

On two subclasses of analytic functions related to Carlson-Shaffer operator defined by subordination

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Abstract

Let $\mathcal{A}$ denote the class of analytic functions defined in the open unit disk, and let $\mathcal{P}$ be the family of analytic and univalent functions with positive real part that map the unit disk onto a convex domain. In this paper, we introduce the new subclasses $\mathcal{K}_{p,q}^{\alpha,\beta}(\psi, \phi)$ and $\mathcal{S}_{p,q}^{\alpha,\beta}(\psi, \phi)$ associated with the Carlson–Shaffer operator. These classes are defined by means of suitable subordination conditions involving analytic functions related to convex and starlike mappings. We investigate several geometric properties of functions belonging to these subclasses and establish coefficient estimates along with other related results. The results obtained in this work generalize and unify various earlier studies in the theory of analytic and univalent functions.

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1 Introduction

Let $\mathcal{A}$ be the class of analytic functions $f(z)$ defined in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and are of the form

$$f(z) = z + \sum_{s=2}^{\infty} a_s z^s, \quad z \in \Delta. \quad (1.1)$$

Let F and G be two analytic functions in Δ with $F(0) = G(0)$. F is said to be subordinate to G , ($F \prec G$) in Δ if there exists an analytic function $w(z)$ on Δ such that

$$w(0) = 0, |w(z)| < 1 \text{ and } F(z) = G(w(z)), \quad z \in \Delta.$$

Also, if G is univalent in Δ , then

$$F(z) \prec G(z) \text{ } z \in \Delta \text{ if and only if } F(0) = G(0) \text{ and } F(\Delta) \subset G(\Delta).$$

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Let $\mathcal{P}$ denote the class of an analytic univalent functions ψ defined in Δ such that $\psi(\Delta)$ is convex with $\psi(0) = 1$ and $\Re(\psi(z)) > 0$, $z \in \Delta$. For $(0 \leq \gamma < 1)$, a function $f \in \mathcal{A}$ is said to be convex of order γ , starlike of order γ , close-to-convex of order γ if satisfies analytic criteria $\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma$, $\Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma$, $\Re(f'(z)) > \gamma$ respectively.

Ma and Minda [14] defined classes $\mathcal{S}^*(\phi)$ and $\mathcal{K}(\phi)$ of functions $f \in \mathcal{A}$ satisfying the conditions $\frac{zf'(z)}{f(z)} \prec \phi(z)$, $\phi \in \mathcal{P}$ and $1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)$, $\phi \in \mathcal{P}$ respectively. We observe that if $\phi(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, $\mathcal{S}^*\left(\frac{1+Az}{1+Bz}\right) = \mathcal{S}^*(A, B)$ and $\mathcal{K}\left(\frac{1+Az}{1+Bz}\right) = \mathcal{K}(A, B)$ are the classes of Janowski starlike functions and Janowski convex functions respectively [10].

Kim et al. [12] introduced the class of functions $f \in \mathcal{A}$ satisfying the analytic condition $\frac{f'(z)}{g'(z)} \prec \psi(z)$, $\psi \in \mathcal{P}$, $g \in \mathcal{K}(\phi)$ and denoted by $\mathcal{C}(\psi, \phi)$. A function $f \in \mathcal{A}$ is said to be in the classes $\mathcal{CS}(\psi, \phi)$ and $\mathcal{QK}(\psi, \phi)$ if satisfies $\frac{f(z)}{g(z)} \prec \psi(z)$, $\psi \in \mathcal{P}$, $g \in \mathcal{S}^*(\phi)$ and $\frac{(zf'(z))'}{g'(z)} \prec \psi(z)$, $\psi \in \mathcal{P}$, $g \in \mathcal{K}(\phi)$ respectively. The classes $\mathcal{CS}(\psi, \phi)$ and $\mathcal{QK}(\psi, \phi)$ are introduced by Bulut [6]. Notice that if $\psi(z) = \phi(z) = \frac{1+z}{1-z}$, the classes $\mathcal{S}^*(\phi)$, $\mathcal{K}(\phi)$, $\mathcal{C}(\psi, \phi)$, $\mathcal{CS}(\psi, \phi)$ and $\mathcal{QK}(\psi, \phi)$ reduces to class of starlike function $\mathcal{S}^*$, class of convex functions $\mathcal{K}$, class of close-to-convex functions $\mathcal{C}$ [11], class of close-to-starlike functions $\mathcal{CS}$ [20] and class of quasi-convex functions $\mathcal{QK}$ [19] respectively.

