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On the convergence, stability, data dependence results and the solution of a nonlinear integral equation by an efficient iterative scheme in Banach spaces

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Abstract

In this paper, we study the convergence of an efficient fixed-point iterative scheme for contraction mappings. The iterative scheme converges strongly to the fixed point of contraction mappings in Banach spaces. Numerical examples are also provided to test the efficiency and effectiveness of the proposed iteration. We also prove that the iteration is the stability and data dependence. Finally, we apply the iteration scheme to find the solution of the nonlinear integral equation.

Keywords: Fixed point, data depence, iteration scheme, nonlinear integral equation, stability, contraction mappings
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1 Introduction

Fixed point problems have an approximate solution or an existence result. One of the most interesting areas of numerical analysis nowadays is the study of convergence results and the approximation of fixed points using iterative approaches. The famous result due to Banach spaces that any contraction operator whose domain is essentially a closed subset of a Banach space will always admit a unique fixed point and the approximate value of this fixed point can be found by implementing the Picard iteration [17] on the operator. The Picard iteration is the simple iteration process and easy as a computation point of view. Since the Picard iteration used in the Banach Contraction Principle cannot converge to the fixed point of nonexpansive mappings, this problem has been tried to be overcome by defining new iteration methods. As a result of this approach, many iteration methods have been brought to the literature and studies on the definition of new iterations have continued to maintain their popularity today. In mathematics, stability is defined as the property that minor modifications to the structure under study will not impair its ability to function. In this context, many studies have been carried out on the stability of fixed-point iteration methods. It is possible to show that the new sequence that will be generated by using another mapping known as the approximation operator for this iteration approach is also convergent to the fixed point of the approximate operator after showing that the iterative sequence converges to the fixed point of the used mapping. Data dependency is brought up in this case by the searches of how close the fixed points of both mappings are to one another and how to calculate this distance. Researchers have conducted comprehensive evaluations of the qualitative properties of these algorithms, focusing on

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factors such as data dependency, algorithmic stability, and comparative convergence performance(see [8, 7, 12, 16]). Many iterative schemes have been developed to approximate fixed point. Some of the well-known iterative schemes Mann [13], Ishikawa [10], Noor [14]. Below are other prominent iteration schemes with which we will numerically compare the convergence behavior of iteration scheme studied in this paper.

J. Ahmad et al. [1] introduced the following iteration scheme called JK-iteration. Let $\mathcal{X}$ be a real Banach space and $\mathcal{K}$ a nonempty subset of $\mathcal{X}$, and $\mathcal{T}:\mathcal{K}\to\mathcal{K}$ be a mapping, for arbitrary $w_1\in\mathcal{K}$ construct a sequence $\{w_n\}$ by

$$\begin{cases} z_n=(1-\tau_n)w_n+\tau_n\mathcal{T}w_n,\\ y_n=\mathcal{T}z_n,\\ w_{n+1}=\mathcal{T}((1-\kappa_n)\mathcal{T}z_n+\kappa_n\mathcal{T}y_n), \end{cases} \tag{1.1}$$

where $\{\kappa_n\},\{\tau_n\}\in[0,1]$. S. Temir and O. Korkut [19] introduced the following iteration scheme(say TK-iteration). Let $\mathcal{X}$ be a real Banach space and $\mathcal{K}$ a nonempty subset of $\mathcal{X}$, and $\mathcal{T}:\mathcal{K}\to\mathcal{K}$ be a mapping, for arbitrary $w_1\in\mathcal{K}$ construct a sequence $\{w_n\}$ by

$$\begin{cases} z_n=\mathcal{T}((1-\tau_n)w_n+\tau_n\mathcal{T}w_n),\\ y_n=\mathcal{T}z_n,\\ w_{n+1}=\mathcal{T}((1-\kappa_n)\mathcal{T}w_n+\kappa_n\mathcal{T}y_n), \end{cases} \tag{1.2}$$

where $\{\kappa_n\},\{\tau_n\}\in[0,1]$. G. A. Okeke et al. [15] introduced the following iteration scheme(say UO-iteration). Let $\mathcal{X}$ be a real Banach space and $\mathcal{K}$ a nonempty subset of $\mathcal{X}$, and $\mathcal{T}:\mathcal{K}\to\mathcal{K}$ be a mapping, for arbitrary $w_1\in\mathcal{K}$ construct a sequence $\{w_n\}$ by

$$\begin{cases} z_n=\mathcal{T}w_n\\ y_n=\mathcal{T}((1-\nu_n)z_n+\nu_n\mathcal{T}z_n),\\ t_n=\mathcal{T}y_n,\\ u_n=((1-\tau_n)t_n+\tau_n\mathcal{T}t_n),\\ w_{n+1}=((1-\kappa_n)u_n+\kappa_n\mathcal{T}u_n), \end{cases} \tag{1.3}$$

where $\{\kappa_n\},\{\tau_n\}$ and $\{\nu_n\}\in[0,1]$. G. A. Okeke et al. [16] introduced the following iteration scheme called modified JK-iteration. Let $\mathcal{X}$ be a real Banach space and $\mathcal{K}$ a nonempty subset of $\mathcal{X}$, and $\mathcal{T}:\mathcal{K}\to\mathcal{K}$ be a mapping, for arbitrary $w_1\in\mathcal{K}$ construct a sequence $\{w_n\}$ by

$$\begin{cases} z_n=\mathcal{T}((1-\tau_n)w_n+\tau_n\mathcal{T}w_n),\\ y_n=\mathcal{T}(\mathcal{T}z_n),\\ w_{n+1}=\mathcal{T}((1-\kappa_n)\mathcal{T}z_n+\kappa_n\mathcal{T}y_n), \end{cases} \tag{1.4}$$

where $\{\kappa_n\},\{\tau_n\}\in[0,1]$. Temir [20] introduced a new iteration scheme for multi-valued generalized nonexpansive mappings. Furthermore, he showed numerically that this iteration scheme converges faster than several existing iteration schemes for multivalued generalized mappings.

Motivated by these, in this paper, we study the modified iteration scheme presented in [20]. Let $\mathcal{X}$ be a real Banach space and $\mathcal{K}$ a nonempty subset of $\mathcal{X}$, and $\mathcal{T}:\mathcal{K}\to\mathcal{K}$ be a mapping. This iterative scheme is defined as: for arbitrary $w_1\in\mathcal{K}$ construct a sequence $\{w_n\}$ by

$$\begin{cases} z_n=(1-\tau_n)w_n+\tau_n\mathcal{T}w_n,\\ y_n=\mathcal{T}(\mathcal{T}z_n),\\ w_{n+1}=\mathcal{T}((1-\kappa_n)\mathcal{T}y_n+\kappa_n\mathcal{T}(\mathcal{T}y_n)),\forall n\in\mathbb{N}, \end{cases} \tag{1.5}$$

where $\{\tau_n\}$ and $\{\kappa_n\}\in[0,1]$.

In this paper, we study the convergence, stability and the data dependence of the iteration scheme defined by (1.5) for contraction mapping in a Banach spaces. This paper contains four sections. In Section 2, we recollect basic definitions and a detailed overview of the fundamental results. In Section 3, The proposed iterative scheme converges strongly to the fixed point of contraction mappings in Banach spaces. Numerical examples are also given to test the efficiency and effectiveness of the proposed iteration. Furthermore we prove the stability and data dependence result of the iterative scheme for contraction mappings. In section 4, we apply the iteration scheme to find solution of nonlinear integral equation.

2 Preliminaries

Let $\mathcal{X}$ be uniformly convex Banach space and $\mathcal{K}$ a nonempty subset of $\mathcal{X}$, and $\mathcal{T}:\mathcal{K}\to\mathcal{K}$ be a mapping. A point $w\in\mathcal{K}$ is called a fixed point of $\mathcal{T}:\mathcal{K}\to\mathcal{K}$ if $w=\mathcal{T}w$. We denote $\mathcal{F}(\mathcal{T})$ the set of a fixed points of $\mathcal{T}$. A mapping

$\mathcal{T} : \mathcal{K} \to \mathcal{K}$ is called contraction if there exists $\theta \in [0, 1)$ such that

$$\|\mathcal{T}w - \mathcal{T}y\| \leq \theta\|w - y\|$$

for all $w, y \in \mathcal{K}$. If $\theta = 1$ in inequality above, then $\mathcal{T}$ is said to be a nonexpansive mapping. A mapping $\mathcal{T}$ is called quasi-nonexpansive if $\mathcal{F}(\mathcal{T}) \neq \emptyset$ and $\|\mathcal{T}w - p\| \leq \|w - p\|$ for all $w \in \mathcal{K}$ and $p \in \mathcal{F}(\mathcal{T})$. It is now well-known that the set $\mathcal{F}(\mathcal{T})$ is nonempty if $\mathcal{T}$ acting on nonempty closed bounded convex subset of a uniformly convex Banach space (see, Browder [4], Gohde [6] and Kirk [11]). In the sequel, we shall need the following results.

