Int. J. Nonlinear Anal. Appl. In Press, 1–7
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.18123.1990

Application of homotopy perturbation method to nonlinear time-fractional fourth order Korteweg–de Vries (KdV) equation

Seyede Zahra Tabatabaei-Nejhad[a], Yousef Bazargan-Lari[b,*], Mohammad Eghtesad[c]

[a]School of Mechanical Engineering, Persian Gulf University, Boushehr, Iran
[b]Department of Mechanical Engineering, Shiraz Branch, Islamic Azad University, Shiraz, Iran
[c]School of Mechanical Engineering, Shiraz University, Shiraz, Iran

(Communicated by Javad Damirchi)

Abstract

The aim of this paper is to find the solution for the nonlinear fractional fourth-order Korteweg–de Vries equation via the homotopy perturbation method (HPM). The solution is compared with one attained by using the homotopy analysis method (HAM). The results show that by using HPM, the same accuracy will be obtained with much less computational effort in comparison with HAM.

Keywords: homotopy perturbation method; partial differential equation; homotopy analysis method; fractional differential equation; nonlinear equation; nonlinear time-fractional KDV equation
2020 MSC: Primary 35R11; Secondary 35Q53, 65M99

1 Introduction

In the last few decades, significant attention has been dedicated to fractional differential equations (FDEs) due to their suitability for description of real phenomenon in engineering, physics and other areas of science. Hereditary and memory properties of several materials and processes can be described very well by utilizing fractional derivatives. This property is the major superiority of fractional derivatives when compared with conventional integer-order ones. The dynamical modelling of systems and processes, based on the variety of their characteristics in terms of fractional derivatives, as expected result in differential equations of fractional order and the requirement to find a solution for such equations. While there are not general methods for solving FDEs, some numerical and analytical methods have been suggested to solve these equations. Differential Transform Method (DTM) [1, 5], Fractional Difference Method (FDM) [25], Variational Iteration Method (VIM) [8, 19, 21, 24], Adomian Decomposition Method (ADM) [2, 15, 16, 26] and Homotopy Perturbation Method (HPM) [17, 22, 28] are the most commonly used methods.

The HPM, suggested first by He [9, 10], is a new analytical technique which provides approximate analytical solution for both linear and nonlinear, ordinary and partial differential equations. This method, which is the combination of homotopy (in topology) and the traditional perturbation method, transfers the equation to some equations which can be solved easily. Also, this method is not restricted to problems which have a small parameter. Accordingly, many authors have been interested to apply this powerful method to find a solution for difficult and complex problems,

*Corresponding author
Email address: bazarganlari@iaushiraz.ac.ir (Yousef Bazargan-Lari)

Received: June 2019 *Accepted:* June 2023

whether integer or non-integer equations. Non-Newtonian flow [27], fractional convection-diffusion equation containing nonlinear source term [23], nonlinear wave equation [11, 12, 13], integro-differential equation [4], space-time fractional advection-dispersion equation [29], time-fractional Navier-Stokes equation in polar coordinate [6], two-dimensional fuzzy Volterra-Fredholm integral equation [3], nonlinear vibration analysis of a rotor supported by magnetic bearings [14] and so on are some problems solved by HPM.

In this paper, a novel modification of HPM for solving fractional equations proposed by Momani [20] is used to solve nonlinear fractional Korteweg–de Vries (KdV) equation and the result is compared with the one obtained by using homotopy analysis method (HAM) by Liu and Li [18]. This equation has a wide range of applications in fluid physics, chemical physics, nonlinear optics, plasma physics and capillary-gravity waves [18].

Section 2 expresses the definition of fractional differentiation used in this article. Sections 3 and 4 outline the basic method of HPM and the modified method applied in this paper, respectively. In Section 5 the result of applying HPM is illustrated. Section 6 provides some conclusions.

2 Basic definitions of fractional differentiation

Fractional differentiation and integration are extension of the integer-order ones. Different expressions have been offered to generalize integrals and derivatives of arbitrary order. Caputo is one of the most commonly used definition of fractional differentiation and is used in this article.

