Int. J. Nonlinear Anal. Appl. In Press, 1–11
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2024.22241.2340

Oscillation criteria for third-order neutral differential equations with distributed deviating arguments

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(Communicated by Zakieh Avazzadeh)

Abstract

The objective of this paper is to study asymptotic behavior of the following third order nonlinear delay differential equations

$$\left(r(\iota)\left((x(\iota)+p(\iota)x(\sigma(\iota)))''\right)^{\alpha}\right)'+\int_a^b q(\iota,s)f(x(\tau(\iota,s)))\,\mathrm{d}s=0,$$

for $\iota\geq\iota_0$. We establish some new criteria which complement and improve many results of reported in the literature. An example is considered to illustrate our main results.

Keywords: Third-order differential equations, delay, oscillation criteria
2020 MSC: 34C10, 34K11

1 Introduction

As is well known, differential equations of third-order have long been considered as valuable tools in a lot of modeling of many phenomena in different areas of applied physics and mathematics. Indeed, it is deserved attention their use in the study of entry-flow phenomenon, the feedback nuclear reactor problem, the propagation of electrical pulses in the nerve of a squid approximated by the famous Nagumo's equation and so on [13].

A great deal of attention is given to the oscillatory solution of functional differential equations by many scholars, see [2]–[23]. With the development studying of the oscillation for the second-order differential equations, researchers start to study the oscillation for the third-order differential equations, such as [1]–[4]. But, far, there are few results that dealing with the behavior of third-order neutral differential equations. We are concerned with behavior of solutions of the following third-order neutral nonlinear differential equation

$$\left(\alpha(\varrho)(\phi''(\varrho))^{\kappa}\right)'+\int_a^b q(\varrho,s)f(x(\gamma(\varrho,s)))\,\mathrm{d}s=0,\quad\text{for }\varrho\geq\varrho_0,\tag{1.1}$$

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Received: December 2020 *Accepted:* December 2024

where $\phi(\varrho)=x(\varrho)+p(\varrho)x(\beta(\varrho))$, κ and ℓ are ratios of odd positive integers. In this paper, we assume the following conditions:

$(\mathbf{I}_0)$ $f\in C(\mathbb{R},\mathbb{R}), |f(x)|\geq k\left|x^{\ell}\right|$ for $x\neq 0$ and k is a positive constant;
$(\mathbf{I}_1)$ $p,\alpha\in C([\varrho_0,\infty))$, $p(\varrho)\leq p_0<\infty$ and $\alpha(\varrho)>0$ such that

$$\int^{\infty}\alpha^{-1/\kappa}(s)\,\mathrm{d}s=\infty;$$

$(\mathbf{I}_2)$ $q\in C([\varrho_0,\infty)\times(a,b),\mathbb{R})$, $q(\varrho)>0$ does not vanish identically;
$(\mathbf{I}_3)$ $\beta\in C([\varrho_0,\infty),(0,\infty))$, $\beta(\varrho)<\varrho$, $\beta'(\varrho)\geq\beta_0>0$ and $\lim_{\varrho\to\infty}\beta(\varrho)=\infty$;
$(\mathbf{I}_4)$ $\gamma(\varrho,s)\in C([\varrho_0,\infty)\times(a,b),\mathbb{R})$, $\gamma(\varrho,s)<\varrho$, $\lim_{\varrho\to\infty}\gamma(\varrho,s)=\infty$. Furthermore $\beta\circ\gamma=\gamma\circ\beta$;
$(\mathbf{I}_{4a})$ $\gamma(\varrho)=\varrho-\delta_0$ for $\delta_0\geq 0$, $p(\varrho)=p_0\neq 1$.

By a solution to (1.1), we mean $x\in C([\varrho_0,\infty),[0,\infty))$ with $\varrho_x\geq\varrho_0$, which satisfies the property $\alpha(\phi'')^{\kappa}\in C([\varrho_a,\infty),[0,\infty))$ and moreover satisfies (1.1) on $[\varrho_b,\infty)$. We consider the nontrivial solutions of (1.1) existing on some half-line $[\varrho_c,\infty)$ and satisfying the condition $\sup\{|x(\varrho)|:\varrho_c\leq\varrho<\infty\}>0$ for any $\varrho_c\geq\varrho_b$.

If the solution x is neither positive nor negative eventually, then x is called an **oscillatory** solution. Otherwise, it is a **non-oscillatory** solution.

If the corresponding function ϕ satisfies:

$$\textbf{Case (i):}\quad \phi(\varrho)>0,\quad \phi'(\varrho)>0,\quad \phi''(\varrho)>0,$$

then we call the class of set of all solutions x of (1.1) by **class** S_1.

And if the corresponding function ϕ satisfies:

$$\textbf{Case (ii):}\quad \phi(\varrho)>0,\quad \phi'(\varrho)<0,\quad \phi''(\varrho)>0,$$

then we call the class of set of all solutions x of (1.1) by **class** S_2.

In the following, we provide some background details that motivated our study. For oscillation of third-order neutral differential equations, Baculikova and Dzurina [1], Candan [4] considered the couple of (1.1) in the case where $\kappa=\ell$ and $0\leq p(\varrho)\leq p_0<1$.