Definition 2.1. ([9]) Let $\{t_n\}_{n=1}^{\infty}$ be an arbitrary sequence in $\mathcal{K}$. Then, an iteration process

$$t_{n+1} = f(\mathcal{T}, t_n), \ for \ n = 1, 2, ...$$

is said to be $\mathcal{T}$-stable or stable with respect to $\mathcal{T}$ for some function f, converging to fixed point p, if $\epsilon_n = \|t_{n+1} - f(\mathcal{T}, t_n)\|$ for $n = 1, 2, ...$, we have $\lim\limits_{n \longrightarrow \infty} \epsilon_n = 0 \Leftrightarrow \lim\limits_{n \longrightarrow \infty} t_n = p$.

In what follows, we shall make use of the following well-known lemmas.

Lemma 2.2. ([3]) Let $\{\sigma_n\}$ and $\{\eta_n\}$ be sequences of positive real numbers satisfying

$$\eta_{n+1} \leq \delta\eta_n + \sigma_n,$$

$n \in \mathbb{N}$ and $\delta \in [0, 1)$. If $\lim\limits_{n\to\infty} \sigma_n = 0$ then $\lim\limits_{n\to\infty} \eta_n = 0$.

Definition 2.3. ([2]) Let $\mathcal{T}, \widetilde{\mathcal{T}} : \mathcal{X} \to \mathcal{X}$ be two operators. We say that $\widetilde{\mathcal{T}}$ is an approximate operator for $\mathcal{T}$ if, for all $w \in \mathcal{X}$ and for a fixed $\epsilon > 0$, we have $\|\mathcal{T}w - \widetilde{\mathcal{T}}w\| \leq \epsilon$.

Now we give a useful definition that is used to determine the faster iteration which converge to the same point. The following definition about the rate of convergence is given by [18] (see also [21]); it directly compares the actual error sequences, avoiding the potential inconsistency of approaches based on arbitrary upper bounds.

Definition 2.4 ([18]). Let $\{u_n\}$ and $\{v_n\}$ be two sequences in a normed space that converge to the same point p. Then $\{u_n\}$ is said to converge faster than $\{v_n\}$ if

$$\lim_{n\to\infty} \frac{\|u_n - p\|}{\|v_n - p\|} = 0.$$

In our analysis we obtain explicit upper bounds for the errors of the three iteration schemes directly from the contraction property and the parameter ranges; these bounds are not arbitrary but reflect the exact asymptotic order. Moreover, in the linear contraction case $\mathcal{T}x = \theta x$ (so that $\|\mathcal{T}x\| = \theta\|x\|$ for all x), these estimates become exact equalities, confirming their sharpness: the true errors cannot be asymptotically smaller than these bounds up to a multiplicative constant. Consequently, showing that the ratio of the upper bounds tends to zero suffices to guarantee that the ratio of the true errors also tends to zero, i.e., faster convergence in the sense of Definition 2.4 is established.

3 Convergence rate, stability and data dependence results

In this section, we discuss the convergence rate, stability and data dependence result the iteration scheme defined by (1.5). First, we prove the following strong convergence theorem.

Theorem 3.1. Let $\mathcal{K}$ be a nonempty closed convex subset of Banach space $\mathcal{X}$, $\mathcal{T}$ be a contraction mapping with $\mathcal{F}(\mathcal{T}) \neq \emptyset$. For arbitrary chosen $w_1 \in \mathcal{K}$, $\{wn\}$ be a sequence generated by (1.5) with real sequences $\{\tau_n\}$ and $\{\kappa_n\} \in [0, 1]$. Then $\{w_n\}_{n=1}^{\infty}$ converges strongly to an unique fixed point of $\mathcal{T}$.

Proof . We will prove that $w_n \to p$ as $n \to \infty$ from (1.5), we have,

$$\begin{aligned} \|z_n - p\| &= \|\Big((1 - \tau_n)w_n + \tau_n\mathcal{T}w_n\Big) - p\| \\ &\leq (1 - \tau_n)\|w_n - p\| + \tau_n\theta\|w_n - p\| \\ &= [1 - \tau_n(1 - \theta)]\|w_n - p\|. \end{aligned} \tag{3.1}$$

Similarly, from (1.5) and (3.1), we get

$$\begin{aligned}\|y_n - p\| &= \|\mathcal{T}(\mathcal{T}z_n) - p\| \\ &\leq \theta^2\|z_n - p\| \\ &\leq \theta^2[(1-\tau_n(1-\theta))\|w_n - p\|] \end{aligned} \tag{3.2}$$

From (1.5) and (3.2) , we get

$$\begin{aligned}\|w_{n+1} - p\| &= \|\mathcal{T}((1-\kappa_n)\mathcal{T}y_n + \kappa_n\mathcal{T}(\mathcal{T}y_n)) - p\| \\ &\leq \theta[(1-\kappa_n)\|\mathcal{T}y_n - p\| + \kappa_n\|\mathcal{T}(\mathcal{T}y_n) - p\|] \\ &\leq \theta[(1-\kappa_n)\theta\|y_n - p\| + \kappa_n\theta^2\|y_n - p\|] \\ &\leq \theta^2[1-\kappa_n\|y_n - p\| + \kappa_n\theta\|y_n - p\|] \\ &\leq \theta^2[(1-\kappa_n(1-\theta)]\|y_n - p\| \\ &\leq \theta^4[(1-\kappa_n(1-\theta))(1-\tau_n(1-\theta))\|w_n - p\|] \end{aligned}$$

Considering that $\{\kappa_n\} \in [0,1]$, $\theta \in [0,1)$ and $(1-\kappa_n(1-\theta)) < 1$, rearranging the above inequality, we get

$$\|w_{n+1} - p\| \leq \theta^4[1-\tau_n(1-\theta)]\|w_n - p\|$$

By induction, we get

$$\begin{aligned}\|w_n - p\| &\leq \theta^4[1-\tau_{n-1}(1-\theta)]\|w_{n-1} - p\| \\ &\vdots \\ \|w_2 - p\| &\leq \theta^4[1-\tau_1(1-\theta)]\|w_1 - p\|. \end{aligned}$$

Therefore, we obtain

$$\|w_{n+1} - p\| \leq \theta^{4n}\prod_{k=1}^{n}[1-\tau_k(1-\theta)]\|w_1 - p\|,$$

$\theta < 1$ and $\tau_k \in [0,1]$ for $k = 1, 2, ...$ Then we have $[1-\tau_k(1-\theta)] \leq 1$ for $k = 1, 2, ...$ So, we know that $1 - w \leq e^{-w}$ for all $w \in [0,1]$. Hence we have

$$\|w_{n+1} - p\| \leq \theta^{4n}e^{-(1-\theta)\sum_{k=1}^{n}\tau_k}\|w_1 - p\|. \tag{3.3}$$

Taking the limit of both sides of the above inequality , $w_n \to p$ as $n \to \infty$. □

We now analytically compare the convergence behavior of the iteration scheme generated by (1.5) with another iteration scheme.

Following the discussion in Zălinescu [21], we emphasize that comparisons based solely on upper error estimates describe the behavior of the associated majorant sequences rather than the exact iterative errors. Therefore, such estimates provide only an asymptotic indication of the convergence speed unless complemented by matching lower bounds or sharpness results.

However, in the present setting, the obtained estimates become exact in the linear contraction case, which ensures that the bounds are asymptotically sharp in that special case.

Theorem 3.2. Let $\mathcal{X}$, $\mathcal{K}$, $\mathcal{T}$ with a fixed point $p \in F(\mathcal{T})$ be the same as in Theorem 3.1. For given $u_1 = w_1 \in \mathcal{K}$, consider the iterative sequences $\{u_n\}$ defined by the iteration (1.4) and $\{w_n\}$ defined by the iteration (1.5), where $\{\kappa_n\}, \{\tau_n\}$ are real sequences in $[\varsigma, 1-\varsigma]$ for all $n \in \mathbb{N}$ and for some $\varsigma \in (0,1)$. Then the iteration scheme generated by (1.5) exhibits a faster asymptotic convergence rate in terms of the corresponding majorant error bounds.