Definition. The Caputo fractional derivative of order α of a continuous function f is defined as follows

$$D^{\alpha}f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}}\, d\tau, & m-1 < \alpha < m, \\ \dfrac{d^m}{dt^m} f(t), & \alpha = m, \end{cases} \tag{2.1}$$

in which m is the first integer larger than α.

3 Basic idea of He's HPM

The following equation is considered in order to explain the main idea of HPM

$$A(u) - g(r) = 0, \qquad r \in \Omega, \tag{3.1}$$

with boundary condition as

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma, \tag{3.2}$$

where A is a general differential operator, B is a boundary operator, $g(r)$ is a known analytical function, and Γ is the boundary of the domain Ω. The operator A can be divided into linear part, named L, and nonlinear part, named N. Therefore, equation (3.1) can be rewritten as following:

$$L(u) + N(u) - g(r) = 0, \qquad r \in \Omega. \tag{3.3}$$

And homotopy can be constructed as

$$H(v(r,t;p),p) = L(v(r,t;p)) - L(u_0(r,t)) + pL(u_0(r,t)) - p[N(v(r,t;p)) - f(r,t)] = 0, \tag{3.4}$$

where $p \in [0,1]$ is an embedding parameter and $u_0(r,t)$ is an initial guess for equation (3.1). One can obviously obtain the following results from equation (3.4):

$$H(v(r,t;p),0) = L(v(r,t;p)) - L(u_0(r,t)) = 0, \tag{3.5}$$

$$H(v(r,t;p),1) = A(v(r,t;p)) - f(r,t) = 0, \tag{3.6}$$

which means by varying the embedding parameter from zero to unity, $v(r,t;p)$ varies from $u_0(r,t)$ to $u(r,t)$. It can be assumed that the solution of equation (3.4) is expressed as a power series in p:

$$v(r,t;p) = \sum_{i=0}^{\infty} p^i v_i(r,t). \tag{3.7}$$

Using equation (3.7) and the resultant of equations (3.5) and (3.6) the approximate solution of equation (3.1) would be:

$$u(r,t) = \lim_{p\to 1} v(r,t;p) = v_0(r,t) + v_1(r,t) + v_2(r,t) + \cdots . \tag{3.8}$$

The convergence of the series in equation (3.8) is discussed by He in [7].

4 Modified HPM for solving FDEs

In this section, the algorithm of the new modification of HPM to solve fractional differential equations suggested by Momani [20] is presented. Consider general nonlinear differential equation of fractional order as following:

$$D^\alpha u(t) + L(u(t)) + N(u(t)) = f(t), \tag{4.1}$$

where L and N are linear and nonlinear operators, respectively. They may contain other fractional derivatives of order less than α. f is a known analytical function and D^α is defined in equation (2.1), subjected to the initial conditions:

$$u^k(0) = c_k, \qquad k = 0, 1, \ldots, m-1. \tag{4.2}$$

Based on the homotopy technique, the following homotopy can be constructed as:

$$u^{(m)} - f(t) = p\left[u^{(m)} - L(u) - N(u) - D^\alpha u\right], \qquad p \in [0,1]. \tag{4.3}$$

5 Application and numerical result

In this section the time-fractional Korteweg–de Vries (KdV) equation is considered. The result of applying modified HPM to this equation is compared with the result obtained by applying homotopy analysis method (HAM) by Liu and Li [18] to illustrate the validation and effectiveness of the method. The governing equation of KdV is as follows [18]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + (q+1)u^q u_x + u_{xxx} = 0, \qquad t > 0, \quad 0 < \alpha < 1, \tag{5.1}$$

with the initial condition

$$u(x,0) = A\left[\operatorname{sech}^2(Kx)\right]^{1/4}, \tag{5.2}$$

where $q > 0$, A and K are constants. $\partial^\alpha/\partial t^\alpha$ is the fractional time derivative operator of order α. By using equation (4.3), the first three terms of u for $q = 4$ and $q = 6$ with $A = 0.2$ and $K = 0.1$ are illustrated in equations (5.5) and (5.8), respectively.

