-\xi}{p_1 - s} \left(\frac{1}{|z|^{p_1-s}} - \frac{1}{|x|^{p_1-s}} \right), \quad \forall |z| \geq |x| > M_\xi, \quad (3.12)$$

for all $t \in [1, d]_{\mathbb{N}}$. By (3.4), for all $t \in [1, d]_{\mathbb{N}}$ and for all $|z| \geq |x| > \max(R, M_\xi)$, we prove that

$$\frac{\hat{\lambda}_1(p_1, a_1)}{p_1} - \frac{\mathcal{F}(t, x)}{|x|^{p_1}} \geq -\frac{\xi}{p_1 - s} \left(\frac{1}{|z|^{p_1-s}} - \frac{1}{|x|^{p_1-s}} \right). \quad (3.13)$$

Letting $|z| \rightarrow \infty$ in (3.13), for all $t \in [1, d]_{\mathbb{N}}$, we find

$$\frac{\hat{\lambda}_1(p_1, a_1)}{p_1} - \frac{\mathcal{F}(t, x)}{|x|^{p_1}} \geq \frac{\xi}{(p_1 - s)|x|^{p_1-s}}, \quad \forall |x| > \max(R, M_\xi);$$

so, for all $t \in [1, d]_{\mathbb{N}}$, we have

$$\hat{\lambda}_1(p_1, a_1)|x|^{p_1} - p_1 \mathcal{F}(t, x) \geq \frac{\xi p_1}{(p_1 - s)} |x|^s, \quad \forall |x| > \max(R, M_\xi). \quad (3.14)$$

Combining (3.3), (3.6) and (3.14) we arrive at

$$\begin{aligned} \sum_{t=1}^d \frac{\xi}{(p_1 - s)} |x_n(t)|^s &\leq \frac{1}{p_1} \sum_{t=1}^d \left(\hat{\lambda}_1(p_1, a_1) |x_n(t)|^{p_1} - p_1 \mathcal{F}(t, x_n(t)) \right) \\ &\leq \frac{1}{p_1} \sum_{t=1}^{d+1} a_1(t) |\Delta x_n(t-1)|^{p_1} - \sum_{t=1}^d \mathcal{F}(t, x_n(t)) \\ &\leq M + \frac{\omega b^+}{p_2} \|x_n\|_{p_2}^{p_2}, \end{aligned} \quad (3.15)$$

and

$$\frac{\xi}{(p_1 - s)} \|x_n\|^s \|y_n\|_s^s \leq M + \frac{\omega b^+}{p_2} \|x_n\|^{p_2} \|y_n\|_{p_2}^{p_2}.$$

According to Lemma 2.1 and taking into account $\|y_n\| = 1$, we deduce that

$$\frac{1}{2^s} \frac{\xi}{(p_1 - s)} \|x_n\|^s \leq M + \frac{\omega b^+}{p_2} d(d+1)^{p_2-1} \|x_n\|^{p_2},$$

and

$$\|x_n\|^{s-p_2} \leq \frac{2^s(p_1 - s)}{\xi} \left(M + \frac{\omega b^+}{p_2} d(d+1)^{p_2-1} \right), \quad (3.16)$$

which contradicts (3.1), since $s > p_2$. Therefore, $\mathcal{E}_\omega$ is coercive. The space X_d is a finite-dimensional Banach space, so weak convergence is equivalent to strong convergence. Therefore, to prove that $\mathcal{E}_\omega$ is sequentially weakly lower semicontinuous, it suffices to show that it is continuous. Each component of $\mathcal{E}_\omega$ (namely $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ and $\mathcal{B}$) is continuous, since they are finite sums of continuous functions involving powers and integrals of continuous functions. Hence, $\mathcal{E}_\omega$ being a combination of continuous functionals, is itself continuous. Consequently, $\mathcal{E}_\omega$ is sequentially weakly lower semicontinuous on X_d .

