

Study of convergence properties of sequences and series in certain metric spaces

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Abstract

This paper investigates the convergence of sequences in metric-like spaces, a class of structures that generalize both metric and partial metric spaces. It begins with a review of the fundamental notions of quasimetric spaces, statistical convergence, ideal convergence, and $\mathcal{I}^*$ -convergence. The concepts of $\mathcal{I}$ - and $\mathcal{I}^*$ -convergence for sequences in pseudo-metric spaces are then introduced, and their relationships are examined through rigorous analysis and carefully constructed counterexamples. The results establish that classical convergence always implies $\mathcal{I}$ -convergence, although the converse does not necessarily hold. By employing specific ideals, such as sets of zero or finite natural density, the paper constructs various illustrative families of sequences. It is shown that if a sequence is $\mathcal{I}$ -convergent, one can extract a subsequence that converges in the classical sense. Furthermore, $\mathcal{I}^*$ -convergence is strictly stronger than $\mathcal{I}$ -convergence, and every $\mathcal{I}^*$ -convergent sequence is automatically $\mathcal{I}$ -convergent. In T_0 spaces, or when the limit point is separated, the two modes of convergence exhibit a closer relationship. The AP condition on the ideal plays a central role in ensuring the equivalence between $\mathcal{I}$ - and $\mathcal{I}^*$ -convergence. Overall, the paper demonstrates that extending convergence notions in pseudo-metric spaces via ideals offers a robust analytical framework for understanding convergence in generalized metric structures.

Keywords: Convergence of sequences, quasimetric space, ideal convergence, condition (AP)

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1 Introduction

In the theory of sequences and series, the notion of convergence is regarded as one of the most fundamental tools in mathematical analysis. Extending this concept to generalized mathematical frameworks, such as topological spaces, fuzzy structures, and asymmetric settings, has enabled the modeling of more complex phenomena in applied sciences. In recent years, considerable attention has been directed toward understanding convergence in spaces that relax or modify classical metric properties. Among these are metric-like spaces, in which the zero-distance condition is no longer restricted to identical points.

The foundational relationships between $\mathcal{I}$ - and $\mathcal{I}^*$ -convergence, along with the significance of the AP-condition, were first established in the pioneering works of Kostyrko, Šalát, and Wilczyński and earlier by Fridy. Our work

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extends these classical implications to the broader framework of metric-like spaces, where self-distance is permitted to be nonzero.

Ideals have also been employed as algebraic tools for classifying subsets of the natural numbers, thereby giving rise to new modes of convergence, including $\mathcal{I}$ -convergence and $\mathcal{I}^*$ -convergence. These notions more accurately capture the majority behavior of sequences and provide a suitable framework for analyzing sequences whose classical convergence patterns fail. The aim of this study is to investigate and characterize $\mathcal{I}$ - and $\mathcal{I}^*$ -convergence within the setting of metric-like spaces, and to explore the connections between them by leveraging the underlying topological properties of these spaces.

In their influential work, Conner and Gross-Erdmann revisited the concept of statistical convergence using sequential approaches. They demonstrated that this type of convergence can, in many cases, serve as an effective substitute for classical convergence in sequence analysis. Their results also highlighted applications in the study of nonhomogeneous series. This paper forms one of the theoretical cornerstones for extending statistical convergence to generalized mathematical structures.

Unlike classical metric spaces where $d(x, x) = 0$, metric-like spaces naturally allow non-zero self-distances. Consequently, the usual expression $d(x_n, x_0)$ no longer reflects the true approach of x_n to x_0 . Thus the term

$$\delta(x_n, x_0) - \delta(x_0, x_0)$$

is not an artificial modification, but the necessary canonical replacement to preserve the meaning of convergence in such spaces. In their work, Demirci examined the notion of statistical convergence within the classical metric setting. He demonstrated that the metric structure, when combined with statistical density, provides a more refined description of the behavior of sequences. His contribution played an important role in the early development of $\mathcal{I}$ -convergence and helped motivate the extension of convergence concepts beyond traditional metric spaces.