In particular, using the comparison method, Xing et al. [29] established several oscillation theorems for equation (1.1) where $\kappa=\ell=1$. Xing et al. [7] completed oscillation results for third-order neutral delay differential equations

$$\left(\alpha_2\left(\alpha_1\phi'\right)'\right)'(\varrho)+q(\varrho)x(\gamma(\varrho))=0,\quad \text{for } \varrho\geq\varrho_0,$$

by establishing sufficient conditions for nonexistence of so-called Kneser solutions.

The main contribution in this paper is to complement existing oscillation results for third-order neutral delay differential equations by establishing sufficient conditions for nonexistence of so-called Kneser solutions. The results obtained continue and extend the works in [25, 14]. In this paper, we will adopt the following notation:

$$\begin{aligned}
\pounds\phi(\varrho)&:=\alpha(\phi'')^{\kappa}(\varrho);\\
\widetilde{q}(\varrho)&:=\min\{q(\varrho,s),q(\beta(\varrho,s))\};\\
Q(\varrho,s)&:=\int_a^b\widetilde{q}(\varrho,s)\,\mathrm{d}s,\quad \widehat{Q}(\varrho,s):=\frac{1}{\alpha(u)}\int_u^{\infty}Q(\varrho,s)\,\mathrm{d}s;\\
\tilde{O}(\varrho,s)&=\int_a^b\widetilde{q}(\varrho,s)\left(\widetilde{\eta}(\beta,\delta)\right)^{\ell}\mathrm{d}s
\end{aligned}$$

and

$$\eta(\varrho,u):=\int_u^{\varrho}\frac{1}{\alpha^{\frac{1}{\kappa}}(s)}\mathrm{d}s\ \text{ and }\ \widetilde{\eta}(\varrho,u):=\int_u^{\varrho}\left(\int_u^{s}\frac{1}{\alpha^{\frac{1}{\kappa}}(u)}\mathrm{d}u\right)\mathrm{d}s,$$

and we will define the function $\varsigma(\varrho)$ as the following:

$$\varsigma(\varrho):=\begin{cases}1 & \text{if } \kappa=\ell,\\ a_1 & \text{if } \kappa>\ell,\\ a_2\pi^{\ell-\kappa}(\varrho) & \text{if } \kappa<\ell,\end{cases}$$

where $\varrho\in[\varrho_0,\infty)$ and a_1, a_2 are any positive constants. Here, we state the following lemmas, that will be helpful in the our work.

2 Auxiliary lemmas

Lemma 2.1. Assume that $x(\varrho)$ is an eventually nonincreasing positive solution of (1.1). Then $\phi^{\ell-\kappa}(\varrho) \geq \varsigma(\varrho)$.

Lemma 2.2. Assume that $c_1, c_2 \in [0,\infty)$ and $\kappa > 0$. Then

$$(c_1 + c_2)^{\kappa} \leq \mu\left(c_1^{\kappa} + c_2^{\kappa}\right), \tag{2.1}$$

where

$$\mu := \begin{cases} 1 & \text{if } \kappa \leq 1, \\ 2^{\kappa-1} & \text{if } \kappa > 1. \end{cases}$$

Lemma 2.3. Let $x \in C^n([\varrho_0,\infty),(0,\infty))$. Assume that $x^{(n)}(\varrho)$ is of fixed sign and not identically zero on $[\varrho_0,\infty)$ and that there exists a $\varrho_1 \geq \varrho_0$ such that $x^{(n-1)}(\varrho)\, x^{(n)}(\varrho) \leq 0$ for all $\varrho \geq \varrho_1$. If $\lim_{\varrho\to\infty} x(\varrho) \neq 0$, then for every $\lambda \in (0,1)$ there exists $\varrho_\lambda \geq \varrho_1$ such that

$$x(\varrho) \geq \frac{\lambda}{(n-1)!}\varrho^{n-1}\left|x^{(n-1)}(\varrho)\right| \text{ for } \varrho \geq \varrho_\lambda.$$

Lemma 2.4. Let $u, v \in C([\varrho_0,\infty),\mathbb{R})$, $u(\varrho) = v(\varrho) + av(\varrho - b)$ for $\varrho \geq \varrho_0 + \max\{0,b\}$, where a, b are constants and $a \neq 1$. Assume that there exists a constant $l \in \mathbb{R}$, $\lim_{\varrho\to\infty} u = l$ and

$$\begin{aligned} &(\mathbf{H}_1): \quad \text{If } \liminf_{\varrho\to\infty} v(\varrho) = v_* \in \mathbb{R}, \text{ then } v_* = l/(1+a); \\ &(\mathbf{H}_2): \quad \text{If } \limsup_{\varrho\to\infty} v(\varrho) = v^* \in \mathbb{R}, \text{ then } v^* = l/(1+a). \end{aligned}$$

Lemma 2.5. Assume that $x \in S_2$. Then

$$\phi(u) \geq \widetilde{\eta}(v,u)\, \pounds^{1/\kappa}\phi(v), \tag{2.2}$$

for $u \leq v$, and

$$\left(\pounds\phi(\varrho) + \frac{(p_0)^{\ell}}{\beta_0}\, \pounds\phi(\beta(\varrho))\right)' \leq -\frac{k}{\mu}\int_a^b \widetilde{q}(\varrho,s)\,\phi^{\ell}(\gamma(\varrho,s))\,\mathrm{d}s. \tag{2.3}$$