$$u_0 = 0.2\sqrt{\operatorname{sech}(0.1x)}, \tag{5.3}$$

$$u_1 = \frac{\sqrt{(\cosh(x/10))^{-1}}\left(-\cosh(2x/5) + 308\cosh(x/5) - 531 + 256\sinh(x/5)\right)t}{800000\cosh(2x/5) + 3200000\cosh(x/5) + 2400000}, \tag{5.4}$$

$$\begin{aligned} u_2 = {} & \frac{1}{\sqrt{\cosh(0.1x)}\cosh^{25}(0.1x)}\Big[2.349853516\times 10^{-26}t^2\Big(4.058441558\times 10^{10}\cosh^{12}(0.2x) \\ & - 3.962824560\times 10^{15}\cosh^{11}(0.2x) - 3.272727274\times 10^{15}\cosh^{10}(0.2x)\sinh(0.2x) \\ & + 1.212202116\times 10^{17}\cosh^{10}(0.2x) + 4.394805178\times 10^{16}\cosh^{9}(0.2x)\sinh(0.2x) \\ & + 4.840378251\times 10^{17}\cosh^{9}(0.2x) + 3.272415578\times 10^{17}\cosh^{8}(0.2x)\sinh(0.2x) \\ & - 4.362052335\times 10^{17}\cosh^{8}(0.2x) + 6.430753243\times 10^{17}\cosh^{7}(0.2x)\sinh(0.2x) \\ & - 5.060813959\times 10^{18}\cosh^{7}(0.2x) - 2.823272716\times 10^{17}\cosh^{6}(0.2x)\sinh(0.2x) \\ & - 9.839880685\times 10^{18}\cosh^{6}(0.2x) - 3.235199998\times 10^{18}\cosh^{5}(0.2x)\sinh(0.2x) \\ & - 5.800099677\times 10^{18}\cosh^{5}(0.2x) - 6.115636363\times 10^{18}\cosh^{4}(0.2x)\sinh(0.2x) \\ & + 6.228359050\times 10^{18}\cosh^{4}(0.2x) - 6.023563635\times 10^{18}\cosh^{3}(0.2x)\sinh(0.2x) \\ & + 1.335893782\times 10^{19}\cosh^{3}(0.2x) - 3.422742856\times 10^{18}\cosh^{2}(0.2x)\sinh(0.2x) \\ & + 9.993020207\times 10^{18}\cosh^{2}(0.2x) - 1.067158441\times 10^{18}\cosh(0.2x)\sinh(0.2x) \\ & + 3.619386526\times 10^{18}\cosh(0.2x) - 1.42160389\times 10^{17}\sinh(0.2x) + 5.3097212\times 10^{17}\Big)\Big]. \end{aligned} \tag{5.5}$$

And for $q = 6$:

$$u_0 = 0.2\sqrt{\operatorname{sech}(0.1x)}, \tag{5.6}$$

$$u_1 = -10^{-14}\frac{t\sqrt{(\cosh(0.1x))^{-1}}\left(0.123456(\cosh(0.2x))^2 - 0.491358\cosh(0.2x) - 0.298667\sinh(0.2x) + 0.890123\right)}{5(\cosh(0.2x)+1)^2}, \tag{5.7}$$