Motivated by the works in [3, 4, 5, 6, 9] and Senguttuvan et al. [23], (see [24] also), we introduce new classes $\mathcal{K}_{p,q}^{\alpha,\beta}(\psi, \phi)$ and $\mathcal{S}_{p,q}^{\alpha,\beta}(\psi, \phi)$ of analytic functions using Carlson Shaffer operator [1, 7, 8, 22], where the Carlson-Shaffer linear operator $\mathcal{R}_{p,q} : \mathcal{A} \rightarrow \mathcal{A}$ is given by

$$\mathcal{R}_{p,q}f(z) = I(p, q; z) * f(z) = z + \sum_{s=2}^{\infty} (v)_{s-1} a_s z^s, \quad (v)_s = \frac{(p)_s}{(q)_s}. \quad (1.2)$$

The function $I(p, q; z) = z + \sum_{s=2}^{\infty} \frac{(p)_{s-1}}{(q)_{s-1}} z^s$, $p \in \mathbb{C}$, $q \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ is known as the incomplete beta function. The term $(\kappa)_s$ is the Pochhammer symbol that can be expanded in Gamma function as

$$(\kappa)_s = \frac{\Gamma(\kappa + s)}{\Gamma(\kappa)} = \begin{cases} 1 & (s = 0) \\ \kappa(\kappa + 1) \dots (\kappa + s - 1) & (s \in \mathbb{N}) \end{cases} \quad (1.3)$$

Definition 1.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{K}_{p,q}^{\alpha,\beta}(\psi, \phi)$ if

$$\frac{\mathcal{R}'_{p,q}f(z) + (\alpha - \beta + 2\alpha\beta)z\mathcal{R}''_{p,q}f(z) + \alpha\beta z^2\mathcal{R}'''_{p,q}f(z)}{g'(z)} \prec \psi(z), \quad 0 \leq \beta \leq \alpha \leq 1, \quad (1.4)$$

where $g \in \mathcal{K}(\phi)$, $\phi, \psi \in \mathcal{P}$ and $\mathcal{R}_{p,q}f(z)$ is given by (1.2).

Definition 1.2. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_{p,q}^{\alpha,\beta}(\psi, \phi)$ if

$$\frac{(1 - \alpha + \beta)\mathcal{R}_{p,q}f(z) + (\alpha - \beta)z\mathcal{R}'_{p,q}f(z) + \alpha\beta z^2\mathcal{R}''_{p,q}f(z)}{g(z)} \prec \psi(z), \quad 0 \leq \beta \leq \alpha \leq 1, \quad (1.5)$$

where $g \in \mathcal{S}^*(\phi)$, $\phi, \psi \in \mathcal{P}$ and $\mathcal{R}_{p,q}f(z)$ is given by (1.2).

Remark 1.3.

- i) The classes $\mathcal{K}_{1,1}^{\alpha,\beta}(\psi, \phi) = \mathcal{K}_{\alpha,\beta}(\psi, \phi)$, $\mathcal{S}_{1,1}^{\alpha,\beta}(\psi, \phi) = \mathcal{S}_{\alpha,\beta}(\psi, \phi)$ were introduced by Bulut [6].
- ii) The class $\mathcal{K}_{1,1}^{0,0}(\psi, \phi) = \mathcal{C}(\psi, \phi)$ was introduced by Kim et al. [12]
- iii) For $-1 \leq B < A \leq 1$, the classes $\mathcal{K}_{1,1}^{0,0}\left(\frac{1+Az}{1+Bz}, \frac{1+z}{1-z}\right) = \mathcal{C}\left(\frac{1+Az}{1+Bz}, \frac{1+z}{1-z}\right) = \mathcal{CCV}(A, B)$, $\mathcal{S}_{1,1}^{0,0}\left(\frac{1+Az}{1+Bz}, \frac{1+z}{1-z}\right) = \mathcal{CS}\left(\frac{1+Az}{1+Bz}, \frac{1+z}{1-z}\right) = \mathcal{CST}(A, B)$ were introduced by Mehrook and Singh [15, 16] and Mehrook et al. [17].
- iv) For $-1 \leq B < A \leq 1$, the classes $\mathcal{K}_{1,1}^{\alpha,0}\left(\frac{1+Az}{1+Bz}, \frac{1+z}{1-z}\right) = \mathcal{QCV}(\alpha, A, B)$, $\mathcal{K}_{1,1}^{1,0}\left(\frac{1+Az}{1+Bz}, \frac{1+z}{1-z}\right) = \mathcal{QK}\left(\frac{1+Az}{1+Bz}, \frac{1+z}{1-z}\right) = \mathcal{QCV}(A, B)$, $\mathcal{S}_{1,1}^{\alpha,0}\left(\frac{1+Az}{1+Bz}, \frac{1+z}{1-z}\right) = \mathcal{QST}(\alpha, A, B)$ were introduced by Altıntaş and Kılıç [2].