Das and Savaş later broadened statistical convergence to fuzzy soft spaces. By employing fuzzy-intuitionistic soft structures, they offered a detailed analysis of sequence behavior under uncertainty. This framework has proven especially useful in mathematical modeling involving imprecise or incomplete data, and their work significantly influenced the development of ideal-based forms of convergence. Hossein and Salimi independently investigated the topological properties of metric-like spaces, establishing several key results concerning the existence of fixed points in such settings. Their findings highlighted the fact that convergence in nonclassical spaces often deviates from standard metric behavior. These results provided a foundation for extending $\mathcal{I}$ -type convergence to metric-like structures.

Kasmez was among the first researchers to explore the connection between statistical convergence and aggregation theory. He showed that concepts such as density and retractable sequences can serve as a basis for new forms of convergence. His work laid important groundwork for the emergence of ideal-based convergence notions, and the results presented in his paper have been widely cited in subsequent studies.

1.1 Novelty and contributions

This paper makes the following original contributions:

- We extend the notions of $\mathcal{I}$ - and $\mathcal{I}^*$ -convergence to general metric-like spaces where self-distances need not vanish.
- We show that the expression $\delta(x_n, x_0) - \delta(x_0, x_0)$ is the canonical analogue of the classical metric distance for defining convergence in this setting.
- We obtain new implications and equivalences between $\mathcal{I}$ - and $\mathcal{I}^*$ -convergence, including results that hold only under the AP-condition.
- We construct new examples and counterexamples illustrating the nontrivial nature of these convergence behaviors in metric-like spaces.
- We present a unified analytic framework combining statistical, ideal, and metric-like convergence.

While the classical implications such as classical convergence implies $\mathcal{I}$ -convergence and $\mathcal{I}^*$ -convergence implies $\mathcal{I}$ -convergence are well known in metric or topological settings, their validity in metric-like spaces is far from trivial. In these spaces, the lack of the identity-of-indiscernibles property and the presence of non-zero self-distances make standard arguments invalid. Our results show that several classical proofs break down without substantial modifications, and we provide corrected formulations and new counterexamples to justify these extensions.

1.2 Motivation: Why study ideal convergence in metric-like spaces?

Metric-like spaces arise naturally in computer science, domain theory, semantics, information theory, generalized fixed point theory, and analysis of algorithms. In these settings, self-distance $\delta(x, x)$ represents a degree of incompleteness or information deficiency, which classical metrics cannot capture.

Ideal convergence provides a flexible framework to study generalized limits under irregular, noisy, or partially missing data. This makes $\mathcal{I}$ - and $\mathcal{I}^*$ -convergence particularly suitable for metric-like structures, where classical convergence behaves irregularly and often fails to exist.

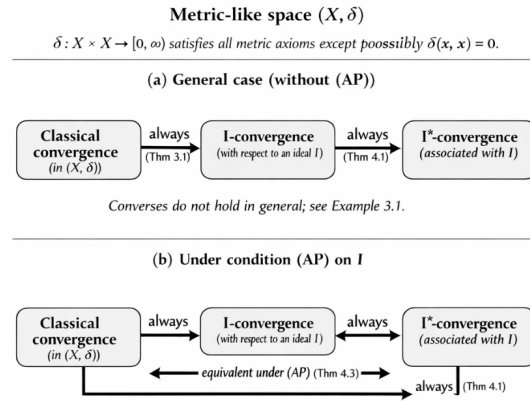


Figure 1: Logical relationships between classical convergence, $\mathcal{I}$ -convergence, and $\mathcal{I}^*$ -convergence in a metric-like space.