Proof . Consider the iteration given in (1.4) and substitute $\{u_n\}$ for $\{w_n\}$. Then we have

$$\begin{aligned} \|z_n - p\| &= \|\mathcal{T}((1-\tau_n)u_n + \tau_n \mathcal{T}u_n) - p\| \\ &\leq \theta\big((1-\tau_n)\|u_n - p\| + \tau_n\theta\|u_n - p\|\big) \\ &= \theta[1-\tau_n(1-\theta)]\|u_n - p\|. \end{aligned} \tag{3.4}$$

Similarly,

$$\begin{aligned} \|y_n - p\| &= \|\mathcal{T}(\mathcal{T}z_n) - p\| \\ &\leq \theta^2\|z_n - p\| \\ &\leq \theta^3[1-\tau_n(1-\theta)]\|u_n - p\|. \end{aligned} \tag{3.5}$$

From (1.4) and (3.5), we obtain

$$\begin{aligned} \|u_{n+1} - p\| &= \|\mathcal{T}((1-\kappa_n)\mathcal{T}z_n + \kappa_n\mathcal{T}y_n) - p\| \\ &\leq \theta\big[(1-\kappa_n)\|\mathcal{T}z_n - p\| + \kappa_n\|\mathcal{T}y_n - p\|\big] \\ &\leq \theta^2\big[(1-\kappa_n)\|z_n - p\| + \kappa_n\|y_n - p\|\big] \\ &\leq \theta^3[1-\tau_n(1-\theta)]\big[(1-\kappa_n) + \kappa_n\theta\big]\|u_n - p\|. \end{aligned}$$

Since $\kappa_n, \tau_n \in [\varsigma, 1-\varsigma]$, we obtain

$$\|u_{n+1} - p\| \leq C\,\theta^{3n}\big[(1-\varsigma(1-\theta))^2\big]^n\|u_1 - p\|,$$

for some constant $C > 0$. Let

$$k_n = \theta^{3n}\big[(1-\varsigma(1-\theta))^2\big]^n\|u_1 - p\|.$$

Next, from (1.5), we have

$$\begin{aligned} \|z_n - p\| &= \|(1-\tau_n)w_n + \tau_n\mathcal{T}w_n - p\| \\ &\leq (1-\tau_n)\|w_n - p\| + \tau_n\theta\|w_n - p\| \\ &= [1-\tau_n(1-\theta)]\|w_n - p\|. \end{aligned} \tag{3.6}$$

Similarly,

$$\|y_n - p\| \leq \theta^2[1-\tau_n(1-\theta)]\|w_n - p\|. \tag{3.7}$$

Thus,

$$\|w_{n+1} - p\| \leq \theta^3[1-\tau_n(1-\theta)]\big[(1-\kappa_n) + \kappa_n\theta\big]\|w_n - p\|.$$

Hence,

$$\|w_{n+1} - p\| \leq C\,\theta^{4n}\big[(1-\varsigma(1-\theta))^2\big]^n\|w_1 - p\|,$$

for some constant $C > 0$. Let

$$m_n = \theta^{4n}\big[(1-\varsigma(1-\theta))^2\big]^n\|w_1 - p\|.$$

Therefore,

$$\frac{m_n}{k_n} = \theta^n \cdot \frac{\|w_1 - p\|}{\|u_1 - p\|} \to 0 \quad \text{as } n \to \infty.$$

This shows that the majorant error bound associated with $\{w_n\}$ decays faster than that of $\{u_n\}$, which completes the proof. □

In order to support the analytical proof of Theorem 3.2 and to verify the efficiency of iterative scheme (1.5), we give a numerical example:

Example 3.3. Let us define the mapping $\mathcal{T} : [0, 100] \to [0, 100]$ by $\mathcal{T}(w) = \sqrt{w^2 - 9w + 54}$. A direct computation shows that $\mathcal{T}$ is a self-map on $[0, 100]$ and admits a unique fixed point $p = 6$. Let $\{\kappa_n\} = \{\nu_n\} = \{\tau_n\} = 0.75$ for all $n \in \mathbb{N}$. The iterative process is initialized with $w_1 = 100$. The iterative values generated by the considered schemes are presented in the numerical comparison tables.

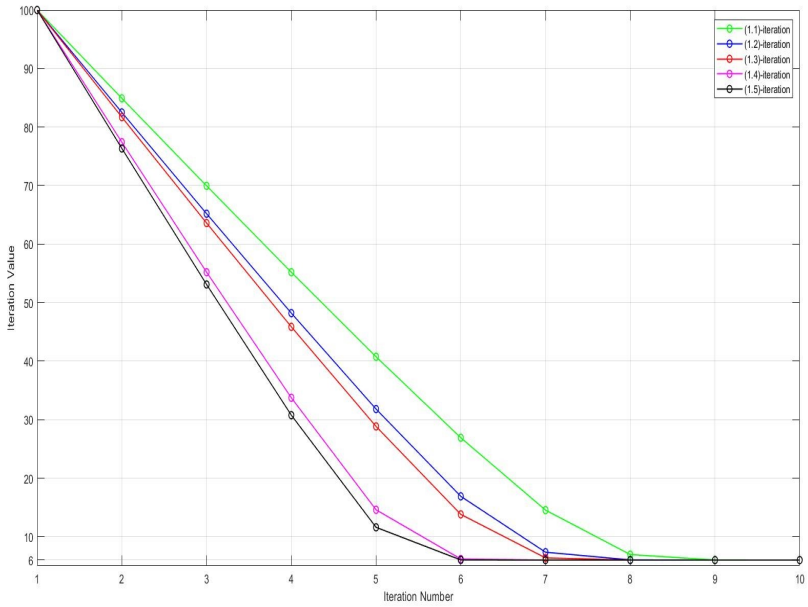


Figure 1: The comparison of the convergences of iteration schemes generated by (1.1),(1.2),(1.3),(1.4), (1.5) to the fixed point $p = 6$ is given in Figure 1 with the initial value $w_1 = 100$ for the mapping defined in Example 3.3.

From Figure 1, it can be observed that the iteration scheme (1.5) demonstrates a faster convergence behaviour compared to several classical iterative schemes. This observation is in agreement with the analytical comparison obtained in the theoretical part of the paper. Next we prove that the iteration defined by (1.5) is stable with respect to $\mathcal{T}$.

Theorem 3.4. Suppose that all conditions of Theorem 3.1 hold. Then the iteration process (1.5) is $\mathcal{T}$-stable.

Proof . Let $\{t_n\}$ be any arbitrary sequence in K. $t_{n+1} = f(\mathcal{T}, t_n)$ is the sequence generated by (1.5) and $\epsilon_n = \|t_{n+1} - f(\mathcal{T}, t_n)\|$ for $n = 1, 2, ...$, in which

$$\begin{cases} r_n = (1 - \tau_n)t_n + \tau_n \mathcal{T} t_n, \\ s_n = \mathcal{T}(\mathcal{T} r_n), \\ t_{n+1} = \mathcal{T}((1 - \kappa_n)\mathcal{T} s_n + \kappa_n \mathcal{T}(\mathcal{T} s_n)) \forall n \in \mathbb{N}. \end{cases}$$

We have to prove that $\lim_{n \longrightarrow \infty} \epsilon_n = 0 \Leftrightarrow \lim_{n \longrightarrow \infty} t_n = p$. Suppose $\lim_{n \longrightarrow \infty} \epsilon_n = 0$. We prove that $\lim_{n \longrightarrow \infty} t_n = p$:

$$\begin{aligned} \|t_{n+1} - p\| &\leq \|t_{n+1} - f(\mathcal{T}, t_n)\| + \|f(\mathcal{T}, t_n) - p\| \\ &\leq \epsilon_n + \|T((1 - \kappa_n)(\mathcal{T} s_n) + \kappa_n \mathcal{T}(\mathcal{T} s_n)) - p\| \\ &\leq \epsilon_n + \theta[(1 - \kappa_n)\|(\mathcal{T} s_n) - p\| + \kappa_n \|\mathcal{T}(\mathcal{T} s_n) - p\|] \\ &\leq \epsilon_n + \theta[(1 - \kappa_n)\theta\|s_n - p\| + \kappa_n \theta^2 \|s_n - p\|] \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \|s_n - p\| &= \|\mathcal{T}(\mathcal{T} r_n) - p\| \\ &\leq \theta^2 \|r_n - p\| \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \|r_n - p\| &= \|(1-\tau_n)t_n + \tau_n \mathcal{T} t_n - p\| \\ &\leq (1-\tau_n)\|t_n - p\| + \tau_n \theta \|t_n - p\| \\ &= (1-\tau_n(1-\theta))\|t_n - p\|. \end{aligned} \tag{3.10}$$

Substituting (3.10) in (3.9), we obtain

$$\|s_n - p\| \leq \theta^2[(1-\tau_n(1-\theta))]\|t_n - p\|. \tag{3.11}$$

So, we get

$$\|t_{n+1} - p\| \leq \epsilon_n + \theta^4[(1-\kappa_n(1-\theta))(1-\tau_n(1-\theta))]\|t_n - p\|.$$

Since $\{\kappa_n\}$ and $\{\tau_n\} \in [0,1]$, $\theta \in [0,1)$ and $\theta^4[(1-\kappa_n(1-\theta))(1-\tau_n(1-\theta))] < 1$, we can easily seen that all conditions of Lemma 2.1 are fulfilled by above inequality. Hence by Lemma 2.1 we get $\lim_{n\longrightarrow\infty} t_n = p$.