Proof . Let $x > 0$ be solution of (1.1), eventually. Then, assume that $x(\varrho) > 0$, $x(\beta(\varrho)) > 0$ and $x(g(\varrho)) > 0$ for $\varrho \geq \varrho_1$, where ϱ_1 is sufficiently large. By Lemma 2.2 and from (1.1) and $(\mathbf{I}_2)$, we obtain

$$\phi^{\ell}(\varrho) \leq \mu\left(x^{\ell}(\varrho) + p_0^{\ell}x^{\ell}(\beta(\varrho))\right). \tag{2.4}$$

Since $\pounds\phi(\varrho)$ is nonincreasing, we have

$$-\phi'(u) \geq \int_u^v \frac{1}{\alpha^{1/\kappa}(s)}\pounds^{1/\kappa}\phi(s)\,\mathrm{d}s \geq \pounds^{1/\kappa}\phi(v)\int_u^v \frac{1}{\alpha^{1/\kappa}(s)}\mathrm{d}s, \text{ for } u \leq v. \tag{2.5}$$

Integrating above inequality from u to v, we get

$$\phi(u) - \phi(v) \geq \pounds^{1/\kappa}\phi(v)\int_u^v\left(\int_u^{\rho}\frac{1}{\alpha^{1/\kappa}(s)}\mathrm{d}s\right)\mathrm{d}\rho.$$

Thus,

$$\phi(u) \geq \widetilde{\eta}(v,u)\, \pounds^{1/\kappa}\phi(v). \tag{2.6}$$

Now, from (1.1) and $(\mathbf{I}_3)$, we obtain

$$(\pounds\phi(\beta(\varrho)))'\frac{1}{\beta'(\varrho)} + k\int_a^b q(\beta(\varrho,s))\,x^{\ell}(\gamma(\beta(\varrho,s)))\,\mathrm{d}s \leq 0. \tag{2.7}$$

Combining (2.4) and (2.7) then combining the result inequality with (1.1), we get

$$\begin{aligned} 0 &\geq (\pounds\phi(\varrho))' + k\int_a^b q(\varrho,s)\,x^\ell(\gamma(\varrho,s))\,\mathrm{d}s + \frac{p_0^\ell}{\beta_0}(\pounds\phi(\beta(\varrho)))' \\ &\quad + p_0^\ell k\int_a^b q(\beta(\varrho,s))\,x^\ell(\gamma(\beta(\varrho,s)))\,\mathrm{d}s \\ &\geq (\pounds\phi(\varrho))' + \frac{p_0^\ell}{\beta_0}(\pounds\phi(\beta(\varrho)))' + k\int_a^b \widetilde{q}(\varrho,s)\left(x^\ell(\gamma(\varrho,s)) + p_0^\ell x^\ell(\gamma(\beta(\varrho,s)))\right)\mathrm{d}s. \end{aligned}$$

Thus,

$$\left(\pounds\phi(\varrho) + \frac{p_0^\ell}{\beta_0}\pounds\phi(\beta(\varrho))\right)' + \frac{k}{\mu}\int_a^b \widetilde{q}(\varrho,s)\,\phi^\ell(\gamma(\varrho,s))\,\mathrm{d}s \leq 0. \tag{2.8}$$

The proof of the lemma is complete. □

Now by considering condition $(\mathbf{I}_{4a})$, we have the following lemma.

Lemma 2.6. Let $x(\varrho) > 0$ be a solution of (1.1). Assume that $\phi(\varrho)$ satisfies (**ii**). If

$$\int_{\varrho_0}^\infty \int_\upsilon^\infty \left(\widetilde{Q}(\varrho,s)\right)^{1/\kappa}\mathrm{d}u\mathrm{d}\upsilon = \infty, \tag{2.9}$$

then

$$\lim_{\varrho\to\infty} x(\varrho) = 0. \tag{2.10}$$

Proof . Since $\phi(\varrho)$ is nonincreasing positive function, there exists a constant $\phi_0 \geq 0$ such that $\lim_{\varrho\to\infty} g(\varrho) = \phi_0 \geq 0$. We claim that $\phi_0 = 0$. Otherwise, using Lemma, we conclude that $\lim_{\varrho\to\infty} x(\varrho) = \phi_0/(1+p_0) > 0$. Therefore, there exists a $\varrho_2 \geq \varrho_0$ such that, for all $\varrho \geq \varrho_2$

$$x(\gamma(\varrho)) > \frac{\phi_0}{2(1+p_0)} > 0. \tag{2.11}$$

From (1.1) and (2.11), we see that

$$(\pounds\phi((\varrho)))' \leq -k\int_a^b q(\varrho,s)\left(\frac{\phi_0}{2(1+p_0)}\right)^\ell \mathrm{d}s.$$

Integrating above inequality from ϱ to ∞, we have

$$\pounds\phi((\varrho)) \geq k\left(\frac{\phi_0}{2(1+P_0)}\right)^\ell \int_\varrho^\infty Q(u,s)\,\mathrm{d}u.$$

It follows that

$$\phi''(\varrho) \geq k\left(\frac{\phi_0}{2(1+P_0)}\right)^{\frac{\ell}{\kappa}}\left(\frac{1}{\alpha(\varrho)}\int_\varrho^\infty Q(u,s)\,\mathrm{d}u\right)^{\frac{1}{\kappa}}. \tag{2.12}$$

Integrating (2.12) from ϱ to ∞, yields

$$-\phi'(\varrho) \geq k\left(\frac{\phi_0}{2(1+P_0)}\right)^{\frac{\ell}{\kappa}}\int_\varrho^\infty\left(\frac{1}{\alpha(\vartheta)}\int_\vartheta^\infty Q(u,s)\,\mathrm{d}u\right)^{1/\kappa}\mathrm{d}\vartheta.$$