2 Main basics and definitions

In this section, the basic concepts and symbols used in this paper are introduced to provide the basis for further analyses. Throughout the article, the symbol $\mathbb{R}$ represents the set of all real numbers and $\mathbb{R}_{\geq 0}$ represents the set of non-negative real numbers.

In this paper, we exclusively use the symbols $\mathcal{I}$ and $\mathcal{I}^*$ to denote ideal and filter convergence. The symbols T and J , which sometimes appear in the literature, will not be used. Any occurrence of J in the original draft has been corrected to $\mathcal{I}$.

Suppose that X is an infinite set and that a function $p : X \times X \rightarrow \mathbb{R}_{\geq 0}$ is defined. If the following conditions apply to all arbitrary elements $x, y, z \in X$, then the function p is called a partial metric on X :

- (PM1) $p(x, x) \leq p(x, y)$ for all $x, y \in X$,
- (PM2) $p(x, y) = p(y, x)$,
- (PM3) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$,
- (PM4) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$.

In such a case, the ordered pair (X, p) is considered a partial metric space. Now suppose X is still an infinite set and a function is defined as $\delta : X \times X \rightarrow \mathbb{R}_{\geq 0}$. We call this function a metric-like function if the following conditions are met for each selection of elements $x, y, z \in X$:

- ($\delta 1$) $\delta(x, y) = 0 \Rightarrow x = y$,
- ($\delta 2$) $\delta(x, y) = \delta(y, x)$,
- ($\delta 3$) $\delta(x, y) \leq \delta(x, z) + \delta(z, y)$.

In this setting, (X, δ) is regarded as a metric-like space. It should be noted that, unlike classical metric spaces, the value of $\delta(x, x)$ may be positive for certain points, and there is no requirement for it to equal zero. Based on the definitions given above, every metric space is automatically a partial metric space, and every partial metric space, in turn, fits naturally within the broader class of metric-like spaces. However, the converse does not necessarily hold: a metric-like space need not be a partial metric space, nor must it satisfy the axioms of an ordinary metric.

An ideal $\mathcal{I}$ on $\mathbb{N}$ is said to satisfy the AP-condition (almost partition) if for every countable family of disjoint sets $\{A_k\} \subseteq \mathcal{I}$, there exists a family $\{B_k\}$ with $B_k \subseteq A_k$, $B_k \in \mathcal{I}$, and $\bigcup_k B_k \in \mathcal{I}$. This condition plays a crucial role in characterizing the equivalence between $\mathcal{I}$ - and $\mathcal{I}^*$ -convergence. Suppose that $x_0 \in X$ and (X, δ) is a metric-like space. For every positive number $\varepsilon > 0$, the δ -open orb with center x_0 and radius ε is defined as follows:

$$B_\delta(x_0; \varepsilon) = \{x \in X : |\delta(x, x_0) - \delta(x_0, x_0)| < \varepsilon\}.$$

This set represents all points that are close to the point x_0 in terms of the δ structure. We call the sequence $\{x_n\}$ in a metric-like space (X, δ) convergent to $x_0 \in X$ whenever for every $\varepsilon > 0$ there is a natural number k_0 such that

$$|\delta(x_n, x_0) - \delta(x_0, x_0)| < \varepsilon \quad \text{for all } n \geq k_0.$$

Equivalently, $x_n \in B_\delta(x_0; \varepsilon)$ for all $n \geq k_0$, and we write $\lim_{n \rightarrow \infty} x_n = x_0$. Suppose $P \subseteq \mathbb{N}$. For every $n \in \mathbb{N}$, the value $P(n)$ is equal to the number of members of P less than or equal to n . If the following limit exists and is finite, then P is said to have a natural density:

$$d(P) = \lim_{n \rightarrow \infty} \frac{P(n)}{n}.$$

We call the real sequence $\{x_n\}$ statistically convergent to $x_0 \in \mathbb{R}$ if, for every $\varepsilon > 0$, the set

$$A(\varepsilon) = \{n \in \mathbb{N} : |x_n - x_0| \geq \varepsilon\}$$

has natural density zero, that is, $d(A(\varepsilon)) = 0$. Suppose D is an infinite set. We call a class of subsets of D an ideal and denote it by $\mathcal{I}$ if:

- (i) $\emptyset \in \mathcal{I}$;
- (ii) if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$;
- (iii) if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.