Conversely, let $\lim_{n\longrightarrow\infty} t_n = p$, we have

$$\begin{aligned} \epsilon_n &= \|t_{n+1} - f(\mathcal{T}, t_n)\| \\ &\leq \|t_{n+1} - p\| + \|f(\mathcal{T}, t_n) - p\| \\ &\leq \|t_{n+1} - p\| + \theta^4[(1-\kappa_n(1-\theta))(1-\tau_n(1-\theta))]\|t_n - p\|. \end{aligned}$$

By taking the limit as $n \to \infty$ in the above inequality we have $\lim_{n\longrightarrow\infty} \epsilon_n = 0$. Hence (1.5) is stable with respect to $\mathcal{T}$. □

Example 3.5. Let $\mathcal{X} = [0,1]$ endowed with the usual metric. Define a operator $\mathcal{T} : [0,1] \to [0,1]$ by $\mathcal{T}w = \frac{w}{2}$ with fixed point $p = 0$. Now we show that $\mathcal{T}$ satisfies condition

$$|\mathcal{T}w - \mathcal{T}y| \leq \theta|w - y|$$

with $\theta = \frac{1}{2}$. Then for all $w, y \in [0,1]$, we have

$$|\mathcal{T}w - \mathcal{T}y| = |\frac{w}{2} - \frac{y}{2}| \leq \frac{1}{2}|w - y|.$$

Let $\{w_n\}_{n=1}^{\infty}$ be the sequence defined by (1.5) with $\{\tau_n\}, \{\kappa_n\} = \frac{1}{n+1}$.

$$\begin{aligned} z_n &= (1-\tau_n)w_n + \tau_n \mathcal{T} w_n \\ &= (1 - \frac{1}{n+1})w_n + \frac{1}{n+1}\frac{w_n}{2} \\ &= (\frac{2n+1}{2n+2})w_n = (1 - \frac{1}{2n+2})w_n. \end{aligned}$$

Similarly, we get

$$y_n = \mathcal{T}(\mathcal{T} z_n) = (1 - \frac{1}{2n+2})\frac{w_n}{4}.$$

From (1.5) we get

$$\begin{aligned} w_{n+1} &= \mathcal{T}\Big((1-\kappa_n)\mathcal{T}y_n + \kappa_n\mathcal{T}(\mathcal{T}y_n)\Big) \\ &= \frac{\Big((1-\frac{1}{n+1})(1-\frac{1}{2n+2})\frac{w_n}{8} + \frac{1}{n+1}(1-\frac{1}{2n+2})\frac{w_n}{16}\Big)}{2} \\ &= (1-\frac{1}{n+1})(1-\frac{1}{2n+2})\frac{w_n}{16} + \frac{1}{n+1}(1-\frac{1}{2n+2})\frac{w_n}{32} \\ &= \Big(2-\frac{2}{2(n+1)}-\frac{2}{n+1}+\frac{1}{(n+1)^2}+\frac{1}{n+1}-\frac{1}{2(n+1)^2}\Big)\frac{w_n}{32} \\ &= \Big(2-\frac{2}{n+1}-\frac{1}{2(n+1)^2}\Big)\frac{w_n}{32}. \end{aligned}$$

Take the sequence $\{v_n\}$ defined by $\{v_n\} = \frac{1}{n+2}$. It is easily to see that the sequence $\{v_n\}$ is an approximation of $\{w_n\}$. Then we have

$$\begin{aligned} \epsilon_n &= |v_{n+1} - f(\mathcal{T}, v_n)| = \Big|v_{n+1} - \Big(2-\frac{2}{n+1}-\frac{1}{2(n+1)^2}\Big)\frac{v_n}{32}\Big| \\ &= \Big|\frac{1}{n+3} - \Big(2-\frac{2}{n+1}-\frac{1}{2(n+1)^2}\Big)\frac{1}{32(n+2)}\Big| \\ &= \Big|\frac{1}{n+3} - \frac{1}{16(n+2)} + \frac{1}{16(n+2)(n+1)} - \frac{1}{64(n+2)(n+1)^2}\Big| \\ &= \Big|\frac{64(n+2)(n+1)^2 - 4(n+3)(n+1)^2 + 4(n+3)(n+1) - n - 3}{64(n+2)(n+3)(n+1)^2}\Big| \\ &= |\frac{60n^3+240n^2+179n-61}{64(n+1)^2(n+2)(n+3)}| \to 0 \quad (as \quad n\to\infty). \end{aligned}$$

Thus we have $\lim_{n\to\infty}\epsilon_n = 0$. Hence (1.5) is stable with respect to $\mathcal{T}$.

By using Definition 2.3, we now prove the data dependence result for the iteration process defined by (1.5).

Theorem 3.6. Suppose $\mathcal{K}, \mathcal{X}, \mathcal{T}$ are as in Theorem 3.1. Consider $\widetilde{\mathcal{T}}$ being an approximate operator for the contraction mapping $\mathcal{T}$ with possible numerical error $\epsilon > 0$. Further, $\{w_n\}$ is the sequence (1.5), for $\mathcal{T}$, and define approximate sequence ξ_n for $\widetilde{\mathcal{T}}$ as follows:

$$\begin{cases} \zeta_n = (1-\tau_n)\xi_n + \tau_n\widetilde{\mathcal{T}}\xi_n, \\ \upsilon_n = \widetilde{\mathcal{T}}(\widetilde{\mathcal{T}}\zeta_n), \\ \xi_{n+1} = \widetilde{\mathcal{T}}((1-\kappa_n)\widetilde{\mathcal{T}}\upsilon_n + \kappa_n\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}\upsilon_n) \forall n \in \mathbb{N}, \end{cases} \tag{3.12}$$

where $\mathcal{K}$ is a nonempty closed convex subset of a Banach space, $w_1 \in \mathcal{K}$, $\{\kappa_n\}$ and $\{\tau_n\} \in [0,1], \forall n \in \mathbb{N}$. If $\mathcal{T}p = p$ and $\widetilde{\mathcal{T}}\widetilde{p} = \widetilde{p}$ such that $\lim_{n\to\infty}\xi_n = \widetilde{p}$ then we have $\|p-\widetilde{p}\| \le \frac{\theta^4+\theta^3+\theta^2+2\theta+1}{1-\theta^4}\epsilon$.

Proof . Let us consider (1.5) and (3.12), we have

$$\begin{aligned} \|z_n - \zeta_n\| &= \|((1-\tau_n)w_n + \tau_n\mathcal{T}w_n) - ((1-\tau_n)\xi_n + \tau_n\widetilde{\mathcal{T}}\xi_n)\| \\ &\le (1-\tau_n)\|w_n - \xi_n\| + \tau_n[\|\mathcal{T}w_n - \widetilde{\mathcal{T}}\xi_n\|] \\ &\le (1-\tau_n)\|w_n - \xi_n\| + \tau_n[\|\mathcal{T}w_n - \mathcal{T}\xi_n\| + \|\mathcal{T}\xi_n - \widetilde{\mathcal{T}}\xi_n\|] \\ &\le (1-\tau_n)\|w_n - \xi_n\| + \theta\tau_n\|w_n - \xi_n\| + \tau_n\epsilon \\ &\le (1-\tau_n(1-\theta))\|w_n - \xi_n\| + \tau_n\epsilon. \end{aligned} \tag{3.13}$$

By using (1.5) and (3.13), we have

$$\begin{aligned}
\|y_n - \upsilon_n\| &= \|\mathcal{T}(\mathcal{T}z_n) - \widetilde{\mathcal{T}}(\widetilde{\mathcal{T}}\zeta_n)\| \\
&\leq \|\mathcal{T}(\mathcal{T}z_n) - \mathcal{T}(\widetilde{\mathcal{T}}\zeta_n)\| + \|\mathcal{T}(\widetilde{\mathcal{T}}\zeta_n) - \widetilde{\mathcal{T}}(\widetilde{\mathcal{T}}\zeta_n)\| \\
&\leq \|\mathcal{T}(\mathcal{T}z_n) - \mathcal{T}(\widetilde{\mathcal{T}}\zeta_n)\| + \epsilon \\
&\leq \theta\|\mathcal{T}z_n - \widetilde{\mathcal{T}}\zeta_n\| + \epsilon \\
&\leq \theta\|\mathcal{T}z_n - \mathcal{T}\zeta_n\| + \|\mathcal{T}\zeta_n - \widetilde{\mathcal{T}}\zeta_n\| + \epsilon \\
&\leq \theta^2\|z_n - \zeta_n\| + \theta\epsilon + \epsilon. \qquad (3.14)
\end{aligned}$$