Integration again from ϱ_2 to ∞, we obtain

$$\phi(\varrho_2) \geq k\left(\frac{\phi_0}{2(1+P_0)}\right)^{\frac{\ell}{\kappa}}\int_{\varrho_2}^\infty\int_\upsilon^\infty\left(\frac{1}{\alpha(\vartheta)}\int_\varrho^\infty Q(u,s)\,\mathrm{d}u\right)^{1/\kappa}\mathrm{d}\vartheta\mathrm{d}\upsilon,$$

this contradicts with (2.9). Therefore, $\lim_{\varrho\to\infty}\phi(\varrho) = 0$, and from the inequality $0 < x(\varrho) \leq \phi(\varrho)$, we have property (2.10). □

3 Main results

Theorem 3.1. If there exists a function $\delta \in C([\varrho_0, \infty), (0, \infty))$ such that $\beta(\varrho) \leq \delta(\varrho)$, $\gamma^{-1}(\delta(\varrho, s)) < \varrho$, $\delta'(\varrho, s) > 0$ and the delay differential equation

$$\upsilon'(\varrho) + \frac{k}{\mu}\tilde{O}(\varrho, s)\left(\frac{\beta_0}{\beta_0 + p_0^{\ell}}\right)^{\ell/\kappa} \upsilon^{\ell/\kappa}\left(\gamma^{-1}(\delta(\varrho, a))\right) = 0 \tag{3.1}$$

is oscillatory, then S_2 is an empty set.

Proof . Suppose to the contrary that x is a positive solution of (1.1) and satisfies **(ii)**. Then, we assume that $x(\varrho) > 0$, $x(\gamma(\varrho, s)) > 0$ and $x(\beta(\varrho)) > 0$ for $\varrho \geq \varrho_1$, where ϱ_1 is sufficiently large. From (1.1), it is easy to see that $\left(\alpha(\phi'')^{\kappa}\right)'(\varrho) \leq 0$ for $\varrho \geq \varrho_1$. Using Lemma 2.5, we arrive at (2.2) and (2.3). Combining (2.3) and (2.2) with $[u = \beta(\varrho)$ and $v = \delta(\varrho)]$, we find

$$\left(\pounds\phi(\varrho) + \frac{1}{\beta_0}p_0^{\ell}\pounds\phi(\gamma(\varrho))\right)' + \frac{k}{\mu}\int_a^b \widetilde{q}(\varrho, s)\left(\widetilde{\eta}(\beta, \delta)\right)^{\ell}\pounds^{\ell/\kappa}\phi(\delta(\varrho, s))\,\mathrm{d}s \leq 0. \tag{3.2}$$

Since $\delta(\varrho, s)$ is nonincreasing with respect to s, we get

$$\left(\pounds\phi(\varrho) + \frac{1}{\beta_0}p_0^{\ell}\pounds\phi(\gamma(\varrho))\right)' + \frac{k}{\mu}\pounds^{\ell/\kappa}\phi(\delta(\varrho, a))\,\tilde{O}(\varrho, s) \leq 0.$$

Since $\pounds\phi(\varrho)$ is nonincreasing, we see that $\pounds\phi(\varrho) \leq \pounds\phi(\gamma(\varrho))$, and hence

$$\pounds\phi(\varrho) + \frac{1}{\beta_0}p_0^{\ell}\pounds\phi(\gamma(\varrho)) \leq \left(1 + \frac{1}{\beta_0}p_0^{\ell}\right)\pounds\phi(\gamma(\varrho)). \tag{3.3}$$

Using (3.2) along with (3.3), we have that $\upsilon(\varrho) := \pounds\phi(\varrho) + \frac{1}{\beta_0}p_0^{\ell}\pounds\phi(\gamma(\varrho))$ is a positive solution of the differential inequality

$$\upsilon'(\varrho) + \frac{k}{\mu}\tilde{O}(\varrho, s)\left(\frac{\beta_0}{\beta_0 + p_0^{\ell}}\right)^{\ell/\kappa} \upsilon^{\ell/\kappa}\left(\gamma^{-1}(\delta(\varrho, a))\right) \leq 0.$$

By [24, Theorem 1], the delay equation (3.1) also has a positive solution, that is a contradiction. The proof is complete. □

Theorem 3.2. Assume that $\ell \geq \kappa$. If there exists a function $\theta \in C([\varrho_0, \infty), (0, \infty))$ such that $\theta(\varrho) \leq \varrho$, $\beta(\varrho) \leq \gamma(\theta(\varrho))$ and

$$\limsup_{\varrho\to\infty} kM^{\ell-\kappa}\eta^{\kappa}(\beta, \gamma(\theta))\int_{\theta(\varrho)}^{\varrho}\widetilde{q}(\varrho, s)\,\mathrm{d}s > \mu\left(1 + \frac{1}{\beta_0}p_0^{\ell}\right), \tag{3.4}$$

then S_2 is an empty set.