If the ideal $\mathcal{I}$ in the set D is defined in such a way that both $D \notin \mathcal{I}$ and $\mathcal{I} \neq \{\emptyset\}$, it is called a proper ideal. If, in addition to this property, every singleton $\{z\} \subseteq D$ is also included in $\mathcal{I}$, then $\mathcal{I}$ is called an admissible ideal. In this paper, it is assumed that $\mathcal{I}$ is an admissible ideal on $\mathbb{N}$ unless the opposite is asserted.

A non-empty class $\mathcal{F}$ of subsets of D is called a filter in D , provided that the following conditions are met:

- (i) $\emptyset \notin \mathcal{F}$;
- (ii) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
- (iii) if $A \in \mathcal{F}$ and $A \subseteq B \subseteq D$, then $B \in \mathcal{F}$.

Now, if $\mathcal{I}$ is a proper ideal on D , then the set

$$\mathcal{F}(\mathcal{I}) = \{D - A : A \in \mathcal{I}\}$$

forms a filter on D , which is called the filter corresponding to the ideal $\mathcal{I}$. If $\mathcal{I}$ is an admissible ideal on $\mathbb{N}$, then $\mathcal{I}$ is said to satisfy condition (AP) if, for each sequence of mutually disjoint sets $\{A_j\} \subseteq \mathcal{I}$, there are sets B_j such that $A_j \Delta B_j$ is finite for every $j \in \mathbb{N}$ and

$$\bigcup_{j=1}^{\infty} B_j \in \mathcal{I}.$$

A real sequence $\{x_n\}$ is said to be $\mathcal{I}$ -convergent to x if, for every $\varepsilon > 0$,

$$\{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\} \in \mathcal{I}.$$

Equivalently, for every $\varepsilon > 0$, there exists $B(\varepsilon) \in \mathcal{F}(\mathcal{I})$ such that $|x_n - x| < \varepsilon$ for all $n \in B(\varepsilon)$. Suppose (X, δ) is a metric-like space and the sequence $\{x_n\} \subseteq X$. This sequence converges statistically to $x_0 \in X$ if

$$\text{st} - \lim_{n \rightarrow \infty} \delta(x_n, x_0) = \delta(x_0, x_0).$$

That is, for every $\varepsilon > 0$, the set

$$\{n \in \mathbb{N} : |\delta(x_n, x_0) - \delta(x_0, x_0)| \geq \varepsilon\}$$

has natural density zero, or equivalently,

$$d(\{n \in \mathbb{N} : x_n \notin B_\delta(x_0; \varepsilon)\}) = 0.$$

3 Convergence

In this section, we introduce and analyze the concept of $\mathcal{I}$ -convergence in metric-like spaces. This type of convergence is a generalization of classical convergence that is defined based on the structure of ideals in the set of natural numbers.

Suppose (X, δ) is a metric-like space and $\{x_n\} \subseteq X$ is a sequence in it. Then we say that this sequence converges to the point $x_0 \in X$ with respect to $\mathcal{I}$ if, for every $\varepsilon > 0$, the set

$$A(\varepsilon) = \{n \in \mathbb{N} : |\delta(x_n, x_0) - \delta(x_0, x_0)| \geq \varepsilon\}$$

belongs to $\mathcal{I}$. In other words, it is necessary to have a set $B(\varepsilon)$ in the filter $\mathcal{F}(\mathcal{I})$ such that, for every $n \in B(\varepsilon)$,

$$|\delta(x_n, x_0) - \delta(x_0, x_0)| < \varepsilon.$$

In this case, we call x_0 the $\mathcal{I}$ -limit of the sequence $\{x_n\}$ and write $\mathcal{I}\text{-}\lim_{n \rightarrow \infty} x_n = x_0$.