v) For $0 \leq \gamma, \delta < 1$, the class $\mathcal{K}_{1,1}^{0,0} \left(\frac{1+(1-2\gamma)z}{1-z}, \frac{1+(1-2\delta)z}{1-z} \right) = \mathcal{C} \left(\frac{1+(1-2\gamma)z}{1-z}, \frac{1+(1-2\delta)z}{1-z} \right) = \mathcal{C}(\gamma, \delta)$ was introduced by Libera [13].

In this paper, we obtain coefficient bounds and associated properties for functions in the classes $\mathcal{K}_{p,q}^{\alpha,\beta}(\psi, \phi)$ and $\mathcal{S}_{p,q}^{\alpha,\beta}(\psi, \phi)$. Now let us consider the preliminary lemmas for proving our main results.

Lemma 1.4. [21] If the analytic function $P(z) = \sum_{s=1}^{\infty} P_s z^s$, $z \in \Delta$ is subordinate to the convex function $H(z) = \sum_{s=1}^{\infty} H_s z^s$, $z \in \Delta$, then $|P_s| \leq |H_1|$ for $(s \geq 1)$.

Lemma 1.5. [25] If a function $f \in \mathcal{K}(\phi)$, $\phi \in \mathcal{P}$ of the form (1.1), then

$$|a_s| \leq \frac{\prod_{k=0}^{s-2} (k + |\phi'(0)|)}{s!}, \quad s = 2, 3, \dots$$

Lemma 1.6. [25] If a function $f \in \mathcal{S}^*(\phi)$, $\phi \in \mathcal{P}$ of the form (1.1), then

$$|a_s| \leq \frac{\prod_{k=0}^{s-2} (k + |\phi'(0)|)}{(s-1)!}, \quad s = 2, 3, \dots$$

2 Main results

In this section, we determine the coefficient bounds for the classes $\mathcal{K}_{p,q}^{\alpha,\beta}(\psi, \phi)$ and $\mathcal{S}_{p,q}^{\alpha,\beta}(\psi, \phi)$.

Theorem 2.1. If $f \in \mathcal{K}_{p,q}^{\alpha,\beta}(\psi, \phi)$ is of the form (1.1), then

$$|a_s| \leq \frac{1}{\Omega(s, \alpha, \beta) |(\psi)_{s-1}|} \left[\frac{\prod_{k=0}^{s-2} (k + |\phi'(0)|)}{s!} + \frac{|\psi'(0)|}{s} \left(1 + \sum_{j=1}^{s-2} \frac{\prod_{k=0}^{s-j-2} (k + |\phi'(0)|)}{(s-j-1)!} \right) \right], \quad (2.1)$$

where $s \geq 2$, $\Omega(s, \alpha, \beta) = [1 + (s-1)(\alpha - \beta + 2\alpha\beta) + (s-1)(s-2)\alpha\beta]$.