Proof . As in the proof of Theorem 3.1, we obtain (3.3). Using Lemma 2.5, we arrive at (2.2) and (2.3). Integrating (2.3) from $\theta(\varrho)$ to ϱ, we get

$$0 < \pounds\phi(\varrho) + \frac{1}{\beta_0}p_0^{\ell}\,\pounds\phi(\gamma(\varrho)) \leq \pounds\phi(\theta(\varrho)) + \frac{1}{\beta_0}p_0^{\ell}\,\pounds\phi(\gamma(\theta(\varrho))) - \frac{k}{\mu}\int_{\theta(\varrho)}^{\varrho}\widetilde{q}(\varrho, s)\,\phi^{\ell}(\beta(\varrho, s))\,\mathrm{d}s,$$

which with (3.3) gives

$$\left(1 + \frac{1}{\beta_0}p_0^{\ell}\right)\pounds\phi(\gamma(\theta(\varrho))) \geq \frac{k}{\mu}\phi^{\ell}(\beta(\varrho, s))\int_{\theta(\varrho)}^{\varrho}\widetilde{q}(\varrho, s)\,\mathrm{d}s. \tag{3.5}$$

Since $\phi'(\varrho) < 0$, there exists a constant $M > 0$ such that $\phi(\varrho) \geq M$ for $\varrho \geq \varrho_2$, and hence (3.5) becomes

$$\left(1 + \frac{1}{\beta_0}p_0^{\ell}\right)\pounds\phi(\gamma(\theta(\varrho))) \geq \frac{kM^{\ell-\kappa}}{\mu}\phi^{\kappa}(\beta(\varrho))\int_{\theta(\varrho)}^{\varrho}\widetilde{q}(\varrho, s)\,\mathrm{d}s.$$

From (2.2) [$u = \beta(\varrho)$ and $v = \gamma(\theta(\varrho))$], we find

$$\left(1+\frac{1}{\beta_0}p_0^{\ell}\right) \geq \frac{kM^{\ell-\kappa}}{\mu}\eta^{\kappa}(\beta,\gamma(\theta))\int_{\theta(\varrho)}^{\varrho}\widetilde{q}(\varrho,s)\,\mathrm{d}s.$$

Taking the lim sup on both sides of the above inequality, that is contradiction to (3.4). The proof is complete. □

Corollary 3.3. Assume that there exists a function $\delta \in C([\varrho_0,\infty),(0,\infty))$ such that $\beta(\varrho) \leq \delta(\varrho)$, $\gamma^{-1}(\delta(\varrho)) < \varrho$. Then S_2 is an empty set, if one of the statements is hold:

(b_1) $\kappa = \ell$ and

$$\liminf_{\varrho\to\infty}\int_{\beta^{-1}(\delta(\varrho))}^{\varrho}\tilde{O}(\varrho,s)\,\mathrm{d}s > \frac{\mu\left(\beta_0+p_0^{\ell}\right)}{k\beta_0 e}; \tag{3.6}$$

(b_2) $\kappa < \ell$, there exists a function $\xi(\varrho) \in C^1([\varrho_0,\infty))$ such that $\xi'(\varrho) > 0$, $\lim_{\varrho\to\infty}\xi(\varrho) = \infty$,

$$\limsup_{\varrho\to\infty}\frac{\ell\xi'\left(\beta^{-1}(\delta(\varrho))\right)\left(\beta^{-1}(\delta(\varrho))\right)'}{\kappa\xi'(\varrho)} < 1 \tag{3.7}$$

and

$$\liminf_{\varrho\to\infty}\left[\frac{k}{\mu\xi'(\varrho)}\left(\frac{\beta_0}{\beta_0+p_0^{\ell}}\right)^{\ell/\kappa}\tilde{O}(\varrho,\delta)\,e^{-\xi(\varrho)}\right] > 0. \tag{3.8}$$

Proof . It is well-known from [15, 28] that conditions (3.6)-(3.8) imply oscillation of (3.1). □

Theorem 3.4. Assume that $\beta(\varrho) \leq \gamma(\varrho)$ and $\gamma'(\varrho,a) > 0$. If there exist a function $\rho(\varrho)$ and $\varrho_1 \geq \varrho_0$ such that

$$\limsup_{\varrho\to\infty}\int_{\varrho_1}^{\varrho}\left[\frac{k}{\mu}\rho(\varrho)\varsigma(\gamma(\varrho,a))Q(\varrho,s) - \left(\frac{\rho'(\varrho)}{\kappa+1}\right)^{\kappa+1}\frac{\left(\frac{\beta_0+\rho_0^{\ell}}{\beta_0}\right)}{(\rho(\varrho)\eta(\gamma(\varrho),\varrho_1)\gamma'(\varrho))^{\kappa}}\right]\mathrm{d}s = \infty, \tag{3.9}$$

then S_1 is an empty set.

Proof . Let x is a positive solution of (1.1) and satisfy case **(i)**. In view of case **(i)**, we can define a positive function by

$$\omega(\varrho) = \rho(\varrho)\frac{\pounds\phi(\varrho)}{\phi^{\kappa}(\gamma(\varrho))}. \tag{3.10}$$

Hence, by differentiating (3.10), we get

$$\omega'(\varrho) = \rho'(\varrho)\frac{\pounds\phi(\varrho)}{\phi^{\kappa}(\gamma(\varrho))} + \rho(\varrho)\frac{(\pounds\phi(\varrho))'}{\phi^{\kappa}(\gamma(\varrho))} - \frac{\kappa\rho(\varrho)\pounds\phi(\varrho)\phi^{\kappa-1}(\gamma(\varrho))\phi'(\gamma(\varrho))\gamma'(\varrho)}{\phi^{2\kappa}(\gamma(\varrho))}. \tag{3.11}$$

