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On a More Accurate Multiple Hilbert-Type Inequality

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Dedicated to the Memory of Charalambos J. Papaioannou

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Abstract

By using Euler-Maclaurin's summation formula and the way of real analysis, a more accurate multiple Hilbert-type inequality and the equivalent form are given. We also prove that the same constant factor in the equivalent inequalities is the best possible.

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1. Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_n, b_n \geq 0, 0 < \sum_{n=1}^{\infty} a_n^p < \infty, 0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then a new inequality with the homogeneous kernel of degree 1 is given as (cf. [12])

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \min\{m,n\} a_m b_n < pq \left\{\sum_{n=1}^{\infty}(na_n)^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty}(nb_n)^q\right\}^{\frac{1}{q}}, \tag{1.1}$$

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where the constant factor pq is the best possible. Hilbert-type inequalities including (1.1) are important in analysis and its applications (cf. [1], [5], [13]).

By introducing another pair of conjugate exponents $(r,s)(r>1,\frac{1}{r}+\frac{1}{s}=1)$ and a parameter $0<\lambda\le\min\{r,s\}$, (1) has been extended as (cf. [12]):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\min\{m,n\})^{\lambda}a_m b_n<\frac{rs}{\lambda}\left\{\sum_{n=1}^{\infty}n^{p(1+\frac{\lambda}{r})-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q(1+\frac{\lambda}{s})-1}b_n^q\right\}^{\frac{1}{q}}, \tag{1.2}$$

where the constant factor $\frac{rs}{\lambda}$ is the best possible. For $\lambda=1, r=p, s=q$, (1.2) reduces to (1.1). Recently, by introducing $\alpha\ge\frac{\sqrt{21}}{12}-\frac{3}{4}=-0.3681^{+}, 0<\lambda\le 1$, Yang gave a more accurate best extension of (1.2) and the equivalent form as (cf. [7]):

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\min\{m,n\}+\alpha)^{\lambda}a_m b_n\\ <\ &\frac{rs}{\lambda}\left\{\sum_{n=1}^{\infty}(n+\alpha)^{p(1+\frac{\lambda}{r})-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}(n+\alpha)^{q(1+\frac{\lambda}{s})-1}b_n^q\right\}^{\frac{1}{q}},\end{aligned} \tag{1.3}$$

$$\begin{aligned}&\left\{\sum_{n=1}^{\infty}\frac{1}{(n+\alpha)^{1+(p\lambda/s)}}\left[\sum_{m=1}^{\infty}(\min\{m,n\}+\alpha)^{\lambda}a_m\right]^p\right\}^{\frac{1}{p}}\\ <\ &\frac{rs}{\lambda}\left\{\sum_{n=1}^{\infty}(n+\alpha)^{p(1+\frac{\lambda}{r})-1}a_n^p\right\}^{\frac{1}{p}}.\end{aligned} \tag{1.4}$$

For $\alpha=0$, inequality (1.3) reduces to (1.2). Another more accurate Hilbert-type inequalities were given by [6], [14], [10], [9], [11], [15]. Yang and Huang also considered the multiple Hilbert-type integral inequality (cf.[8]-[3]). Recently, Huang gave a more accurate multiple Hilbert's inequality(cf. [2]).

In this paper, by using Euler-Maclaurin's summation formula and the way of real analysis, a more accurate multiple Hilbert-type inequality and the equivalent form are given, which are the best extensions of (1.3) and (1.4).

2. Some lemmas

Lemma 2.1. *If* $n\in\mathbf{N}\backslash\{1\}, p_i, r_i>1$ $(i=1,\cdots,n), \sum_{i=1}^{n}\frac{1}{p_i}=1$, $\sum_{i=1}^{n}\frac{1}{r_i}=1$, $0<\lambda\le 1$, $\alpha\ge\frac{\sqrt{21}}{12}-\frac{3}{4}$, *then*

$$A:=\prod_{i=1}^{n}\left[(m_i+\alpha)^{(\frac{\lambda}{r_i}+1)(p_i-1)}\prod_{j=1(j\ne i)}^{n}\frac{1}{(m_j+\alpha)^{1+(\lambda/r_j)}}\right]^{\frac{1}{p_i}}=1. \tag{2.1}$$

Proof. *We find*

$$A=\prod_{i=1}^{n}\left[(m_i+\alpha)^{(\frac{\lambda}{r_i}+1)(p_i-1)+1+\frac{\lambda}{r_i}}\prod_{j=1}^{n}\frac{1}{(m_j+\alpha)^{1+(\lambda/r_j)}}\right]^{\frac{1}{p_i}}$$

$$= \prod_{i=1}^{n}\left[(m_i+\alpha)^{(\frac{\lambda}{r_i}+1)p_i}\prod_{j=1}^{n}\frac{1}{(m_j+\alpha)^{1+(\lambda/r_j)}}\right]^{\frac{1}{p_i}}$$

$$= \prod_{i=1}^{n}(m_i+\alpha)^{(\frac{\lambda}{r_i}+1)}\left[\prod_{j=1}^{n}\frac{1}{(m_j+\alpha)^{1+(\lambda/r_j)}}\right]^{\sum_{i=1}^{n}\frac{1}{p_i}},$$

and then (2.1) is valid. □

Lemma 2.2. *If $n\in\mathbf{N}$,$r>1,\frac{1}{r}+\frac{1}{s}=1,0<\lambda\le 1,\alpha\ge\frac{\sqrt{21}}{12}-\frac{3}{4}$, then*

$$\frac{rs}{\lambda}\left[1-\frac{1}{r}(\frac{1+\alpha}{n+\alpha})^{\frac{\lambda}{s}}\right]<\sum_{m=1}^{\infty}\frac{(\min\{n,m\}+\alpha)^{\lambda}}{(n+\alpha)^{\lambda/s}(m+\alpha)^{1+(\lambda/r)}}<\frac{rs}{\lambda}. \tag{2.2}$$