Putting (3.13) in (3.14), then we have

$$\begin{aligned}
\|y_n - \upsilon_n\| &\leq \theta^2[(1-\tau_n(1-\theta))\|w_n - \xi_n\| + \tau_n\epsilon] + \theta\epsilon + \epsilon \\
&\leq \theta^2[(1-\tau_n(1-\theta))]\|w_n - \xi_n\| + \theta^2\tau_n\epsilon + \theta\epsilon + \epsilon. \qquad (3.15)
\end{aligned}$$

By using (1.5) and (3.15), we have

$$\begin{aligned}
\|w_{n+1} - \xi_{n+1}\| =& \|\mathcal{T}((1-\kappa_n)\mathcal{T}y_n + \kappa_n\mathcal{T}\mathcal{T}y_n) - \widetilde{\mathcal{T}}((1-\kappa_n)\widetilde{\mathcal{T}}\upsilon_n + \kappa_n\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}\upsilon_n)\| \\
\leq& \|\mathcal{T}((1-\kappa_n)\mathcal{T}y_n + \kappa_n\mathcal{T}\mathcal{T}y_n) - \mathcal{T}((1-\kappa_n)\widetilde{\mathcal{T}}\upsilon_n + \kappa_n\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}\upsilon_n)\| \\
&+ \|\mathcal{T}((1-\kappa_n)\widetilde{\mathcal{T}}\upsilon_n + \kappa_n\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}\upsilon_n) - \widetilde{\mathcal{T}}((1-\kappa_n)\widetilde{\mathcal{T}}\upsilon_n + \kappa_n\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}\upsilon_n)\| \\
\leq& \theta\|(1-\kappa_n)\mathcal{T}y_n + \kappa_n\mathcal{T}\mathcal{T}y_n - ((1-\kappa_n)\widetilde{\mathcal{T}}\upsilon_n + \kappa_n\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}\upsilon_n)\| + \epsilon \\
\leq& \theta(1-\kappa_n)\|\mathcal{T}y_n - \widetilde{\mathcal{T}}\upsilon_n\| + \theta\kappa_n\|\mathcal{T}\mathcal{T}y_n - \widetilde{\mathcal{T}}\widetilde{\mathcal{T}}\upsilon_n)\| + \epsilon \\
=& \theta(1-\kappa_n)\|\mathcal{T}y_n - \mathcal{T}\upsilon_n + \mathcal{T}\upsilon_n - \widetilde{\mathcal{T}}\upsilon_n\| + \theta\kappa_n\|\mathcal{T}\mathcal{T}y_n - \mathcal{T}\widetilde{\mathcal{T}}\upsilon_n + \mathcal{T}\widetilde{\mathcal{T}}\upsilon_n - \widetilde{\mathcal{T}}\widetilde{\mathcal{T}}\upsilon_n\| + \epsilon \\
\leq& \theta(1-\kappa_n)[\|\mathcal{T}y_n - \mathcal{T}\upsilon_n\| + \|\mathcal{T}\upsilon_n - \widetilde{\mathcal{T}}\upsilon_n\|] + \theta\kappa_n[\|\mathcal{T}\mathcal{T}y_n - \mathcal{T}\widetilde{\mathcal{T}}\upsilon_n\| + \|\mathcal{T}\widetilde{\mathcal{T}}\upsilon_n - \widetilde{\mathcal{T}}\widetilde{\mathcal{T}}\upsilon_n\|] + \epsilon \\
\leq& \theta(1-\kappa_n)[\theta\|y_n - \upsilon_n\| + \epsilon] + \theta\kappa_n[\theta\|\mathcal{T}y_n - \widetilde{\mathcal{T}}\upsilon_n\| + \epsilon] + \epsilon \\
\leq& \theta\Big[(1-\kappa_n)\Big(\theta^3(1-\tau_n(1-\theta))\|w_n - \xi_n\| + \theta^3\tau_n\epsilon + \theta^2\epsilon + \theta\epsilon + \epsilon\Big) + \kappa_n\Big(\theta^2\|y_n - \upsilon_n\| + \theta\epsilon + \epsilon\Big)\Big] + \epsilon \\
\leq& \theta^4(1-\kappa_n)(1-\tau_n(1-\theta))\|w_n - \xi_n\| + \theta^4(1-\kappa_n)\tau_n\epsilon + \theta^3(1-\kappa_n)\epsilon + \theta^2(1-\kappa_n)\epsilon + \theta(1-\kappa_n)\epsilon \\
&+ \theta\Big[\kappa_n\theta^2\Big(\theta^2(1-\tau_n(1-\theta))\|w_n - \xi_n\| + \theta^2(1-\tau_n\epsilon + \epsilon] + \theta\epsilon + \epsilon\Big) + \kappa_n\theta\epsilon + \kappa_n\epsilon\Big] + \epsilon \\
\leq& \theta^4\Big[(1-\kappa_n)(1-\tau_n(1-\theta)) + \kappa_n\theta(1-\tau_n(1-\theta))\Big]\|w_n - \xi_n\| \\
&+ \theta^4\Big((1-\kappa_n)\tau_n(1-\theta)\Big)\epsilon + \theta^3\Big(1-\kappa_n(1-\theta)\Big)\epsilon + \theta^2\Big(1-\kappa_n(1-\theta)\Big)\epsilon + \theta\kappa_n\epsilon + \epsilon \\
=& \theta^4\Big[(1-\kappa_n(1-\theta))(1-\tau_n(1-\theta))\Big]\|w_n - \xi_n\| + (1-\kappa_n(1-\theta))\epsilon\Big(\theta^4 + \theta^3 + \theta^2 + \theta\Big) + \epsilon\Big(1 + \kappa_n\theta\Big) \\
\leq& \theta^4\Big[(1-\kappa_n(1-\theta))(1-\tau_n(1-\theta))\Big]\|w_n - \xi_n\| + \epsilon\Big(\theta^4 + \theta^3 + \theta^2 + 2\theta + 1\Big). \qquad (3.16)
\end{aligned}$$

By Theorem 3.1, we have $\lim_{n\to\infty} w_n = p$, and by assumption, we have $\lim_{n\to\infty} \xi_n = \widetilde{p}$. By using these facts together with (3.16), we obtain

$$\|p - \widetilde{p}\| \leq \frac{\epsilon\Big(\theta^4 + \theta^3 + \theta^2 + 2\theta + 1\Big)}{1 - \theta^4}.$$

□

It is analytically challenging to determine the fixed point of the mapping $\widetilde{\mathcal{T}}$. Instead of computing this fixed point directly, estimating the norm of the difference between this fixed point and the fixed point of another mapping $\mathcal{T}$, whose fixed point can be easily calculated, provides an approximation for the location of the fixed point of $\widetilde{\mathcal{T}}$. Let us now illustrate this with an example.

Example 3.7. Let us define a function $\widetilde{\mathcal{T}}w = \frac{w}{2} + \frac{1}{3}sinw + \frac{1}{4}$. Let us examine the convergence to the fixed point of $\widetilde{\mathcal{T}}$ using the iteration defined in (1.5), as shown in the table below and in Figure 2. The convergence of the mapping

Table 1: Sequences generated by (1.5)-iteration processes for mapping $\widetilde{\mathcal{T}}$.

	(1.5)-iteration
w_1	0.2000000000
w_2	0.8983625793
w_3	1.0624726423
w_4	1.0875696035
w_5	1.0910254682
w_6	1.0914938087
w_7	1.0915571393
w_8	1.0915657005
w_9	1.0915668578
w_{10}	1.0915670142
w_{11}	1.0915670354
w_{12}	1.0915670382
w_{13}	1.0915670386
w_{14}	1.0915670387
w_{15}	1.0915670387

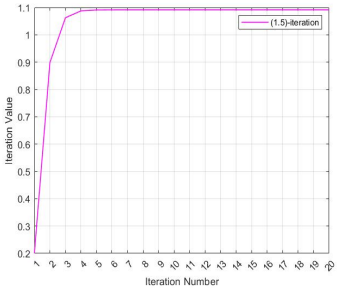


Figure 2: Convergence of (1.5)-iteration processes to the fixed point 1.0915670387 of the mapping $\widetilde{\mathcal{T}}$.

$\widetilde{\mathcal{T}}$ to its fixed point is analyzed using the iteration presented in (1.5). Let $\{\kappa_n\} = \{\tau_n\} = 0.75\ \forall n \in \mathbb{N}$, $w_1 = 0.2$ initial point. 1.0915670387 is the fixed point of the mapping $\widetilde{\mathcal{T}}$.