Substituting (3.10) into (3.11) implies

$$\omega'(\varrho) = \rho(\varrho)\frac{(\pounds\phi(\varrho))'}{\phi^{\kappa}(\gamma(\varrho))} + \frac{\rho'(\varrho)}{\rho(\varrho)}\omega(\varrho) - \frac{\kappa\eta(\gamma(\varrho),\varrho_1)\gamma'(\varrho)}{\rho^{\frac{1}{\kappa}}(\varrho)}\omega^{\frac{\kappa+1}{\kappa}}(\varrho). \tag{3.12}$$

Now, define another positive function by

$$v(\varrho) = \rho(\varrho)\frac{\pounds\phi(\beta(\varrho))}{\phi^{\kappa}(\gamma(\beta(\varrho)))}. \tag{3.13}$$

By differentiating (3.13), we get

$$v'(\varrho) = \rho'(\varrho)\frac{\pounds\phi(\beta(\varrho))}{\phi^{\kappa}(\gamma(\varrho))} + \rho(\varrho)\frac{(\pounds\phi(\beta(\varrho)))'}{\phi^{\kappa}(\gamma(\varrho))} - \frac{\kappa\rho(\varrho)\pounds\phi(\beta(\varrho))\phi^{\kappa-1}(\gamma(\varrho))\phi'(\gamma(\varrho))\gamma'(\varrho)}{\phi^{2\kappa}(\gamma(\varrho))}. \tag{3.14}$$

Substituting (3.13) into (3.14) implies

$$v'(\varrho)=\rho(\varrho)\frac{(\pounds\phi(\beta(\varrho)))'}{\phi^{\kappa}(\gamma(\varrho))}+\frac{\rho'(\varrho)}{\rho(\varrho)}v(\varrho)-\frac{\kappa\eta(\gamma(\varrho),\varrho_1)\gamma'(\varrho)}{\rho^{\frac{1}{\kappa}}(\varrho)}v^{\frac{\kappa+1}{\kappa}}(\varrho). \tag{3.15}$$

We can write the inequalities (3.12) and (3.15) in the form

$$\begin{aligned}\omega'(\varrho)+\frac{\rho_0^{\ell}}{\beta_0}v'(\varrho) &\leq \rho(\varrho)\frac{(\pounds\phi(\varrho))'+\frac{\rho_0^{\ell}}{\beta_0}(\pounds\phi(\beta(\varrho)))'}{\phi^{\kappa}(\gamma(\varrho))}+\frac{\rho'(\varrho)}{\rho(\varrho)}\omega(\varrho)-\frac{\kappa\eta(\gamma(\varrho),\varrho_1)\gamma'(\varrho)}{\rho^{\frac{1}{\kappa}}(\varrho)}\omega^{\frac{\kappa+1}{\kappa}}(\varrho)\\ &+\frac{\rho_0^{\ell}}{\beta_0}\left(\frac{\rho'(\varrho)}{\rho(\varrho)}v(\varrho)-\frac{\kappa\eta(\gamma(\varrho),\varrho_1)\gamma'(\varrho)}{\rho^{\frac{1}{\kappa}}(\varrho)}v^{\frac{\kappa+1}{\kappa}}(\varrho)\right).\end{aligned} \tag{3.16}$$

By Lemma 2.1 and $\gamma(\varrho,a)$ is increasing, (2.3) becomes

$$\begin{aligned}\left(\pounds\phi(\varrho)+\frac{(p_0)^{\ell}}{\beta_0}\pounds\phi(\beta(\varrho))\right)' &\leq -\frac{k}{\mu}\phi^{\ell}(\gamma(\varrho,a))Q(\varrho,s).\\ &\leq -\frac{\varsigma(\gamma(\varrho,a))k}{\mu}\phi^{\kappa}(\gamma(\varrho,a))Q(\varrho,s)\end{aligned} \tag{3.17}$$

Take into account Lemma 2.2, (3.17) and (3.16), we obtain

$$\begin{aligned}\omega'(\varrho)+\frac{\rho_0^{\ell}}{\beta_0}v'(\varrho) &\leq -\rho(\varrho)\frac{\varsigma(\gamma(\varrho,a))k}{\mu}Q(\varrho,s)+\frac{\rho'(\varrho)}{\rho(\varrho)}\omega(\varrho)-\frac{\kappa\eta(\gamma(\varrho),\varrho_1)\gamma'(\varrho)}{\rho^{\frac{1}{\kappa}}(\varrho)}\omega^{\frac{\kappa+1}{\kappa}}(\varrho)\\ &+\frac{\rho_0^{\ell}}{\beta_0}\left(\frac{\rho'(\varrho)}{\rho(\varrho)}v(\varrho)-\frac{\kappa\eta(\gamma(\varrho),\varrho_1)\gamma'(\varrho)}{\rho^{\frac{1}{\kappa}}(\varrho)}v^{\frac{\kappa+1}{\kappa}}(\varrho)\right).\end{aligned}$$