$$\begin{aligned}
u_2 = \Big[& -12.673441t\left(1.015423t - 74.701870 + 1.400595 - 1.338379t^{10} - 0.021205t\right)\sinh(0.2x) \\
& + \left(2.240951 - 0.031763t - 2.141407t^{10}\right)\sinh(0.6x) + \left(-31.499457 + 30.100223t^{10} - 0.075028t\right)\cosh(0.8x) \\
& + \left(-63.502260 + 60.681432t^{10} + 0.306753\right)\cosh(0.6x) + \left(102.710006 + 98.147534t^{10} + t\right)\cosh(0.4x) \\
& + \left(-136.145794 + 130.098075t^{10} + 1.719798t\right)\cosh(0.2x) \\
& + \left(2\times10^{-6} - 2.802358\times10^{-9}t - 2\times10^{-6}t^{10}\right)\sinh(2.6x) \\
& + \left(4.5\times10^{-5} + 7.677158\times10^{-8}t - 4.3\times10^{-5}t^{10}\right)\sinh(2.4x) \\
& + \left(5.18\times10^{-4} - 4.95\times10^{-4}t^{10} + 2\times10^{-6}t\right)\sinh(2.2x) + \left(0.019513 - 0.018646t^{10} + 3.2\times10^{-5}\right)\sinh(1.8x) \\
& + \left(0.076317 - 2.8\times10^{-5}t - 0.072927t^{10}\right)\sinh(1.6x) + \left(0.233722 - 6.62\times10^{-4}t - 0.223339t^{10}\right)\sinh(1.4x) \\
& + \left(0.572379 - 0.546953t^{10} - 0.00315t\right)\sinh(1.2x) \\
& + \left(-3.896551\times10^{-8} + 3.723463\times10^{-8}t^{10} + 2.405545\times10^{-14}t\right)\cosh(2.8x) \\
& + \left(-3.111715 - 0.101132t + 2.972490t^{10}\right)\cosh(1.2x) + \left(-0.289044 - 0.039383t + 0.276205t^{10}\right)\cosh(1.4x) \\
& + \left(0.180908 - 0.009450t - 0.172872t^{10}\right)\cosh(1.6x) + \left(0.108877 - 0.001097t - 0.104041t^{10}\right)\cosh(1.8x) \\
& + \left(0.005814 + 3.5\times10^{-5}t - 0.005556t^{10}\right)\cosh(2.2x) + \left(6.21\times10^{-4} - 5.93\times10^{-4}t^{10} + 3\times10^{-6}t\right)\cosh(2.4x) \\
& + \left(3\times10^{-6} - 4.62\times10^{-8}t - 2.9\times10^{-5}t^{10}\right)\cosh(2.6x) + (1.132834 - 0.009230t)\sinh(x) \\
& + (71.383546 + 11.454595\cosh(x) - 1.082512\sinh(x))\,t^{10} + (-11.987071 - 0.156232t)\cosh(x) \\
& + \left(0.003770 + 1.1\times10^{-5}t - 0.003603t^{1/10}\right)\sinh(2x) + \left(1.812534 - 0.0191640t - 1.732020t^{1/10}\right)\sinh(0.8x) \\
& + \left(0.032070 + 5.9\times10^{-5}t - 0.0306458t^{1/10}\right)\cosh(2x)\Big]\frac{1}{\Psi}.
\end{aligned} \tag{5.8}$$

In which

$$\begin{aligned}
\Psi = \frac{1}{\cosh^3(0.1x)}\Big(& 7.755876\times10^7\cosh(0.1x) + 4.292145\times10^6\cosh(1.3x) \\
& + 1.0015\times10^7\cosh(1.1x) + \cosh(2.9x) + 29\cosh(2.7x) \\
& + 6.786391\times10^7\cosh(0.3x) + 406\cosh(2.5x) + 3654\cosh(2.3x) \\
& + 1.18755\times10^5\cosh(1.9x) + 23751\cosh(2.1x) \\
& + 4.75020\times10^5\cosh(1.7x) + 5.189593\times10^7\cosh(0.5x) \\
& + 3.459729\times10^7\cosh(0.7x) + 1.560780\times10^6\cosh(1.5x) + 2.003\times10^7\cosh(0.9x)\Big).
\end{aligned} \tag{5.9}$$

Therefore, the approximate solution is

$$u = u_0 + u_1 + u_2. \tag{5.10}$$

The convergence of the solution obtained in this article is verified by comparing it with the solution obtained by HAM for $q = 6$ [18].

Fig. 1 shows that the solution obtained in this paper, for $t = 0.1$, by assuming only three first terms are very close to solution obtained by HAM by assuming 15 terms. Also, a good agreement of obtained solution and the solution via HAM is demonstrated in Fig. 2.

6 Conclusion

In this paper, the solution of nonlinear fractional KdV equation by using modified homotopy perturbation method is obtained and the result is compared with one attained by homotopy analysis method. It was illustrated that by