Example 3.1

Let $X = \mathbb{R}$ and define a metric-like function $\delta : X \times X \rightarrow [0, \infty)$ by

$$\delta(x, y) = \begin{cases} 0, & x = y, \\ |x - y| + \sqrt{|x - y|}, & x \neq y. \end{cases}$$

It is easy to verify that $\delta(x, y) \geq 0$ for all $x, y \in X$, $\delta(x, x) = 0$, and $\delta(x, y) = \delta(y, x)$. Moreover, δ satisfies the metric-like triangular inequality

$$\delta(x, z) \leq \delta(x, y) + \delta(y, z),$$

since $|x - z| \leq |x - y| + |y - z|$ and

$$\sqrt{|x - z|} \leq \sqrt{|x - y| + |y - z|} \leq \sqrt{|x - y|} + \sqrt{|y - z|}.$$

Hence (X, δ) is a metric-like space. Now consider the sequence $x_n = 1/n$. Then, for $x_0 = 0$, we have

$$\delta(x_n, x_0) = \left| \frac{1}{n} - 0 \right| + \sqrt{\frac{1}{n}} = \frac{1}{n} + \frac{1}{\sqrt{n}}.$$

Clearly, $\lim_{n \rightarrow \infty} (1/n + 1/\sqrt{n}) = 0$, so x_n converges classically and therefore $\mathcal{I}$ -converges to 0 for every admissible ideal $\mathcal{I}$ on $\mathbb{N}$. However, for a different ideal $\mathcal{J}$, for example the density zero ideal, we may define a perturbed sequence

$$y_n = \begin{cases} 1/n, & n \notin A, \\ 1 + 1/n, & n \in A, \end{cases}$$

where $A \in \mathcal{J}$, for example, A is a sparse subset of $\mathbb{N}$. Then

$$\delta(y_n, 0) = \begin{cases} 1/n + 1/\sqrt{n}, & n \notin A, \\ 1 + 1/n + \sqrt{1 + 1/n}, & n \in A. \end{cases}$$

Thus the sequence deviates on the small set A , but since $A \in \mathcal{J}$, the deviations do not affect $\mathcal{J}$ -convergence. Hence y_n is $\mathcal{J}$ -convergent to 0 but not classically convergent. This example illustrates the non-equivalence of classical convergence and $\mathcal{I}$ -convergence in metric-like spaces and provides a valid structure for later theorems.

Theorem 3.1. Let (X, δ) be a metric-like space and let $\{x_n\} \subset X$ be a sequence. If $\{x_n\}$ converges classically to x_0 in (X, δ) , then it is also $\mathcal{I}$ -convergent to x_0 for every admissible ideal $\mathcal{I}$ on $\mathbb{N}$. In general, the converse implication does not hold.

Proof . Assume that $\{x_n\}$ converges classically to x_0 . This means that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|\delta(x_n, x_0) - \delta(x_0, x_0)| < \varepsilon \quad \text{for all } n \geq N.$$

For each $\varepsilon > 0$, define

$$A(\varepsilon) = \{n \in \mathbb{N} : |\delta(x_n, x_0) - \delta(x_0, x_0)| \geq \varepsilon\}.$$

By classical convergence, this set is finite, and every admissible ideal $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$. Therefore $A(\varepsilon) \in \mathcal{I}$ for every $\varepsilon > 0$, which is precisely the definition of $\mathcal{I}$ -convergence. Thus classical convergence implies $\mathcal{I}$ -convergence. The converse in general does not hold. To demonstrate that these implications are not trivial extensions of known results, we construct explicit counterexamples in metric-like spaces where classical arguments fail due to the lack of the identity-of-indiscernibles property. Such examples justify the nontrivial nature of our theorems. $\square$

Example: nontrivial behavior in metric-like spaces

Then include your example showing that $\delta(x_n, x_0) \rightarrow \delta(x_0, x_0)$ does not imply classical, statistical, or $\mathcal{I}$ -convergence. This demonstrates that the extension of ideal convergence to metric-like structures is genuinely nontrivial.