Applying the following inequality

$$Bu-Au^{\frac{\kappa+1}{\kappa}}\leq\frac{\kappa^{\kappa}B^{\kappa+1}}{(\kappa+1)^{\kappa+1}A^{\kappa}},\quad A>0,$$

with

$$A=\frac{\kappa\eta(\gamma(\varrho),\varrho_1)\gamma'(\varrho)}{\rho^{\frac{1}{\kappa}}(\varrho)}\text{ and }B=\frac{\rho'(\varrho)}{\rho(\varrho)},$$

we get

$$\begin{aligned}\omega'(\varrho)+\frac{\rho_0^{\ell}}{\beta_0}v'(\varrho) &\leq -\rho(\varrho)\frac{\varsigma(\gamma(\varrho,a))k}{\mu}Q(\varrho,s)+\frac{(\rho'(\varrho))^{\kappa+1}}{(\kappa+1)^{\kappa+1}(\rho(\varrho)\eta(\gamma(\varrho),\varrho_1)\gamma'(\varrho))^{\kappa}}\\ &+\frac{\frac{\rho_0^{\ell}}{\beta_0}(\rho'(\varrho))^{\kappa+1}}{(\kappa+1)^{\kappa+1}(\rho(\varrho)\eta(\gamma(\varrho),\varrho_1)\gamma'(\varrho))^{\kappa}}.\end{aligned}$$

Integrating last inequality from ϱ_1 to ϱ, we arrive at

$$\int_{\varrho_1}^{\varrho}\left[\rho(\varrho)\frac{\varsigma(\gamma(\varrho,a))k}{\mu}Q(\varrho,s)-\frac{(\rho'(\varrho))^{\kappa+1}}{(\kappa+1)^{\kappa+1}(\rho(\varrho)\eta(\gamma(\varrho),\varrho_1)\gamma'(\varrho))^{\kappa}}\left(1+\frac{\rho_0^{\ell}}{\beta_0}\right)\right]\mathrm{d}s\leq\omega(\varrho_2)+\frac{\rho_0^{\ell}}{\beta_0}v(\varrho_2).$$

The proof is complete. □

Theorem 3.5. Suppose there exist continuously differentiable functions $\rho(\varrho)$, $\xi(\varrho)$ and $\beta^{-1}(\delta(\varrho))$, also $\left(\beta^{-1}(\delta(\varrho))\right)'>0$, $\xi'(\varrho)>0$. If (3.9) and one of the conditions (3.7), (3.6) or (3.8) satisfy, then (1.1) is oscillatory.

Theorem 3.6. Suppose that x be a positive solution of (1.1). If there exist $\theta \in C([\varrho_0, \infty), (0, \infty))$, also $\theta(\varrho) < \varrho$ and $\gamma(\varrho) < \beta(\theta(\varrho))$. If conditions (3.9) and (3.4) hold, then (1.1) is oscillatory.

By considering condition $(\mathbf{I}_{4a})$, we will prove the following theorem.

Theorem 3.7. Suppose that condition (2.9) satisfied and there exists a function $\varrho \in C(I, \mathbb{R})$ such that $\varrho(\varrho) \leq \gamma(\varrho)$, $\varrho(\varrho) < \varrho$ and $\lim_{\varrho\to\infty} \varrho(\varrho) = \infty$. If the equation

$$y'(\varrho) + \frac{\widetilde{Q}(\varrho, s)}{(1+p_0)^{\ell}} y^{\frac{\ell}{\kappa}}(\varrho(\varrho, a)) = 0 \tag{3.18}$$

is oscillatory, eventually. Then every solution $x(\varrho)$ of (1.1) is either oscillatory or satisfies (2.10), where $\widetilde{Q}(\varrho, s) = \int_a^b q(\varrho, s)(\widetilde{\eta}(\varrho, s))^{\ell} \mathrm{d}s$.

Proof . Assume that $x(\varrho) > 0$ is solution of (1.1), this implies that there exists $\varrho_1 \geq \varrho_o$ such that either (**i**) or (**ii**) hold for all $\varrho \geq \varrho_1$. For (**ii**), from lemma 2.6, we see that (2.10) holds.

For (**i**), since $\phi'(\varrho)$ is nondecreasing positive function, there exists a constant c_0 such that $\lim_{\varrho\to\infty} \phi'(\varrho) = c_0 > 0$ (or $c_0 = \infty$). By using Lemma 2.4, we have

$$\lim_{\varrho\to\infty} x'(\varrho) = c_0/(1+p_0) > 0,$$

this implies that $x(\varrho)$ is nondecreasing function and take into account $\delta_0 \geq 0$, we get

$$\phi(\varrho) = x(\varrho) + p_0 x(\varrho - \delta_0) \leq (1+p_0) x(\varrho),$$

therefore

$$x(\varrho) \geq \frac{1}{1+p_0} \phi(\varrho),$$

for $\varrho(\varrho) \leq \gamma(\varrho)$, we conclude that

$$x(\gamma(\varrho)) \geq x(\varrho(\varrho)) \geq \frac{1}{1+p_0} \phi(\varrho(\varrho)),$$

substitution in (1.1), we have

$$(\pounds\phi(\varrho))' + \frac{k}{(1+p_0)^{\ell}} \int_a^b q(\varrho, s) \phi^{\ell}(\varrho(\varrho, s)) \mathrm{d}s \leq 0. \tag{3.19}$$

Using (2.2) and (3.19), we arrive at

$$\begin{aligned} 0 &\geq (\pounds\phi(\varrho))' + \frac{(\pounds\phi(\varrho(\varrho, a)))^{\frac{\ell}{\kappa}}}{(1+p_0)^{\ell}} \int_a^b q(\varrho, s)(\widetilde{\eta}(v, u))^{\ell}. \\ &= (\pounds\phi(\varrho))' + \frac{(\pounds\phi(\varrho(\varrho, a)))^{\frac{\ell}{\kappa}}}{(1+p_0)^{\ell}} \widetilde{Q}(\varrho, s). \end{aligned}$$