Proof . Let $f \in \mathcal{K}_{p,q}^{\alpha,\beta}(\psi, \phi)$. Setting

$$g(z) = z + \sum_{s=2}^{\infty} b_s z^s \in \mathcal{K}(\phi), \quad \phi \in \mathcal{P},$$

by Lemma 1.5, we have

$$|b_s| \leq \frac{\prod_{k=0}^{s-2} (k + |\phi'(0)|)}{s!}, \quad s = 2, 3, \dots \quad (2.2)$$

Define a function $h(z)$ by

$$h(z) = \frac{\mathcal{R}'_{p,q} f(z) + (\alpha - \beta + 2\alpha\beta) z \mathcal{R}''_{p,q} f(z) + \alpha\beta z^2 \mathcal{R}'''_{p,q} f(z)}{g'(z)} = 1 + \sum_{m=1}^{\infty} c_m z^m, \quad z \in \Delta. \quad (2.3)$$

In the view of Definition 1.1, we get

$$h(z) \prec \psi(z), \quad z \in \Delta.$$

Upon using Lemma 1.4, we obtain

$$\left| \frac{h^{(m)}(0)}{m!} \right| = |c_m| \leq |\psi'(0)|, \quad (m = 1, 2, 3, \dots) \quad (2.4)$$

From (2.3), we have

$$\begin{aligned} \mathcal{R}'_{p,q}f(z) + (\alpha - \beta + 2\alpha\beta)z\mathcal{R}''_{p,q}f(z) + \alpha\beta z^2\mathcal{R}'''_{p,q}f(z) &= h(z)g'(z) \\ 1 + \sum_{s=2}^{\infty} s\Omega(s, \alpha, \beta)(v)_{s-1}a_s z^{s-1} &= 1 + \sum_{s=2}^{\infty} \left[\sum_{j=1}^{s-1} (s-j)c_j b_{s-j} \right] z^{s-1}. \end{aligned}$$

Since $a_1 = b_1 = c_0 = 1$, by comparing the coefficient of z^{s-1} , we get

$$s\Omega(s, \alpha, \beta)(v)_{s-1}a_s = \sum_{j=0}^{s-1} (s-j)c_j b_{s-j} = sb_s + (s-1)c_1 b_{s-1} + \dots + 2c_{s-2}b_2 + c_{s-1}.$$

The above inequality gives

$$s\Omega(s, \alpha, \beta)(v)_{s-1}|a_s| \leq sb_s + \sum_{j=1}^{s-1} (s-j)|c_j||b_{s-j}| \quad (2.5)$$

By applying (2.2) and (2.4) in (2.5), we have

$$\begin{aligned} s\Omega(s, \alpha, \beta)(v)_{s-1}|a_s| &\leq s \frac{\prod_{k=0}^{s-2} (k + |\phi'(0)|)}{s!} + |\psi'(0)| \left(\sum_{j=1}^{s-2} (s-j) \frac{\prod_{k=0}^{s-j-2} (k + |\phi'(0)|)}{(s-j)!} + 1 \right) \\ &= s \frac{\prod_{k=0}^{s-2} (k + |\phi'(0)|)}{s!} + |\psi'(0)| \left(\sum_{j=1}^{s-2} \frac{\prod_{k=0}^{s-j-2} (k + |\phi'(0)|)}{(s-j-1)!} + 1 \right) \end{aligned} \quad (2.6)$$

This gives the desired result. $\square$

Theorem 2.2. If $f \in \mathcal{S}_{p,q}^{\alpha,\beta}(\psi, \phi)$ is of the form (1.1), then

$$|a_s| \leq \frac{1}{\Theta(s, \alpha, \beta)(v)_{s-1}} \left[\frac{\prod_{k=0}^{s-2} (k + |\phi'(0)|)}{(s-1)!} + |\psi'(0)| \left(1 + \sum_{j=1}^{s-2} \frac{\prod_{k=0}^{s-j-2} (k + |\phi'(0)|)}{(s-j-1)!} \right) \right], \quad (2.7)$$

where $s \geq 2$, $\Theta(s, \alpha, \beta) = [1 - \alpha + \beta + s(\alpha - \beta) + s(s-1)\alpha\beta]$.