This fact, the direct method of the calculus of variations (minimum principle) [29] yields the existence of a minimizer $\tilde{x}_\omega \in X_d$ such that

$$\mathcal{E}_\omega(\tilde{x}_\omega) = \inf_{x \in X_d} \mathcal{E}_\omega(x). \quad (3.17)$$

Moreover, $\mathcal{E}'_\omega(\tilde{x}_\omega) = 0$. It must still be demonstrated that the solution is non-trivial. For a given $\varepsilon > 0$, hypothesis (H_3) leads to the conclusion that there exists a constant $\rho > 0$ such that

$$|\mathcal{F}(t, x)| \leq \frac{\varepsilon}{p_2} |x|^{p_2}, \quad \text{for all } t \in [1, d]_{\mathbb{N}} \text{ and } |x| < \rho. \quad (3.18)$$

Let $\hat{u}_1 = \hat{x}_b(p_2, a_2)$ be the normalized eigenfunction ($\|\hat{u}_1\| = 1$) corresponding to $\hat{\lambda}_b(p_2, a_2)$. For a sufficiently small real number $\gamma \in]0, 1[$, we define $x_\gamma = \gamma \hat{u}_1$ so that

$$0 \leq x_\gamma(t) \leq \rho, \quad \text{for all } t \in [1, d]_{\mathbb{N}}.$$

Then from (2.4), (2.11) and (3.18), we get

$$\begin{aligned} \mathcal{E}_\omega(x_\gamma) &\leq \frac{\gamma^{p_1}}{p_1} \sum_{t=1}^{d+1} a_1(t) |\Delta \hat{u}_1(t-1)|^{p_1} + \frac{\gamma^{p_2}}{p_2} \left(\sum_{t=1}^{d+1} a_2(t) |\Delta \hat{u}_1(t-1)|^{p_2} - \omega \sum_{t=1}^d b(t) |\hat{u}_1|^{p_2} \right) \\ &\quad + \varepsilon \frac{\gamma^{p_2}}{p_2} \|\hat{u}_1\|_{p_2}^{p_2} \\ &\leq a_1^+ \frac{\gamma^{p_1}}{p_1} \sum_{t=1}^{d+1} |\Delta \hat{u}_1(t-1)|^{p_1} - \frac{\gamma^{p_2}}{p_2} \left(\omega - \hat{\lambda}_b(p_2, a_2) \right) \sum_{t=1}^d b(t) |\hat{u}_1(t)|^{p_2} + \varepsilon \frac{\gamma^{p_2}}{p_2} \|\hat{u}_1\|_{p_2}^{p_2}, \end{aligned}$$

therefore

$$\begin{aligned} \mathcal{E}_\omega(x_\gamma) &\leq a_1^+ \frac{\gamma^{p_1}}{p_1} \|\hat{u}_1\|^{p_1} - \frac{\gamma^{p_2}}{p_2} \left(\omega - \hat{\lambda}_b(p_2, a_2) \right) b_0 \|\hat{u}_1\|_{p_2}^{p_2} + \varepsilon \frac{\gamma^{p_2}}{p_2} \|\hat{u}_1\|_{p_2}^{p_2} \\ &\leq a_1^+ \frac{\gamma^{p_1}}{p_1} \|\hat{u}_1\|^{p_1} - \frac{\gamma^{p_2}}{p_2} \left[\left(\omega - \hat{\lambda}_b(p_2, a_2) \right) b_0 - \varepsilon \right] \|\hat{u}_1\|_{p_2}^{p_2}. \end{aligned}$$

By lemma 2.1 and the fact that $\|\hat{u}_1\| = 1$, we have

$$\mathcal{E}_\omega(x_\gamma) \leq \frac{1}{p_1} a_1^+(t) \gamma^{p_1} - \frac{1}{2^{p_2} p_2} \left[\left(\omega - \hat{\lambda}_b(p_2, a_2) \right) b_0 - \varepsilon \right] \gamma^{p_2}.$$

Therefore, for each $\omega > \hat{\lambda}_1(p_2, a_2)$, if we choose ε such that $\varepsilon < (\omega - \hat{\lambda}_b(p_2, a_2)) b_0$, thus

$$\mathcal{E}_\omega(x_\gamma) \leq c_1 \gamma^{p_1} - c_2 \gamma^{p_2}, \quad (3.19)$$

with $c_1 = \frac{1}{p_1}a_1^+ > 0$ and $c_2 = \frac{1}{2^{p_2}p_2} \left[(\omega - \hat{\lambda}_b(p_2, a_2)) b_0 - \varepsilon \right] > 0$. As $p_2 < p_1$, and γ is assumed to be sufficiently small, then from (3.19) and recall (3.17), we deduce that

$$\mathcal{E}_\omega(x_\gamma) < 0 \quad \implies \quad \mathcal{E}_\omega(\tilde{x}_\omega) \leq \mathcal{E}_\omega(x_\gamma) < 0 = \mathcal{E}_\omega(0),$$

and so $\tilde{x}_\omega$ is nontrivial solution to problem (P_ω) . This completes the proof. $\square$

Theorem 3.2. Assume that hypothesis (H_4) holds. Then for $\omega < \hat{\lambda}_b(p_2, a_2)$, problem (P_ω) admits no nontrivial solution in X_d .