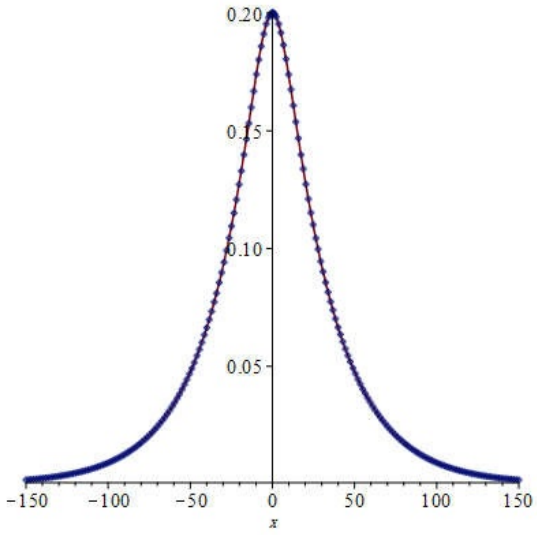


Figure 1: Comparison of the solutions obtained by modified HPM with one obtained by HAM in $t = 0.1$, for $q = 6$ and fixed values $K = 0.1$, $A = 0.2$ and $\alpha = 0.5$.

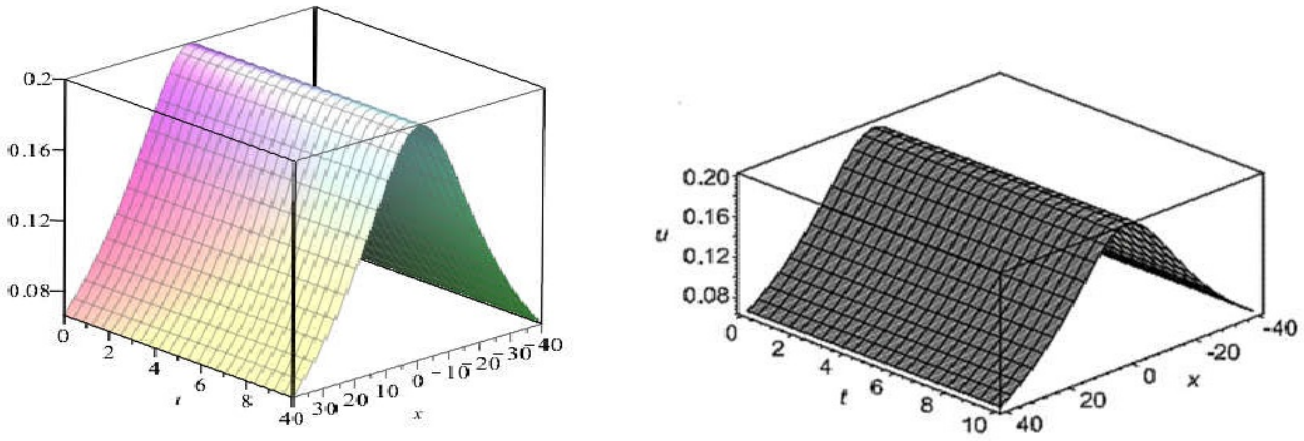


Figure 2: The solution obtained by a. HPM and b. HAM [18].

assuming only first three terms while using HPM, the solution is very close to one found via HAM and by assuming fifteen terms. Consequently, HPM presents the solution which converges very faster than HAM.

References

[1] A. Arikoglu and I. Ozkol, *Solution of fractional differential equations by using differential transform method*, Chaos, Solitons & Fractals **34** (2007), 1473–1481.

[2] V. Daftardar-Gejji and H. Jafari, *Adomian decomposition: a tool for solving a system of fractional differential equations*, J. Math. Anal. Appl. **301** (2005), 508–518.

[3] A. Ebadian, S. Najafzadeh, F. Farahrooz, and A.A. Khajehnasiri, *On the convergence and numerical computation of two-dimensional fuzzy Volterra-Fredholm integral equation by the homotopy perturbation method*, SeMA J. **75** (2017), no. 1, 17–34.

[4] M. El-Shahed, *Application of He's homotopy perturbation method to Volterra's integro-differential equation*, Int. J. Nonlinear Sci. Numer. Simul. **6** (2005), 163–168.

[5] V.S. Erturk, S. Momani, and Z. Odibat, *Application of generalized differential transform method to multi-order fractional differential equations*, Commun. Nonlinear Sci. Numer. Simul. **13** (2008), 1642–1654.

[6] Z. Ganji, D.D. Ganji, A.D. Ganji, and M. Rostamian, *Analytical solution of time-fractional Navier-Stokes equation in polar coordinate by homotopy perturbation method*, Numer. Methods Partial Differential Equations **26** (2010), 117–124.