Proof . Let $f \in \mathcal{S}_{p,q}^{\alpha,\beta}(\psi, \phi)$. Setting

$$g(z) = z + \sum_{s=2}^{\infty} b_s z^s \in \mathcal{S}^*(\phi), \quad \phi \in \mathcal{P},$$

by Lemma 1.6, we have

$$|b_s| \leq \frac{\prod_{k=0}^{s-2} (k + |\phi'(0)|)}{(s-1)!}, \quad s = 2, 3, \dots \quad (2.8)$$

Define a function $h(z)$ by

$$h(z) = \frac{(1 - \alpha + \beta)\mathcal{R}_{p,q}f(z) + (\alpha - \beta)z\mathcal{R}'_{p,q}f(z) + \alpha\beta z^2\mathcal{R}''_{p,q}f(z)}{g(z)} = 1 + \sum_{m=1}^{\infty} c_m z^m, \quad z \in \Delta \quad (2.9)$$

In the view of Definition 1.2, we get $h(z) \prec \psi(z)$, $z \in \Delta$. Upon using Lemma 1.4, we obtain

$$\left| \frac{h^{(m)}(0)}{m!} \right| = |c_m| \leq |\psi'(0)|, \quad (m = 1, 2, 3, \dots) \quad (2.10)$$

From (2.9), we have

$$(1 - \alpha + \beta)\mathcal{R}_{p,q}f(z) + (\alpha - \beta)z\mathcal{R}'_{p,q}f(z) + \alpha\beta z^2\mathcal{R}''_{p,q}f(z) = h(z)g(z)$$

$$1 + \sum_{s=2}^{\infty} \Theta(s, \alpha, \beta)(v)_{s-1} a_s z^{s-1} = 1 + \sum_{s=2}^{\infty} \left[\sum_{j=1}^{s-1} c_j b_{s-j} \right] z^{s-1}$$

By comparing the coefficient of z^{s-1} , we get

$$\Theta(s, \alpha, \beta)(v)_{s-1} a_s = \sum_{j=0}^{s-1} c_j b_{s-j}$$

Since $a_1 = b_1 = c_0 = 1$, the above inequality gives

$$\Theta(s, \alpha, \beta)(v)_{s-1} |a_s| \leq |b_s| + \sum_{j=1}^{s-1} |c_j| |b_{s-j}| \quad (2.11)$$

By applying (2.8) and (2.10) in (2.11), we have

$$\Theta(s, \alpha, \beta)(v)_{s-1} |a_s| \leq \frac{\prod_{k=0}^{s-2} (k + |\phi'(0)|)}{(s-1)!} + |\psi'(0)| \left(\sum_{j=1}^{s-2} \frac{\prod_{k=0}^{s-j-2} (k + |\phi'(0)|)}{(s-j-1)!} + 1 \right) \quad (2.12)$$

This gives the desired result. $\square$

Corollary 2.3. If $f \in \mathcal{K}_{\alpha, \beta}(\psi, \phi)$ is of the form (1.1), then

$$|a_s| \leq \frac{1}{\Omega(s, \alpha, \beta)} \left[\frac{\prod_{k=0}^{s-2} (k + |\phi'(0)|)}{s!} + \frac{|\psi'(0)|}{s} \left(1 + \sum_{j=1}^{s-2} \frac{\prod_{k=0}^{s-j-2} (k + |\phi'(0)|)}{(s-j-1)!} \right) \right], \quad (2.13)$$

where $s \geq 2$, $\Omega(s, \alpha, \beta) = [1 + (s-1)(\alpha - \beta + 2\alpha\beta) + (s-1)(s-2)\alpha\beta]$.

Proof . If we let $p = 1$ and $q = 1$ in Theorem 2.1, we get the desired result. $\square$

Corollary 2.4. If $f \in \mathcal{QK}(\psi, \phi)$ is of the form (1.1), then

$$|a_s| \leq \left[\frac{\prod_{k=0}^{s-2} (k + |\phi'(0)|)}{s^2(s-1)!} + \frac{|\psi'(0)|}{s^2} \left(1 + \sum_{j=1}^{s-2} \frac{\prod_{k=0}^{s-j-2} (k + |\phi'(0)|)}{(s-j-1)!} \right) \right], s \geq 2. \quad (2.14)$$

Proof . If we let $p = 1, q = 1, \alpha = 1$ and $\beta = 0$, in Theorem 2.1, we get the desired result. $\square$

Corollary 2.5. If $f \in \mathcal{QCV}(\alpha, A, B)$ is of the form (1.1), then

$$|a_s| \leq \frac{1}{1 + (s-1)\alpha} \left[1 + \frac{(s-1)(A-B)}{2} \right], s \geq 2.$$