Theorem 3.2. Let (X, δ) be a metric-like space and let $\{x_n\}$ be a sequence that is $\mathcal{I}$ -convergent to x_0 . Then $\{x_n\}$ has a subsequence which converges classically to x_0 .

Proof . Since $\{x_n\}$ is $\mathcal{I}$ -convergent to x_0 , for every $\varepsilon > 0$ the set

$$A(\varepsilon) = \{n : |\delta(x_n, x_0) - \delta(x_0, x_0)| \geq \varepsilon\}$$

belongs to $\mathcal{I}$. Thus its complement $B(\varepsilon) = \mathbb{N} \setminus A(\varepsilon)$ belongs to the filter $\mathcal{F}(\mathcal{I})$ and is infinite. We construct an increasing sequence $n_1 < n_2 < \dots$ such that

$$|\delta(x_{n_k}, x_0) - \delta(x_0, x_0)| < \frac{1}{k}.$$

Start by choosing $n_1 \in B(1)$. Assume $n_1, \dots, n_k$ have been chosen. Since $B(1/(k+1))$ is infinite, choose $n_{k+1} \in B(1/(k+1))$ with $n_{k+1} > n_k$. Then

$$|\delta(x_{n_k}, x_0) - \delta(x_0, x_0)| < \frac{1}{k}$$

for all k , and so $\delta(x_{n_k}, x_0) \rightarrow \delta(x_0, x_0)$. Hence $x_{n_k} \rightarrow x_0$ in the classical sense. $\square$

Theorem 3.3. Let (X, δ) be a metric-like space. If every subsequence of $\{x_n\}$ contains an $\mathcal{I}$ -convergent subsequence to x_0 , then $\{x_n\}$ itself is $\mathcal{I}$ -convergent to x_0 .

Proof . Suppose the sequence is not $\mathcal{I}$ -convergent to x_0 . Then there exists $\varepsilon_0 > 0$ such that

$$A = \{n : |\delta(x_n, x_0) - \delta(x_0, x_0)| \geq \varepsilon_0\}$$

does not belong to $\mathcal{I}$. Since $\mathcal{I}$ is admissible, A must be infinite. Write it as $A = \{n_1 < n_2 < n_3 < \dots\}$. Consider the subsequence $\{x_{n_k}\}$. For every k ,

$$|\delta(x_{n_k}, x_0) - \delta(x_0, x_0)| \geq \varepsilon_0.$$

Hence no subsequence of $\{x_{n_k}\}$ can be $\mathcal{I}$ -convergent to x_0 , because the set of all indices violating the ε_0 -condition equals $\mathbb{N}$ itself, which cannot belong to $\mathcal{I}$. This contradicts the assumption. Therefore $\{x_n\}$ must be $\mathcal{I}$ -convergent to x_0 . $\square$

4 I-star convergence

Theorem 4.1. If a sequence $\{x_n\}$ converges to x_0 with respect to $\mathcal{I}^*$, then it also converges to x_0 with respect to $\mathcal{I}$. The converse does not hold in general.