Let $\mathcal{T}w = \frac{w}{2}$ and choose $\theta = \frac{1}{2}$. We then compute the difference between the fixed points to which the mappings $\widetilde{\mathcal{T}}$ and $\mathcal{T}$ converge. $\|p - \widetilde{p}\| = 1.0915670387$. This example will now be applied to Theorem 3.6. We get

$$\|\mathcal{T}w - \widetilde{\mathcal{T}}w\| \leq \left\|\frac{1}{3}\sin w + \frac{1}{4}\right\| \leq \epsilon = 0.54785.$$

We also get

$$\begin{aligned}
\|p - \widetilde{p}\| &\leq \frac{\epsilon\Big(\theta^4 + \theta^3 + \theta^2 + 2\theta + 1\Big)}{1 - \theta^4} \\
&= \frac{0.54785\Big((\frac{1}{2})^4 + (\frac{1}{2})^3 + (\frac{1}{2})^2 + 2\Big)}{1 - (\frac{1}{2})^4} \\
&= \frac{0.54785\Big(0.0625 + 0.125 + 0.25 + 2\Big)}{1 - (\frac{1}{16})} = \frac{0.54785\Big((2.43375)16\Big)}{15} \\
&= 2.6(0.54785) = 1.42441.
\end{aligned}$$

Thus we obtain

$$\|p - \widetilde{p}\| = 1.0915670387 \leq 1.42441.$$

4 Application to nonlinear integral equation

In this article, our interest is to approximate the solution of the following integral equation (4.3) via of the iteration scheme defined by (1.5). Let $C([a,b])$ denote the space of all continuous functions on the interval $[a,b]$, endowed with maximum norm, $\|w-y\|_\infty = max_{u\in[a,b]}|w(u)-y(u)|; \forall w,y \in C([a,b])$. In this section we will be interested in the following delay differential equation:

$$\acute{w}(t) = \phi(t, w(t), w(t-\lambda)), t \in [t_0, b], \tag{4.1}$$

with initial condition

$$w(t) = \psi(t), t \in [t_0-\lambda, t_0]. \tag{4.2}$$

To achieve our aim, the following axioms are considered:

(1) $t_0, b \in \mathbb{R}, \lambda > 0$;

(2) $\phi \in C([t_0,b]\times\mathbb{R}^2, \mathbb{R})$;

(3) $\psi \in C([t_0-\lambda, b], \mathbb{R})$;

(4) there exists $L_\phi > 0$, such that;

$$|\phi(t,w_1,w_2) - \phi(t,y_1,y_2)| \leq L_\phi(|w_1-y_1| + |w_2-y_2|, \forall w_i, y_i \in \mathbb{R}, i = 1,2, t \in [t_0,b];$$

(5) $2L_\phi(b-t_0) < 1$.

Next, the (4.1) and (4.2) are reformulated as the following integral equation:

$$w(t) = \begin{cases} \psi(t), & t \in [t_0-\delta, t_0] \\ \psi(t_0) + \int_{t_0}^t \phi(u, w(u), w(u-\lambda))du, & t \in [t_0, b]. \end{cases} \tag{4.3}$$

The following result can be found in [5].

Theorem 4.1. Assume that conditions (1-5) are hold. Then the problem (4.1)-(4.2) has a unique solution, say $p \in C([t_0-\lambda, b], \mathbb{R}) \cap C^1([t_0,b], \mathbb{R})$ and $p = \lim_{n\to\infty} \mathcal{T}^n(w)$ for any $w \in C([t_0-\lambda, b], \mathbb{R})$.

Next, we will prove that the iteration defined by (1.5) converges strongly to the unique solution of integral equation (4.3). For this, we give our main result in this section as follows:

Theorem 4.2. Assume that conditions (1-5) are hold. Then the problem (4.1)-(4.2) has a unique solution, say $p \in C([t_0-\lambda, b], \mathbb{R}) \cap C^1([t_0,b], \mathbb{R})$. Let $\{w_n\}$ be a sequence generated by the algorithm (1.5) with real sequences $\{\kappa_n\}$ and $\{\tau_n\} \in [0,1]$ with $\sum_{n=1}^\infty \tau_n = \infty$. Then $\{w_n\}$ converges to p.

Proof . Let

$$\mathcal{T}w(t) = \begin{cases} \psi(t), & t \in [t_0-\lambda, t_0] \\ \psi(t_0) + \int_{t_0}^t \phi(u, w(u), w(u-\lambda))du, & t \in [t_0, b]. \end{cases} \tag{4.4}$$

be an integral operator with respect to (4.3). Let $\{w_n\}$ be a sequence defined by (1.5) iteration scheme for the operator (4.4). Denote by p be the fixed point of $\mathcal{T}$. We want to show $w_n \to p$ as $n \to \infty$. For $t \in [t_0-\lambda, t_0]$, it is easy to see that $w_n \to p$ as $n \to \infty$. Assume $t \in [t_0, b]$. We define $z_n = (1-\tau_n)w_n + \tau_n \mathcal{T}w_n$; $\varpi_n = \mathcal{T}z_n$; $y_n = \mathcal{T}\varpi_n$; $\alpha_n =$

$\mathcal{T}y_n;\ \beta_n = (1-\kappa_n)\alpha_n + \kappa_n\mathcal{T}\alpha_n;\ w_{n+1} = \mathcal{T}\beta_n, n \in \mathbb{N}$. Using (1.5) and (4), we obtain

$$\begin{aligned}
\|z_n - p\|_\infty =& \|(1-\tau_n)w_n + \tau_n\mathcal{T}w_n - p\|_\infty \\
\leq& (1-\tau_n)\|w_n - p\|_\infty + \tau_n\|\mathcal{T}w_n - p\|_\infty \\
\leq& (1-\tau_n)\|w_n - p\|_\infty + \max_{t\in[t_0-\lambda,b]}\Big|\psi(t_0) + \tau_n\int_{t_0}^{t}\phi(u, w_n(u), w_n(u-\lambda))du \\
& - \psi(t_0) - \int_{t_0}^{t}\phi(u, p(u), p(u-\lambda))du\Big| \\
\leq& (1-\tau_n)\|w_n - p\|_\infty + \tau_n\max_{t\in[t_0-\lambda,b]}\int_{t_0}^{t}|\phi(u, w_n(u), w_n(u-\lambda))du - \phi(u, p(u), p(u-\lambda))du| \\
\leq& (1-\tau_n)\|w_n - p\|_\infty + \tau_n\max_{t\in[t_0-\mu,b]}\int_{t_0}^{t}L_\phi(|w_n(u) - p(u)| + |w_n(u-\lambda) - p(u-\lambda)|)du \\
\leq& (1-\tau_n)\|w_n - p\|_\infty + \tau_n\int_{t_0}^{t}L_\phi\max_{t\in[t_0-\lambda,b]}(|w_n(u) - p(u)| + \max_{t\in[t_0-\lambda,b]}|w_n(u-\lambda) - p(u-\lambda)|)du \\
\leq& (1-\tau_n)\|w_n - p\|_\infty + \tau_n\int_{t_0}^{t}L_\phi(\|w_n - p\|_\infty + \|w_n - p\|_\infty)du \\
\leq& (1-\tau_n)\|w_n - p\|_\infty + 2\tau_nL_\phi(b-t_0)\|w_n - p\|_\infty \\
\leq& (1-\tau_n(1-2L_\phi(b-t_0)))\|w_n - p\|_\infty.
\end{aligned}$$

Thus,

$$\|z_n - p\|_\infty \leq (1-\tau_n(1-2L_\phi(b-t_0)))\|w_n - p\|_\infty. \tag{4.5}$$

Using (1.5), (4) and (4.5), we obtain

$$\begin{aligned}
\|\varpi_n - p\|_\infty =& \|\mathcal{T}z_n - p\|_\infty = \|\mathcal{T}z_n - \mathcal{T}p\|_\infty \\
=& \max_{t\in[t_0-\lambda,b]}\big|\mathcal{T}z_n - \mathcal{T}p\big| \\
=& \max_{t\in[t_0-\lambda,b]}\Big|\psi(t_0) + \int_{t_0}^{t}\phi(u, z_n(u), z_n(u-\lambda))du - \psi(t_0) - \int_{t_0}^{t}\phi(u, p(u), p(u-\lambda))du\Big| - \phi(u, p(u), p(u-\lambda))du\Big| \\
\leq& \max_{t\in[t_0-\mu,b]}\int_{t_0}^{t}L_\phi\big(|z_n(u) - p(u)| + |z_n(u-\lambda) - p(u-\lambda)|\big)du \\
\leq& \int_{t_0}^{t}L_\phi\big(|\max_{t\in[t_0-\lambda,b]}|z_n(u) - p(u)| + \max_{t\in[t_0-\lambda,b]}|z_n(u-\lambda) - p(u-\lambda)|\big)du \\
=& \int_{t_0}^{t}L_\phi\big(\|z_n(u) - p(u)\|_\infty + \|z_n(u-\lambda) - p(u-\lambda)\|_\infty\big)du \\
=& \int_{t_0}^{t}L_\phi\big(\|z_n(u) - p(u)\|_\infty + \|z_n(u-\lambda) - p(u-\lambda)\|_\infty\big)du \\
\leq& 2L_\phi(b-t_0)\|z_n(u) - p(u)\|_\infty \\
\leq& 2L_\phi(b-t_0)(1-\tau_n(1-2L_\phi(b-t_0)))\|w_n(u) - p(u)\|_\infty.
\end{aligned}$$