Proof . If we let $p = 1, q = 1, \beta = 0, \psi(z) = \frac{1+Az}{1+Bz}$ ($-1 \leq B < A \leq 1$) and $\phi(z) = \frac{1+z}{1-z}$, in Theorem 2.1, we get the desired result. $\square$

Corollary 2.6. [20] If $f \in \mathcal{C}$ is of the form (1.1), then

$$|a_s| \leq s, s \geq 2.$$

Proof . If we let $\alpha = 0, A = 1$ and $B = -1$ in Corollary 2.5, we get the desired result. $\square$

Corollary 2.7. [18] If $f \in \mathcal{Q}$ is of the form (1.1), then

$$|a_s| \leq 1, s \geq 2.$$

Proof . If we let $\alpha = 1, A = 1$ and $B = -1$ in Corollary 2.5, we get the desired result. $\square$

Corollary 2.8. If $f \in \mathcal{S}_{\alpha, \beta}(\psi, \phi)$ is of the form (1.1), then

$$|a_s| \leq \frac{1}{\Theta(s, \alpha, \beta)} \left[\frac{\prod_{k=0}^{s-2} (k + |\phi'(0)|)}{(s-1)!} + |\psi'(0)| \left(1 + \sum_{j=1}^{s-2} \frac{\prod_{k=0}^{s-j-2} (k + |\phi'(0)|)}{(s-j-1)!} \right) \right], \quad (2.15)$$

where $s \geq 2, \Theta(s, \alpha, \beta) = [1 - \alpha + \beta + s(\alpha - \beta) + s(s-1)\alpha\beta]$.

Proof . If we let $p = 1$ and $q = 1$, in Theorem 2.2, we get the desired result. $\square$

Corollary 2.9. If $f \in \mathcal{C}(\psi, \phi)$ is of the form (1.1), then

$$|a_s| \leq \left[\frac{\prod_{k=0}^{s-2} (k + |\phi'(0)|)}{s!} + \frac{|\psi'(0)|}{s} \left(1 + \sum_{j=1}^{s-2} \frac{\prod_{k=0}^{s-j-2} (k + |\phi'(0)|)}{(s-j-1)!} \right) \right], s \geq 2. \quad (2.16)$$

Proof . If we let $p = 1, q = 1, \alpha = 1$ and $\beta = 0$, in Theorem 2.2, we get the desired result. $\square$

Corollary 2.10. If $f \in \mathcal{CS}(\psi, \phi)$ is of the form (1.1), then

$$|a_s| \leq \left[\frac{\prod_{k=0}^{s-2} (k + |\phi'(0)|)}{(s-1)!} + |\psi'(0)| \left(1 + \sum_{j=1}^{s-2} \frac{\prod_{k=0}^{s-j-2} (k + |\phi'(0)|)}{(s-j-1)!} \right) \right], s \geq 2. \quad (2.17)$$

Proof . If we let $p = 1, q = 1, \alpha = 0$ and $\beta = 0$, in Theorem 2.2, we get the desired result. $\square$

Corollary 2.11. [13] If $f \in \mathcal{C}(\gamma, \delta)$ ($0 \leq \gamma, \delta < 1$) is of the form (1.1), then

$$|a_s| \leq \frac{2(3-2\delta)(4-2\delta)\dots(s-2\delta)}{s!} [s(1-\delta) + (\gamma-\delta)], s \geq 2. \quad (2.18)$$

Proof . If we let $\psi(z) = \frac{1+(1-2\gamma)z}{1-z}$ ($0 \leq \gamma < 1$) and $\phi(z) = \frac{1+(1-2\delta)z}{1-z}$ ($0 \leq \delta < 1$) in Corollary 2.9, we get the desired result. $\square$

Corollary 2.12. If $f \in \mathcal{Q}_{ST}(\alpha, A, B)$ is of the form (1.1), then

$$|a_s| \leq \frac{s}{1+(s-1)\alpha} \left[1 + \frac{(s-1)(A-B)}{2} \right], s \geq 2.$$

Proof . If we let $p = 1, q = 1, \beta = 0, \psi(z) = \frac{1+Az}{1+Bz}$ ($-1 \leq B < A \leq 1$) and $\phi(z) = \frac{1+z}{1-z}$, in Theorem 2.2, we get the desired result. $\square$

Corollary 2.13. [20] If $f \in \mathcal{CS}$ is of the form (1.1), then

$$|a_s| \leq s^2, s \geq 2.$$

Proof . If we let $\alpha = 0, A = 1, B = -1$ in Corollary 2.12, we get the desired result. $\square$

Motivated by the works of Akbarally et al. [1], we obtain the radius problems for classes $\mathcal{K}_{p,q}^{\alpha,\beta}(\psi, \phi)$ and $\mathcal{S}_{p,q}^{\alpha,\beta}(\psi, \phi)$ in the following theorems.