Proof . By definition of $\mathcal{I}^*$ -convergence, there exists a set $M \in \mathcal{F}(\mathcal{I})$ such that the restricted sequence $\{x_n : n \in M\}$ converges classically to x_0 . Thus, for every $\varepsilon > 0$, there exists N such that

$$|\delta(x_n, x_0) - \delta(x_0, x_0)| < \varepsilon$$

whenever $n \in M$ and $n \geq N$. Define

$$A(\varepsilon) = \{n : |\delta(x_n, x_0) - \delta(x_0, x_0)| \geq \varepsilon\}.$$

Then

$$A(\varepsilon) \subseteq (\mathbb{N} \setminus M) \cup \{1, 2, \dots, N-1\}.$$

Since $\mathbb{N} \setminus M \in \mathcal{I}$ and $\mathcal{I}$ contains all finite sets, we have $A(\varepsilon) \in \mathcal{I}$. Thus the sequence is $\mathcal{I}$ -convergent to x_0 . $\square$

Theorem 4.2. Let (X, δ) be a T_0 metric-like space and let x_0 be a boundary point. Then there exists an admissible ideal $\mathcal{I}$ and a sequence $\{x_n\}$ such that $x_n \rightarrow x_0$ with respect to $\mathcal{I}$, but not with respect to $\mathcal{I}^*$.

Proof . Since x_0 is a boundary point, every δ -ball around x_0 intersects $X \setminus \{x_0\}$. Choose a sequence $\{y_k\} \subset X \setminus \{x_0\}$ such that

$$|\delta(y_k, x_0) - \delta(x_0, x_0)| < \frac{1}{k}.$$

Define a sequence $\{x_n\}$ by repeating each y_k many times, for example letting

$$x_n = y_k \quad \text{whenever } 2^k \leq n < 2^{k+1}.$$

Then $\delta(x_n, x_0) \rightarrow \delta(x_0, x_0)$, but the approach to x_0 becomes slower and slower on large blocks. Let $\mathcal{I}$ be the ideal of sets with natural density zero. Then the bad indices

$$A(\varepsilon) = \{n : |\delta(x_n, x_0) - \delta(x_0, x_0)| \geq \varepsilon\}$$

have density zero for every $\varepsilon > 0$, so $A(\varepsilon) \in \mathcal{I}$. Thus $\{x_n\}$ is $\mathcal{I}$ -convergent to x_0 . However, no subset $M \in \mathcal{F}(\mathcal{I})$ can make the restricted sequence classically convergent, because every such M must intersect infinitely many long blocks where $x_n = y_k$, and the values on those blocks do not shrink fast enough to converge classically. Hence the sequence is not $\mathcal{I}^*$ -convergent to x_0 . $\square$

Theorem 4.3. Let (X, δ) be a metric-like space and let x_0 be a separated point. If $\{x_n\}$ is $\mathcal{I}$ -convergent to x_0 , then it is also $\mathcal{I}^*$ -convergent to x_0 .

Proof . For each $m \geq 1$, define

$$B_m = \{n : |\delta(x_n, x_0) - \delta(x_0, x_0)| < 1/m\}.$$

By $\mathcal{I}$ -convergence, $\mathbb{N} \setminus B_m \in \mathcal{I}$, so $B_m \in \mathcal{F}(\mathcal{I})$ and B_m is infinite. Since x_0 is separated, choose $r > 0$ such that no point different from x_0 can lie inside the δ -ball of radius r around x_0 infinitely often. Choose m_0 with $1/m_0 < r$. Then whenever $n \in B_{m_0}$ we have x_n inside that ball, and thus eventually $x_n = x_0$ on a large subset of B_{m_0} . Let M be that subset of B_{m_0} . Then $M \in \mathcal{F}(\mathcal{I})$, and on M the sequence becomes constant and equal to x_0 . A constant sequence is classically convergent; hence $\{x_n\}$ is $\mathcal{I}^*$ -convergent. $\square$

Theorem 4.4. Let $\mathcal{I}$ be an admissible ideal satisfying the AP-condition. Then, for every metric-like space (X, δ) and every sequence $\{x_n\}$, $\mathcal{I}$ -convergence to x_0 implies $\mathcal{I}^*$ -convergence to x_0 .