Proof . Assume by contradiction that there exists a nontrivial solution $\tilde{x}_\omega \in X_d$ of problem (P_ω) with $\omega < \hat{\lambda}_b(p_2, a_2)$. Then, testing the weak formulation with $y = \tilde{x}_\omega$, we deduce that

$$\sum_{t=1}^{d+1} (a_1(t)|\Delta \tilde{x}_\omega(t-1)|^{p_1} + a_2(t)|\Delta \tilde{x}_\omega(t-1)|^{p_2}) - \omega \sum_{t=1}^d b(t)|\tilde{x}_\omega(t)|^{p_2} = \sum_{t=1}^d \ell(t, \tilde{x}_\omega(t))\tilde{x}_\omega(t).$$

According to hypothesis (H_4) , we obtain

$$\sum_{t=1}^d \ell(t, \tilde{x}_\omega(t))\tilde{x}_\omega(t) \leq \hat{\lambda}_1(p_1, a_1) \|\tilde{x}_\omega\|_{p_1}^{p_1},$$

implies that,

$$\left(\sum_{t=1}^{d+1} a_1(t)|\Delta \tilde{x}_\omega(t-1)|^{p_1} - \hat{\lambda}_1(p_1, a_1) \|\tilde{x}_\omega\|_{p_1}^{p_1} \right) + \left(\sum_{t=1}^{d+1} a_2(t)|\Delta \tilde{x}_\omega(t-1)|^{p_2} - \omega \sum_{t=1}^d b(t)|\tilde{x}_\omega(t)|^{p_2} \right) \leq 0.$$

As

$$\sum_{t=1}^{d+1} a_1(t)|\Delta \tilde{x}_\omega(t-1)|^{p_1} \geq \hat{\lambda}_1(p_1, a_1) \|\tilde{x}_\omega\|_{p_1}^{p_1}$$

therefore,

$$\left(\hat{\lambda}_b(p_2, a_2) - \omega \right) \sum_{t=1}^d |\tilde{x}_\omega(t)|^{p_2} \leq 0.$$

We deduce that $\hat{\lambda}_b(p_2, a_2) - \omega \leq 0$, which leads to contradiction with $\omega < \hat{\lambda}_b(p_2, a_2)$. Therefore, problem (P_ω) does not have nontrivial solution in X_d for every $\omega < \hat{\lambda}_b(p_2, a_2)$. $\square$

We present some examples of functions $\ell : [1, d]_{\mathbb{N}} \times \mathbb{R} \rightarrow \mathbb{R}$ that satisfy the hypotheses $(H_1 - H_4)$.

Example 3.3. Let $1 < p_2 < s < p_1$ and let $\alpha, \beta : [1, d]_{\mathbb{N}} \rightarrow (0, +\infty)$ be functions such that ,

$$0 < \alpha(t) \leq \hat{\lambda}_1(p_1, a_1), \quad \text{and} \quad 0 < \beta(t) \leq 1, \quad \text{for all } t \in [1, d]_{\mathbb{N}}.$$

The function $\ell : [1, d]_{\mathbb{N}} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\ell(t, x) = \alpha(t)|x|^{p_1-2}x - |x|^{s-2}x + \beta(t)|x|^{p_2-2}x \ln(1+x^2),$$

and its primitive given by

$$\mathcal{F}(t, x) = \frac{\alpha(t)}{p_1}|x|^{p_1} - \frac{1}{s}|x|^s + \beta(t) \int_0^x |u|^{p_2-2}u \ln(1+u^2) du,$$

satisfy all the assumptions $(H_1 - H_4)$.