3 Radius of convexity, starlikeness and close-to-convexity

Theorem 3.1. Let $f \in \mathcal{K}_{p,q}^{\alpha,\beta}(\psi, \phi)$ be in the form (1.1). If

$$|z| \leq \left[\frac{(1-\gamma)\Omega(s, \alpha, \beta)|v|_{s-1}}{s(s-\gamma) \left(\frac{\prod_{k=0}^{s-2} (k+|\phi'(0)|)}{s!} + \frac{|\psi'(0)|}{s} \left(1 + \sum_{j=1}^{s-2} \frac{\prod_{k=0}^{s-j-2} (k+|\phi'(0)|)}{(s-j-1)!} \right) \right)} \right]^{\frac{1}{s-1}}, s > 1, \quad (3.1)$$

then f is convex of order γ .

Proof . It is enough to show that the values for $\left[1 + \frac{zf''(z)}{f'(z)} \right]$ lie in a circle centered at $w = 1$ with radius $1 - \gamma$. We have

$$\begin{aligned} \left| \left(1 + \frac{zf''(z)}{f'(z)} \right) - 1 \right| &\leq 1 - \gamma \\ \frac{\sum_{s=2}^{\infty} s(s-1)|a_s||z|^{s-1}}{1 - \sum_{s=2}^{\infty} s|a_s||z|^{s-1}} &\leq 1 - \gamma \\ \sum_{s=2}^{\infty} s(s-1)|a_s||z|^{s-1} &\leq (1-\gamma) \left(1 - \sum_{s=2}^{\infty} s|a_s||z|^{s-1} \right) \\ \frac{\sum_{s=2}^{\infty} s(s-\gamma)|a_s||z|^{s-1}}{1-\gamma} &\leq 1 \end{aligned}$$

This implies $\frac{s(s-\gamma)|z|^{s-1}}{1-\gamma} \leq \frac{1}{|a_s|}$. Since $f \in \mathcal{K}_{p,q}^{\alpha,\beta}(\psi, \phi)$, applying Theorem 2.1 yields condition (3.1). So the proof of Theorem 3.1 is completed. $\square$

Theorem 3.2. Let $f \in \mathcal{K}_{p,q}^{\alpha,\beta}(\psi, \phi)$ be in the form (1.1). If

$$|z| \leq \left[\frac{(1-\gamma)\Omega(s, \alpha, \beta)|(v)_{s-1}|}{(s-\gamma) \left(\frac{\prod_{k=0}^{s-2} (k + |\phi'(0)|)}{s!} + \frac{|\psi'(0)|}{s} \left(1 + \sum_{j=1}^{s-2} \frac{\prod_{k=0}^{s-j-2} (k + |\phi'(0)|)}{(s-j-1)!} \right) \right)} \right]^{\frac{1}{s-1}}, \quad s > 1, \quad (3.2)$$

then f is starlike of order γ .

Proof . We must show that, $\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \gamma$. We have

$$\frac{\sum_{s=2}^{\infty} (s-1)|a_s||z|^{s-1}}{1 - \sum_{s=2}^{\infty} |a_s||z|^{s-1}} \leq 1 - \gamma$$

This implies $\frac{(s-\gamma)|z|^{s-1}}{1-\gamma} \leq \frac{1}{|a_s|}$. Since $f \in \mathcal{K}_{p,q}^{\alpha,\beta}(\psi, \phi)$, applying Theorem 2.1 yields condition (3.2). So the proof of Theorem 3.2 is completed. $\square$

Theorem 3.3. Let $f \in \mathcal{K}_{p,q}^{\alpha,\beta}(\psi, \phi)$ be in the form (1.1). If

$$|z| \leq \left[\frac{(1-\gamma)\Omega(s, \alpha, \beta)|(v)_{s-1}|}{s \left(\frac{\prod_{k=0}^{s-2} (k + |\phi'(0)|)}{s!} + \frac{|\psi'(0)|}{s} \left(1 + \sum_{j=1}^{s-2} \frac{\prod_{k=0}^{s-j-2} (k + |\phi'(0)|)}{(s-j-1)!} \right) \right)} \right]^{\frac{1}{s-1}}, \quad s > 1, \quad (3.3)$$

then f is close-to-convex of order γ .