[7] M. Ganjiani and H. Ganjiani, *Solution of coupled system of nonlinear differential equations using homotopy analysis method*, Nonlinear Dyn. **56** (2009), 159–167.

[8] J.-H. He, *Approximate analytical solution for seepage flow with fractional derivatives in porous media*, Comput. Methods Appl. Mech. Eng. **167** (1998), 57–68.

[9] J.-H. He, *Homotopy perturbation technique*, Comput. Methods Appl. Mech. Eng. **178** (1999), 257–262.

[10] J.-H. He, *A coupling method of a homotopy technique and a perturbation technique for non-linear problems*, Int. J. Non-Linear Mech. **35** (2000), 37–43.

[11] J.-H. He, *Limit cycle and bifurcation of nonlinear problems*, Chaos, Solitons & Fractals **26** (2005), 827–833.

[12] J.-H. He, *Periodic solutions and bifurcations of delay-differential equations*, Phys. Lett. A **347** (2005), 228–230.

[13] J.-H. He, *Application of homotopy perturbation method to nonlinear wave equations*, Chaos Solitons Fractals **26** (2005), 695–700.

[14] A. Heydari, M. Mirparizi, F. Shakeriaski, F.S. Samani, and M. Keshavarzi, *Nonlinear vibration analysis of a rotor supported by magnetic bearings using homotopy perturbation method*, Propuls. Power Res. **6** (2017), 223–232.

[15] H. Jafari and V. Daftardar-Gejji, *Solving a system of nonlinear fractional differential equations using Adomian decomposition*, J. Comput. Appl. Math. **196** (2006), 644–651.

[16] I. Javed, A. Ahmad, M. Hussain, and S. Iqbal, *Some solutions of fractional order partial differential equations using Adomian decomposition method*, arXiv preprint arXiv:171209207 (2017).

[17] M. Khader, *Application of homotopy perturbation method for solving nonlinear fractional heat-like equations using Sumudu transform*, Sci. Iran. Trans. B Mech. Eng. **24** (2017), 648.

[18] Y.-P. Liu and Z.-B. Li, *Homotopy analysis method for nonlinear differential equations with fractional orders*, Z. Naturforsch. A **63** (2008), 241–247.

[19] Y. Molliq, M.S.M. Noorani, and I. Hashim, *Variational iteration method for fractional heat- and wave-like equations*, Nonlinear Anal. Real World Appl. **10** (2009), 1854–1869.

[20] S. Momani and S. Abuasad, *Application of He's variational iteration method to Helmholtz equation*, Chaos, Solitons & Fractals **27** (2006), 1119–1123.

[21] S. Momani and Z. Odibat, *Analytical approach to linear fractional partial differential equations arising in fluid mechanics*, Phys. Lett. A **355** (2006), 271–279.

[22] S. Momani and Z. Odibat, *Homotopy perturbation method for nonlinear partial differential equations of fractional order*, Phys. Lett. A **365** (2007), 345–350.

[23] S. Momani and A. Yıldırım, *Analytical approximate solutions of the fractional convection-diffusion equation with nonlinear source term by He's homotopy perturbation method*, Int. J. Comput. Math. **87** (2010), 1057–1065.

[24] Z. Odibat and S. Momani, *The variational iteration method: an efficient scheme for handling fractional partial differential equations in fluid mechanics*, Comput. Math. Appl. **58** (2009), 2199–2208.

[25] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Elsevier, 1998.

[26] S.S. Ray and R. Bera, *An approximate solution of a nonlinear fractional differential equation by Adomian decomposition method*, Appl. Math. Comput. **167** (2005), 561–571.

[27] A. Siddiqui, R. Mahmood, and Q. Ghori, *Thin film flow of a third grade fluid on a moving belt by He's homotopy perturbation method*, Int. J. Nonlinear Sci. Numer. Simul. **7** (2006), 7–14.

[28] M.K. Singh and A. Chatterjee, *Solution of one-dimensional space- and time-fractional advection-dispersion equation by homotopy perturbation method*, Acta Geophys. **65** (2017), 353–361.

[29] A. Yıldırım and H. Koçak, *Homotopy perturbation method for solving the space-time fractional advection-dispersion equation*, Adv. Water Resour. **32** (2009), 1711–1716.