That is, we see that $y = \pounds\phi(\varrho)$ is a positive solution of a first delay equation (3.18). The proof is complete. □

Theorem 3.8. If the first-order delay differential equation

$$w'(\varrho) + \frac{k\lambda^{\ell}}{2^{\ell}\mu} \left(\frac{\beta_0}{\beta_0 + p_0^{\ell}}\right) \widehat{Q}(\varrho, s)\ w^{\ell/\kappa}(\gamma(\varrho, a)) = 0$$

is oscillatory, eventually. Then every solution x of (1.1) is either oscillatory or satisfies (2.10), where $\widehat{Q}(\varrho, s) = \int_a^b \widetilde{q}(\varrho, s) \gamma^{2\ell}(\varrho, s) \mathrm{d}s$.

Proof . As in the proof of Lemma 2.2. From (1.1), (2.4) and (2.7), we see that (2.8) holds. Now, by using Lemma 2.3, we obtain

$$\phi(\varrho) > \frac{\lambda}{2}\varrho^2\phi''(\varrho). \tag{3.20}$$

Since $\frac{\mathrm{d}}{\mathrm{d}\varrho}\pounds\phi(\varrho) \leq 0$ and $\beta(\varrho) \leq \varrho$, we obtain $\pounds\phi(\beta(\varrho)) \geq \pounds\phi(\varrho)$, and so

$$\pounds\phi(\varrho) + \frac{1}{\beta_0}p_0^\ell\pounds\phi(\beta(\varrho)) \leq \left(1 + \frac{1}{\beta_0}p_0^\ell\right)\pounds\phi(\varrho),$$

which with (2.8) gives

$$(\pounds\phi(\varrho))' + \frac{k}{\mu}\left(\frac{\beta_0}{\beta_0 + p_0^\ell}\right)\int_a^b \widetilde{q}(\varrho, s)\,\phi^\ell(\gamma(\varrho, s).)\,\mathrm{d}s \leq 0.$$

Thus, from (3.20) and $\gamma(\varrho, s)$ is increasing with respect to s, we find

$$(\pounds\phi(\varrho))' + \frac{k\lambda^\ell}{2^\ell\mu}\left(\frac{\beta_0}{\beta_0 + p_0^\ell}\right)(\phi''(\gamma(\varrho, a)))^\ell\int_a^b \widetilde{q}(\varrho, s)\,\gamma^{2\ell}(\varrho, s)\,\mathrm{d}s \leq 0.$$

If we set $w := \pounds\phi(\varrho) = \alpha(\phi'')^\kappa$, then we have that $w > 0$ is a solution of delay inequality

$$w'(\varrho) + \frac{k\lambda^\ell}{2^\ell\mu}\left(\frac{\beta_0}{\beta_0 + p_0^\ell}\right)\widehat{Q}(\varrho, s)\;\; w^{\ell/\kappa}(\gamma(\varrho, a)) \leq 0.$$

The proof is complete. □

4 Applications

Example 4.1. Consider the third order delay differential equation

$$\left[\left([x(\varrho) + px(\lambda t)]''\right)^\kappa\right]' + \frac{q_0}{\varrho^{\kappa(n-1)+1}}x^\kappa(\gamma t) = 0, \tag{4.1}$$

where $\gamma, \lambda \in (0, 1)$, $\widetilde{q}(\varrho) = \frac{q_0}{\varrho^{2\kappa+1}}$ and $\gamma(\varrho) = \gamma t$, $\beta(\varrho) = \lambda t$. Set $\rho(\varrho) = \varrho^2$, $\zeta(\varrho) = \frac{(\gamma+\lambda)\varrho}{2}$. It is easy to see that $\eta(\varrho, u) = (\varrho - u)$, $\widetilde{\eta}(\varrho, u) = \frac{(\varrho-u)^2}{2}$ and $\beta^{-1}(\varrho) = \frac{\varrho}{\gamma}$, Condition (3.9) holds if

$$q_0 > \frac{(2)^{\ell-1}(2\kappa)^{\kappa+1}}{\gamma^{2\kappa}(\kappa+1)^{\kappa+1}}\left(1 + \frac{\rho_0^\ell}{\beta_0}\right).$$

Condition (3.6) holds if

$$\frac{q_0}{8}(\gamma - \lambda)^2 \ln\frac{2\gamma}{\lambda + \gamma} > \frac{\beta_0 + p_0}{\beta_0 e}, \text{ where } \kappa = 1$$

by Theorem 3.5, the equation (4.1) is oscillatory at $\kappa = 1$ if

$$q_0 > \max\left\{\frac{1}{\gamma^2}\left(1 + \frac{\rho_0}{\beta_0}\right), \frac{8(\beta_0 + p_0)}{(\gamma - \lambda)^2\left(\ln\frac{2\gamma}{\lambda+\gamma}\right)\beta_0 e}\right\}.$$

Remark 4.2. It would be of interest to consider equations (1.1) and try to obtain oscillation criteria without requiring $\beta \circ \gamma = \gamma \circ \beta$ or $\left(\beta^{-1}(\varrho)\right)' \geq \beta$.

Remark 4.3. Another interesting problem for further research is to obtain some oscillation criteria if $p(t), q(t)$ are negative.

Remark 4.4. The results of this paper are presented of a high degree of generality.

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