Thus,

$$\|\varpi_n - p\|_\infty \leq 2L_\phi(b-t_0)(1-\tau_n(1-2L_\phi(b-t_0)))\|w_n - p\|_\infty. \tag{4.6}$$

Using (1.5) and (4), we obtain

$$\begin{aligned}
\|y_n - p\|_\infty =& \|\mathcal{T}\varpi_n - p\|_\infty = \|\mathcal{T}\varpi_n - \mathcal{T}p\|_\infty \\
=& \max_{t\in[t_0-\lambda,b]}\big|\mathcal{T}\varpi_n - \mathcal{T}p\big| \\
=& \max_{t\in[t_0-\lambda,b]}\Big|\psi(t_0) + \int_{t_0}^{t}\phi(u, \varpi_n(u), \varpi_n(u-\lambda))du - \psi(t_0) - \int_{t_0}^{t}\phi(u, p(u), p(u-\lambda))du\Big|
\end{aligned}$$

$$
\begin{aligned}
&= \max_{t\in[t_0-\lambda,b]} \Big| \int_{t_0}^{t} \phi(u,\varpi_n(u),\varpi_n(u-\lambda))du - \phi(u,p(u),p(u-\lambda))du \Big| \\
&\leq max_{t\in[t_0-\mu,b]} \int_{t_0}^{t} L_\phi\big(|\varpi_n(u)-p(u)| + |\varpi_n(u-\lambda)-p(u-\lambda)|\big)du \\
&\leq \int_{t_0}^{t} L_\phi\big(|max_{t\in[t_0-\lambda,b]}|\varpi_n(u)-p(u)| + \max_{t\in[t_0-\lambda,b]}|\varpi_n(u-\lambda)-p(u-\lambda)|\big)du \\
&= \int_{t_0}^{t} L_\phi\big(\|\varpi_n(u)-p(u)\|_\infty + \|\varpi_n(u-\lambda)-p(u-\lambda)\|_\infty\big)du \\
&= \int_{t_0}^{t} L_\phi\big(\|\varpi_n(u)-p(u)\|_\infty + \|\varpi_n(u-\lambda)-p(u-\lambda)\|_\infty\big)du \\
&\leq 2L_\phi(b-t_0)\|\varpi_n(u)-p(u)\|_\infty \\
&\leq 4L_\phi^2(b-t_0)^2(1-\tau_n(1-2L_\phi(b-t_0)))\|w_n-p\|_\infty.
\end{aligned}
$$

Thus,

$$
\|y_n-p\|_\infty \leq 4L_\phi^2(b-t_0)^2(1-\tau_n(1-2L_\phi(b-t_0)))\|w_n-p\|_\infty. \tag{4.7}
$$

Using (1.5), (4) and (4.7), we obtain

$$
\begin{aligned}
\|\alpha_n-p\|_\infty &= \|\mathcal{T}y_n-p\|_\infty = \|\mathcal{T}y_n-\mathcal{T}p\|_\infty \\
&= \max_{t\in[t_0-\lambda,b]} \big|\mathcal{T}y_n-\mathcal{T}p\big| \\
&= \max_{t\in[t_0-\lambda,b]} \big|\psi(t_0)+\int_{t_0}^{t}\phi(u,y_n(u),y_n(u-\lambda))du - \psi(t_0) - \int_{t_0}^{t}\phi(u,p(u),p(u-\lambda))du\big| \\
&= \max_{t\in[t_0-\lambda,b]} \Big|\int_{t_0}^{t}\phi(u,y_n(u),y_n(u-\lambda))du - \phi(u,p(u),p(u-\lambda))du\Big| \\
&\leq \max_{t\in[t_0-\mu,b]} \int_{t_0}^{t} L_\phi\big(|y_n(u)-p(u)| + |y_n(u-\lambda)-p(u-\lambda)|\big)du \\
&\leq \int_{t_0}^{t} L_\phi\big(|max_{t\in[t_0-\lambda,b]}|y_n(u)-p(u)| + \max_{t\in[t_0-\lambda,b]}|y_n(u-\lambda)-p(u-\lambda)|\big)du \\
&= \int_{t_0}^{t} L_\phi\big(\|y_n(u)-p(u)\|_\infty + \|y_n(u-\lambda)-p(u-\lambda)\|_\infty\big)du \\
&= \int_{t_0}^{t} L_\phi\big(\|y_n(u)-p(u)\|_\infty + \|y_n(u-\lambda)-p(u-\lambda)\|_\infty\big)du \\
&\leq 2L_\phi(b-t_0)\|y_n(u)-p(u)\|_\infty \\
&\leq 2L_\phi(b-t_0)(1-\tau_n(1-2L_\phi(b-t_0)))\|w_n(u)-p(u)\|_\infty.
\end{aligned}
$$

Thus,

$$
\|\alpha_n-p\|_\infty \leq (2L_\phi(b-t_0))^3(1-\tau_n(1-2L_\phi(b-t_0)))\|w_n-p\|_\infty. \tag{4.8}
$$

Using (1.5) and (4), we obtain

$$
\begin{aligned}
\|\beta_n-p\|_\infty &= \|(1-\kappa_n)\alpha_n+\kappa_n\mathcal{T}\alpha_n-p\|_\infty \\
&\leq (1-\kappa_n)\|\alpha_n-p\|_\infty + \kappa_n\|\mathcal{T}\alpha_n-p\|_\infty \\
&\leq (1-\kappa_n)\|\alpha_n-p\|_\infty + max_{t\in[t_0-\lambda,b]}\big|\psi(t_0)+\kappa_n\int_{t_0}^{t}\phi(u,\alpha_n(u),\alpha_n(u-\lambda))du \\
&\quad -\psi(t_0)-\int_{t_0}^{t}\phi(u,p(u),p(u-\lambda))du\big| \\
&\leq (1-\kappa_n)\|\alpha_n-p\|_\infty + \kappa_n max_{t\in[t_0-\lambda,b]}\int_{t_0}^{t}|\phi(u,\alpha_n(u),\alpha_n(u-\lambda))du \\
&\quad -\phi(u,p(u),p(u-\lambda))du|
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\kappa_n)\|\alpha_n - p\|_\infty + \kappa_n max_{t\in[t_0-\mu,b]} \int_{t_0}^{t} L_\phi(|\alpha_n(u)-p(u)| + |\alpha_n(u-\lambda)-p(u-\lambda)|)du \\
&\leq (1-\kappa_n)\|\alpha_n - p\|_\infty + \kappa_n \int_{t_0}^{t} L_\phi max_{t\in[t_0-\lambda,b]}(|\alpha_n(u)-p(u)| + \max_{t\in[t_0-\lambda,b]} |\alpha_n(u-\lambda)-p(u-\lambda)|)du \\
&\leq (1-\kappa_n)\|\alpha_n - p\|_\infty + \kappa_n \int_{t_0}^{t} L_\phi(\|\alpha_n - p\|_\infty + \|\alpha_n - p\|_\infty)du \\
&\leq (1-\kappa_n)\|\alpha_n - p\|_\infty + 2\kappa_n L_\phi(b-t_0)\|\alpha_n - p\|_\infty \\
&\leq (1-\kappa_n(1-2L_\phi(b-t_0)))\|\alpha_n - p\|_\infty.
\end{aligned}
$$

Thus,

$$
\|\beta_n - p\|_\infty \leq (2L_\phi(b-t_0))^4(1-\tau_n(1-2L_\phi(b-t_0)))(1-\kappa_n(1-2L_\phi(b-t_0)))\|w_n - p\|_\infty. \tag{4.9}
$$