Example 3.4. Let $1 < p_2 < s < p_1$ and $r > 0$ such that $p_1 < r < 2p_1 - s$. Let $\alpha, \beta : [1, d]_{\mathbb{N}} \rightarrow (0, +\infty)$ be two functions such that,

$$0 < \alpha(t) + 1 \leq \hat{\lambda}_1(p_1, a_1), \quad \text{and} \quad 0 < \beta(t) \leq 1, \quad \text{for all } t \in [1, d]_{\mathbb{N}}.$$

The function $\ell : [1, d]_{\mathbb{N}} \times \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\ell(t, x) = \alpha(t)|x|^{p_1-2}x + \beta(t)\frac{|x|^{r-2}x}{1 + |x|^{r-p_1}}.$$

satisfy all the assumptions $(H_1 - H_4)$.

4 Conclusion

In this paper, we investigated a class of nonlinear discrete Dirichlet boundary value problems driven by a weighted double-phase (p_1, p_2) -Laplacian operator. By combining variational methods with the spectral properties of discrete p -Laplacian operators, we established both existence and nonexistence results depending on the range of the real parameter ω .

More precisely, under suitable growth and structural assumptions on the nonlinear term, we proved that the associated energy functional is coercive and sequentially weakly lower semicontinuous whenever ω exceeds a critical eigenvalue. This allowed us to apply the direct method of the calculus of variations to obtain the existence of at least one nontrivial solution. The analysis highlights the fundamental role played by the first eigenvalues of the weighted discrete p -Laplacian operators and the interaction between the two different growth exponents p_1 and p_2 . Two examples were discussed to illustrate the assumptions imposed on the nonlinear term.

On the other hand, we showed that no nontrivial solution exists when the parameter ω lies below the corresponding spectral threshold. This nonexistence result underlines the sharpness of the obtained conditions and reveals a threshold-type behavior of the problem with respect to the parameter ω .

The results presented in this work extend and complement existing studies on continuous and discrete problems with double-phase structure. They provide a rigorous variational framework for the analysis of nonlinear difference equations with competing growth conditions. Possible directions for future research include multiplicity and sign-changing solutions, anisotropic and variable exponent discrete operators, as well as extensions to time-dependent discrete systems.

Acknowledgements

The author would like to thank the reviewers for their careful reading, thorough review, and constructive comments and suggestions.

References

- [1] R. P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, 2000.
- [2] R. P. Agarwal, K. Perera, and D. O'Regan, *Multiple positive solutions of singular discrete p -Laplacian problems via variational methods*, Adv. Difference Equ. **2** (2005), 93–99.
- [3] D.G. Afonso and G.M. Bisci, *Multiple Solutions for Sublinear Double Phase Problems*, Asymptot. Anal. **142** (2025), no. 2, 432–446.
- [4] B. Anouar, V.D. Rădulescu, and D.D. Repovš, *Nonvariational and singular double phase problems for the Baouendi-Grushin operator*, J. Differ. Equ. **303** (2021), 645–666.
- [5] A. Bahrouni, V.D. Rădulescu, and P. Winkert, *Double phase problems with variable growth and convection for the Baouendi-Grushin operator*, Z. Angew. Math. Phys. **71** (2020), no. 6, 183.
- [6] A. Bahrouni and V.D. Rădulescu, *Singular double-phase systems with variable growth for the Baouendi-Grushin operator*, Discrete Contin. Dyn. Syst. **41** (2021), 4283–4296.
- [7] G. Bonanno and P. Candito, *Nonlinear difference equations investigated via critical point methods*, Nonlinear Anal. **70** (2009), 3180–3186.