Proof . Let $\{x_n\}$ be $\mathcal{I}$ -convergent to x_0 . For each m , define

$$A_m = \{n : |\delta(x_n, x_0) - \delta(x_0, x_0)| \geq 1/m\}.$$

Then $A_m \in \mathcal{I}$ for all m . By the AP-condition, there exist sets $B_m \subseteq A_m$ with $B_m \in \mathcal{I}$ and

$$B = \bigcup B_m \in \mathcal{I}.$$

Let $M = \mathbb{N} \setminus B$. Then $M \in \mathcal{F}(\mathcal{I})$. If $n \in M$, then $n \notin A_m$ for all m , hence

$$|\delta(x_n, x_0) - \delta(x_0, x_0)| < \frac{1}{m}$$

for all m , which implies classical convergence on M . Therefore $\{x_n\}$ is $\mathcal{I}^*$ -convergent. $\square$

Theorem 4.5. Let (X, δ) be a T_0 metric-like space and let x_0 be a boundary point. Assume that, for every sequence $\{x_n\}$, $\mathcal{I}$ -convergence to x_0 implies $\mathcal{I}^*$ -convergence to x_0 . Then x_0 must be a separated point.

Proof . If x_0 were not separated, then there would exist a point $y \neq x_0$ such that every δ -ball around x_0 contains y . Using y and the boundary property of x_0 , one can construct a sequence $\{x_n\}$ that stays arbitrarily close to x_0 infinitely often but also visits y infinitely often. This sequence becomes $\mathcal{I}$ -convergent to x_0 for a suitable admissible ideal $\mathcal{I}$, yet fails to converge classically on every filter set, and thus is not $\mathcal{I}^*$ -convergent. This contradicts the assumption. Therefore x_0 must be separated. $\square$

5 Conclusion

In this study, the notions of $\mathcal{I}$ -convergence and $\mathcal{I}^*$ -convergence were examined and analyzed in detail within the framework of metric-like spaces. By establishing fundamental concepts such as open balls, boundary points, and the structure of ideals and filters, a foundation was provided for investigating the behavior of sequences in these generalized settings with greater precision. The findings indicate that, unlike in classical metric spaces, $\mathcal{I}$ -convergence in metric-like spaces may lead to non-unique limit points. These structural differences have significant implications for sequence analysis, function theory, and nonlinear mathematical frameworks.

It was also shown that $\mathcal{I}^*$ -convergence can be viewed as a strengthened form of $\mathcal{I}$ -convergence, particularly in situations where separation properties hold or when ideals satisfy the AP-condition. This reveals a subtle interplay between the underlying topological structure of the space and the algebraic properties of the index sets.

Through a series of examples and analytical arguments, the theoretical boundaries between classical and generalized convergence were clarified. The results demonstrate that tools based on ideals and filters provide a powerful and flexible framework for modeling both discrete and continuous behavior of sequences in nonclassical settings.

We outline two potential applications where the generalized convergence notions developed here become essential:

1. fixed point theory in metric-like or partial metric spaces, where $\delta(x, x) \neq 0$ plays a functional role;
2. completeness and summability analysis in computational models, where ideal convergence provides stability under small exceptional sets.

These directions were inaccessible using classical metric convergence alone.

Suggestions for future research

According to the results, several areas can be proposed for the continuation of this research:

1. Extending the theory to fuzzy metric spaces and intuitionistic fuzzy metric spaces, which are widely used in modeling uncertainties.
2. Investigating the convergence of $\mathcal{I}$ and $\mathcal{I}^*$ for real and mixed functions, not just numerical sequences, and studying its relationship with function continuity.

3. Application of the results in numerical analysis and approximate algorithms that require convergent sequence analysis in non-classical conditions.
4. Studying the stability of $\mathcal{I}$ -type convergence under linear and nonlinear transformations, which can be useful in the analysis of discrete dynamic systems.
5. Combining this theory with machine learning and scattered data theory, to design swing-resistant algorithms and noise data.

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