Proof . We must show that, $|f'(z) - 1| \leq 1 - \gamma$. We have

$$\sum_{s=2}^{\infty} s|a_s||z|^{s-1} \leq 1 - \gamma$$

This implies $\frac{s|z|^{s-1}}{1-\gamma} \leq \frac{1}{|a_s|}$. Since $f \in \mathcal{K}_{p,q}^{\alpha,\beta}(\psi, \phi)$, applying Theorem 2.1 yields condition (3.3). So the proof of Theorem 3.3 is completed. $\square$

Theorem 3.4. Let $f \in \mathcal{S}_{p,q}^{\alpha,\beta}(\psi, \phi)$ be in the form (1.1). If

$$|z| \leq \left[\frac{(1-\gamma)\Theta(s, \alpha, \beta)|(v)_{s-1}|}{s(s-\gamma) \left(\frac{\prod_{k=0}^{s-2} (k + |\phi'(0)|)}{(s-1)!} + |\psi'(0)| \left(1 + \sum_{j=1}^{s-2} \frac{\prod_{k=0}^{s-j-2} (k + |\phi'(0)|)}{(s-j-1)!} \right) \right)} \right]^{\frac{1}{s-1}}, \quad s > 1,$$

then f is convex of order γ .

Proof . By applying the same procedures used in Theorem 3.1 and also using Theorem 2.2, the proof of theorem is completed. $\square$

Theorem 3.5. Let $f \in \mathcal{S}_{p,q}^{\alpha,\beta}(\psi, \phi)$ be in the form (1.1). If

$$|z| \leq \left[\frac{(1-\gamma)\Theta(s, \alpha, \beta)|(v)_{s-1}|}{(s-\gamma) \left(\frac{\prod_{k=0}^{s-2} (k + |\phi'(0)|)}{(s-1)!} + |\psi'(0)| \left(1 + \sum_{j=1}^{s-2} \frac{\prod_{k=0}^{s-j-2} (k + |\phi'(0)|)}{(s-j-1)!} \right) \right)} \right]^{\frac{1}{s-1}}, \quad s > 1,$$

then f is starlike of order γ .

Proof . By applying the same procedures used in Theorem 2.2, the proof of theorem is completed. $\square$

Theorem 3.6. Let $f \in \mathcal{S}_{p,q}^{\alpha,\beta}(\psi, \phi)$ be in the form (1.1). If

$$|z| \leq \left[\frac{(1-\gamma)\Theta(s, \alpha, \beta)|(v)_{s-1}|}{s \left(\frac{\prod_{k=0}^{s-2} (k + |\phi'(0)|)}{(s-1)!} + |\psi'(0)| \left(1 + \sum_{j=1}^{s-2} \frac{\prod_{k=0}^{s-j-2} (k + |\phi'(0)|)}{(s-j-1)!} \right) \right)} \right]^{\frac{1}{s-1}}, \quad s > 1,$$

then f is close-to-convex of order γ .

Proof . By applying the same procedures used in Theorem 3.3 and also using Theorem 2.2, the proof of theorem is completed. $\square$

4 Conclusion

In the current paper, we defined two new classes $\mathcal{K}_{p,q}^{\alpha,\beta}(\psi, \phi)$ and $\mathcal{S}_{p,q}^{\alpha,\beta}(\psi, \phi)$ of analytic functions using Carlson-Shaffer operator. Results on radius of convexity, starlikeness and close-to-convexity are investigated for functions in

these new classes. Also, the connection between these new classes and other known subclasses of analytic functions are obtained as corollaries of our main theorems. As a future work our results may motivate researchers in geometric function theory to extend this study to other existing and new subclasses of close-to-convex and close-to-starlike functions.

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