- [8] G.M. Bisci and V. Rădulescu, *Variational Analysis of Nonlinear Discrete Problems*, Springer, 2016.
- [9] M. Chehab and O. Chakrone, *Properties of the first eigenvalue with sign changing weight of the discrete p -Laplacian and applications*, Bol. Soc. Parana. Mat. **36** (2018), no. 2, 151–167.
- [10] L. Cherfils and Y. Il'Yasov, *On the stationary solutions of generalized reaction diffusion equations with (p, q) -Laplacian*, Commun. Pure Appl. Anal. **29** (2005), no. 1, 9–22.
- [11] G. D'Agù, J. Mawhin, and A. Sciammetta, *Positive solutions for a discrete two point nonlinear boundary value problem with p -Laplacian*, J. Math. Appl. **447** (2017), 383–397.
- [12] J. Diblík, M. Galewski, I. Kossowski, and D. Motreanu, *On competing (p, q) -Laplacian Dirichlet problem with unbounded weight*, Differ. Integral Equ. **38** (2025), no. 1-2, 23–42.
- [13] S. Elaydi, *An Introduction to Difference Equations*, Springer, 2005.
- [14] M. Galewski and J. Smejda, *On the dependence on parameters for mountain pass solutions of second order discrete BVP's*, Appl. Math. Comput. **219** (2013), no. 11, 5963–5971.
- [15] T. Gou and V.D. Rădulescu, *Non-autonomous double phase eigenvalue problems with indefinite weight and lack of compactness*, Bull. Lond. Math. Soc. **46** (2024), 734–755.
- [16] J. Henderson and H.B. Thompson, *Existence of multiple solutions for second-order discrete boundary value problems*, Comput. Math. Appl. **43** (2002), no. 10-11, 1239–1248.
- [17] K. Kefi and M.M. Al-Shomrani, *Multiple solutions for double-phase elliptic problem with non-local interaction*, Mathematics **13** (2025), 1281.
- [18] W.G. Kelly and A.C. Peterson, *Difference Equations : An Introduction with Application*, Academic Press, New York, 1991.
- [19] S.A. Marano, S.J.N. Mosconi, and N.S. Papageorgiou, *Multiple solutions to (p, q) -Laplacian problems with resonant concave nonlinearity*, Adv. Nonlinear Stud. **16** (2016), no. 1, 51–65.
- [20] D. Motreanu and A. Razani, *Competing anisotropic and Finsler (p, q) -Laplacian problems*, Bound Value Probl. **2024** (2024), no. 39.
- [21] A. Nastasi, C. Vetro, and F. Vetro, *Positive solutions of discrete boundary value problems with the (p, q) -Laplacian operator*, Electron. J. Differ. Equ. **2017** (2017), no. 225, 1–12.
- [22] M. Ousbika and Z. El Allali, *A discrete problem involving the $p(k)$ -Laplacian operator with three variable exponents*, Int. J. Nonlinear Anal. Appl. **11** (2021), no. 2, 728–740.
- [23] M. Ousbika, H. He, Z. El Allali, and J. Zuo, *Non-trivial solutions for a partial discrete Dirichlet nonlinear problem with p -Laplacian*, Commun. Anal. Mech. **15** (2023), no. 2, 728–740.
- [24] N.S. Papageorgiou, V.D. Rădulescu, and D.D. Repovš, *Double-Phase problems and a discontinuity property of the spectrum*, Proc. Amer. Math. Soc. **147** (2019), 2899–2910.
- [25] A. Razani and S. Baraket, *Non-variational Solutions for Anisotropic (p, q) -Laplacian Systems with Nonhomogeneous Growth*, Bull. Iran. Math. Soc. **51** (2025), no. 38.
- [26] A. Razani, F. Safari, and G.M. Figueiredo, *Existence and multiplicity of solutions for a weighted (p, q) -Laplacian problem on the Heisenberg Lie groups*, Bull. Belg. Math. Soc. Simon Stevin **30** (2025), no. 3, 281–296.
- [27] A. Razani, F. Safari, and G.M. Figueiredo, *Degenerated and Competing Horizontal (p, q) -Laplacians with Weights on the Heisenberg Group*, Numer. Funct. Anal. Optim. **44** (2023), no. 3, 179–201.
- [28] A. Razani and F. Safari, *Existence of solutions for a (p, q) -Laplace equation with Steklov boundary conditions*, Int. J. Nonlinear Anal. Appl. **14** (2023), no. 7, 119–122.
- [29] M. Struwe, *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Ergeb. Math. Grenzgeb. (3)/A Series of Modern Surveys in Mathematics, MATHE3 vol **34**, Springer Berlin, Heidelberg, Fourth edition, 2008.
- [30] J.S. Yu and B. Zheng, *Modeling Wolbachia infection in mosquito population via discrete dynamical model*, J. Difference Equ. Appl. **25** (2019), no. 11, 1549–1567.

- [31] F. Xiong and Z. Zhou, *Three solutions to Dirichlet problems for second-order self-adjoint difference equations involving p -Laplacian*, Adv Differ. Equ. **2021** (2021), no. 192.
- [32] V.V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Ross. Akad. Nauk Ser. Mat. **29** (1987), 33–66.
- [33] Z. Zhang and Z. Zhou, *Infinitely many solutions for discrete boundary value problems with the (p, q) -Laplacian operator*, J. Funct. Spaces **2021** (2021), Article ID 1980285.