Using (1.5), (4) and (4.9), we obtain

$$
\begin{aligned}
\|w_{n+1} - p\|_\infty &= \|\mathcal{T}\beta_n - p\|_\infty = \|\mathcal{T}\beta_n - \mathcal{T}p\|_\infty \\
&= \max_{t\in[t_0-\lambda,b]} \big|\mathcal{T}\beta_n - \mathcal{T}p\big| \\
&= \max_{t\in[t_0-\lambda,b]} \Big|\psi(t_0) + \int_{t_0}^{t} \phi(u,\beta_n(u),\beta_n(u-\lambda))du - \psi(t_0) - \int_{t_0}^{t} \phi(u,p(u),p(u-\lambda))du\Big| \\
&= \max_{t\in[t_0-\lambda,b]} \Big| \int_{t_0}^{t} \phi(u,\beta_n(u),\beta_n(u-\lambda))du - \phi(u,p(u),p(u-\lambda))du\Big| \\
&\leq \max_{t\in[t_0-\mu,b]} \int_{t_0}^{t} L_\phi\big(|\beta_n(u)-p(u)| + |\beta_n(u-\lambda)-p(u-\lambda)|\big)du \\
&\leq \int_{t_0}^{t} L_\phi\big(|max_{t\in[t_0-\lambda,b]}|\beta_n(u)-p(u)| + \max_{t\in[t_0-\lambda,b]} |\beta_n(u-\lambda)-p(u-\lambda)|\big)du \\
&= \int_{t_0}^{t} L_\phi\big(\|\beta_n(u)-p(u)\|_\infty + \|\beta_n(u-\lambda)-p(u-\lambda)\|_\infty\big)du \\
&= \int_{t_0}^{t} L_\phi\big(\|\beta_n(u)-p(u)\|_\infty + \|\beta_n(u-\lambda)-p(u-\lambda)\|_\infty\big)du \\
&\leq 2L_\phi(b-t_0)\|\beta_n(u)-p(u)\|_\infty.
\end{aligned}
$$

Thus, from (4.9) we obtain

$$
\|w_{n+1} - p\|_\infty \leq (2L_\phi(b-t_0))^5(1-\tau_n(1-2L_\phi(b-t_0)))(1-\kappa_n(1-2L_\phi(b-t_0)))\|w_n - p\|_\infty. \tag{4.10}
$$

From assumption (5) and the fact that $(1-\kappa_n(1-2L_\phi(b-t_0))) < 1$, (4.10) reduces

$$
\|w_{n+1} - p\|_\infty \leq (1-\tau_n(1-2L_\phi(b-t_0)))\|w_n - p\|_\infty. \tag{4.11}
$$

Via induction, we obtain

$$
\|w_{n+1} - p\|_\infty \leq \prod_{k=1}^{n}[1-\tau_k((1-2L_\phi(b-t_0)))]\|w_1 - p\|_\infty.
$$

So, we know that $1-w \leq e^{-w}$ for all $w \in [0,1]$. Hence we have

$$
\|w_{n+1} - p\|_\infty \leq e^{-(1-2L_\phi(b-t_0))\sum_{k=1}^{n}\tau_k}\|w_1 - p\|_\infty.
$$

Taking the limit of both sides of the above inequality, $w_n \to p$ as $n \to \infty$. Hence, the proof is complete. □

Example 4.3. Consider the first-order linear delay differential equation in a Banach space equipped with the max norm. For $t \in [0,3]$,

$$\dot{w}(t) = \frac{2}{19} w(t-2), \tag{4.12}$$

with initial condition, for $t \in [-2,0]$,

$$w(t) = \psi(t) = e^t. \tag{4.13}$$

The following conditions are satisfied:

(1) $t_0 = 0,\ b = 3$;

(2) $\phi \in C([0,3] \times \mathbb{R}^2, \mathbb{R})$;

(3) $\psi \in C([-2,0], \mathbb{R})$;

(4) there exists $L_\phi = \frac{2}{19} > 0$ such that

$$|\phi(t, w_1, w_2) - \phi(t, y_1, y_2)| \leq L_\phi(|w_1 - y_1| + |w_2 - y_2|),$$

for all $t \in [0,3]$;

(5) $2L_\phi(3-0) < 1$.

The problem can be reformulated as the integral equation

$$\mathcal{T}w(t) = \begin{cases} \psi(t), & t \in [-2,0], \\ \psi(0) + \displaystyle\int_0^t \frac{2}{19} w(u-2)\, du, & t \in [0,3]. \end{cases} \tag{4.14}$$

For $t \in [0,2]$, since $u - 2 \in [-2,0]$, we have $w(u-2) = e^{u-2}$, and thus

$$w(t) = 1 + \frac{2}{19} e^{-2}(e^t - 1), \quad t \in [0,2].$$

For $t \in [2,3]$, the solution is given recursively by

$$w(t) = w(2) + \frac{2}{19} \int_2^t w(u-2)\, du.$$

No global closed-form expression exists due to the delay term; the solution is therefore characterized via the method of steps.

Moreover, define $\mathcal{T} : C([-2,3], \mathbb{R}) \to C([-2,3], \mathbb{R})$. Then

$$|\mathcal{T}w(t) - \mathcal{T}y(t)| \leq \int_0^t \frac{2}{19} |w(u-2) - y(u-2)|\, du,$$

which implies

$$\|\mathcal{T}w - \mathcal{T}y\|_\infty \leq \frac{6}{19} \|w - y\|_\infty.$$

Hence $\mathcal{T}$ is a contraction with constant $q = \frac{6}{19} < 1$ and admits a unique fixed point, which coincides with the solution of the problem.

5 Conclusions

We get some results on the strong of a new modified iteration process (1.5) for contractive mappings in Banach spaces. In addition, numerical examples are also provided to test the efficiency and effectiveness of the proposed iteration. We also prove that the iteration is the stability and data dependent. Moreover, we apply the iteration scheme to find the solution of the nonlinear integral equation. The modified iteration process proposed in this study can serve as an effective method for nonexpansive and generalized nonexpansive mappings in future research. Further, in future studies, iteration process can be developed and iteration that converges faster than prominent iterations can be presented.

References

[1] J. Ahmad, K. Ullah, M. Arshad, and Z. Ma, *A new iterative method for Suzuki mappings in Banach spaces*, J. Math. **2021** (2021), Article ID 6622931.

[2] V. Berinde, *Iterative Approximation of Fixed Points*, Springer, 2007.

[3] V. Berinde, *On the stability of some fixed point procedures*, Bul. Stiint. Univ. Baia Mare Ser. B Mat.-Inform. **18** (2002), no. 1, 7–14.

[4] F.E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Natl. Acad. Sci. USA **54** (1965), 1041–1044.

[5] G.H. Coman, G. Pavel, I. Rus, and I.A. Rus, *Introduction in the theory of operatorial equation*, Ed. Dacia, Cluj-Napoca, 1976.

[6] D. Gohde, *Zum Prinzip der Kontraktiven Abbildung*, Math. Nachr. **30** (1965), 251–258.

[7] E. Hacıoglu, *A comparative study on iterative algorithms of almost contractions in the context of convergence, stability and data dependency*, Comput. Appl. Math. **40** (2021), no. 8.

[8] E. Hacıoglu, F. Gursoy, S. Maldar, Y. Atalan, and G.V. Milovanovic, *Iterative approximation of fixed points and applications to two-point second-order boundary value problems and to machine learning*, Appl. Numer. Math. **167** (2021), 143–172.

[9] A.M. Harder and T.L. Hicks, *Stability results for fixed point iteration procedures*, Math. Japon. **33** (1988), no. 5, 693–706.

[10] I. Ishikawa, *Fixed point by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147–150.

[11] W.A. Kirk, *A fixed point theorem for mappings which do not increase distance*, Amer. Math. Monthly **72** (1965), 1004–1006.

[12] S. Maldar, F. Gursoy, Y. Atalan, and M. Abbas, *On a three-step iteration process for multivalued Reich-Suzuki type α-nonexpansive and contractive mappings*, J. Appl. Math. Comput. **68** (2022), no. 2, 863–883.

[13] W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[14] M.A. Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl. **251** (2000), 217–229.

[15] G. A. Okeke, A. V. Udo, N. H. Alharthi, and R. T. Alqahtani, *A new robust iterative scheme applied in solving a fractional diffusion model for Oxygen delivery via a capillary of tissues*, Mathematics **12** (2024), no. 9, 1339.

[16] G.A. Okeke, A.V. Udo, R.T. Alqahtani, M. Kaplan, and W.E. Ahmed, *A novel iterative scheme for solving delay differential equations and third order boundary value problems via Green's functions*, AIMS Math. **9** (2024), no. 3, 6468–6498.

[17] E. Picard, *Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives*, J. Math. Pures Appl. **6** (1890), 149–210.

[18] O. Popescu, *Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators*, Math. Commun. **12** (2007), no. 2, 195–202.

[19] S. Temir and O. Korkut, *Approximating fixed points of generalized α-nonexpansive mapping by the new iteration process*, J. Math. Sci. Model. **5** (2022), no. 1, 35–39.

[20] S. Temir, *Approximating of fixed points for multi-valued generalized α-nonexpansive mappings in Banach spaces*, Ann. Acad. Rom. Sci. Ser. Math. Appl. **15** (2023), no. 1-2, 554–574.

[21] C. Zălinescu, *On Berinde's method for comparing iterative processes*, Fixed Point Theory Algorithms Sci. Eng. **2021** (2021), no. 1, 2.