Proof . *For $x\in(-\alpha,\infty)$,Setting $f(x):=\frac{(\min\{n,x\}+\alpha)^{\lambda}}{(n+\alpha)^{\lambda/s}(x+\alpha)^{1+(\lambda/r)}}$, $f_1(x):=(n+\alpha)^{-\frac{\lambda}{s}}(x+\alpha)^{\frac{\lambda}{s}-1}$, $f_2(x):=(n+\alpha)^{\frac{\lambda}{r}}(x+\alpha)^{-\frac{\lambda}{r}-1}$. We find $(-1)^i f_j^{(i)}(x)>0, f_j^{(i)}(\infty)=0(i=0,1,2,3,4;j=1,2)$. By Euler-Maclaurin's summation formula (cf. [13]), we obtain*

$$\sum_{m=1}^{n}f_1(m) < \int_1^n f_1(x)dx+\frac{1}{2}[f_1(1)+f_1(n)]+\frac{1}{12}f_1'(x)\Big|_1^n,$$

$$\sum_{m=n}^{\infty}f_2(m) < \int_n^{\infty} f_2(x)dx+\frac{1}{2}f_2(n)-\frac{1}{12}f_2'(n).$$

□

For $f_1(n)=f_2(n)$, $0<\lambda,\frac{\lambda}{s}\le 1$, we have the following:

$$\sum_{m=1}^{\infty}\frac{(\min\{n,m\}+\alpha)^{\lambda}}{(n+\alpha)^{\lambda/s}(m+\alpha)^{1+(\lambda/r)}}=\sum_{m=1}^{\infty}f(m)=\sum_{m=1}^{n}f_1(m)+\sum_{m=n}^{\infty}f_2(m)-f_1(n)$$

$$< \int_1^{\infty}f(x)dx+\frac{1}{2}f_1(1)-\frac{1}{12}f_1'(1)+\frac{1}{12}(f_1'(n)-f_2'(n))$$

$$= \int_{-\alpha}^{\infty}f(x)dx-\left[\int_{-\alpha}^{1}f_1(x)dx-\frac{1}{2}f_1(1)+\frac{1}{12}f_1'(1)-\frac{1}{12}(f_1'(n)-f_2'(n))\right],$$

$$\int_{-\alpha}^{\infty}f(x)dx = \int_{-\alpha}^{n}f_1(x)dx+\int_n^{\infty}f_2(x)dx=\frac{s}{\lambda}+\frac{r}{\lambda}=\frac{rs}{\lambda},$$

$$\int_{-\alpha}^{1}f_1(x)dx-\frac{1}{2}f_1(1)+\frac{1}{12}f_1'(1)-\frac{1}{12}(f_1'(n)-f_2'(n))$$

$$= \frac{s}{\lambda}(n+\alpha)^{-\frac{\lambda}{s}}(1+\alpha)^{\frac{\lambda}{s}}-\frac{1}{2}(n+\alpha)^{-\frac{\lambda}{s}}(1+\alpha)^{\frac{\lambda}{s}-1}$$

$$+\frac{1}{12}(\frac{\lambda}{s}-1)(n+\alpha)^{-\frac{\lambda}{s}}(1+\alpha)^{\frac{\lambda}{s}-2}-\frac{\lambda}{12}(n+\alpha)^{-2}$$

$$= \frac{(\frac{1+\alpha}{n+\alpha})^{\lambda/s}(s/\lambda)}{12(1+\alpha)^2}$$

$$\times\left\{(\frac{\lambda}{s})^2-[6(1+\alpha)+1](\frac{\lambda}{s})+12(1+\alpha)^2-\frac{\lambda^2}{s}(\frac{1+\alpha}{n+\alpha})^{2-\frac{\lambda}{s}}\right\}$$

$$\geq \frac{(\frac{1+\alpha}{n+\alpha})^{\lambda/s}(s/\lambda)}{12(1+\alpha)^2}\left\{(\frac{\lambda}{s})^2-[6(1+\alpha)+1](\frac{\lambda}{s})+12(1+\alpha)^2-(\frac{\lambda}{s})\right\}$$

$$= \frac{(\frac{1+\alpha}{n+\alpha})^{\lambda/s}(s/\lambda)}{12(1+\alpha)^2}\left\{(\frac{\lambda}{s})^2-[6(1+\alpha)+2](\frac{\lambda}{s})+12(1+\alpha)^2\right\}$$

$$> \frac{(\frac{1+\alpha}{n+\alpha})^{\frac{\lambda}{s}}\frac{s}{\lambda}}{12(1+\alpha)^2}\left\{1-[6(1+\alpha)+2]+12(1+\alpha)^2\right\}\geq 0\,(1+\alpha\geq\frac{3+\sqrt{21}}{12}).$$

Hence we have the right-hand side inequality of (2.2). Since $f(x)$ *is decreasing and strictly decreasing in* (n,∞)*, we still have the following:*

$$\sum_{m=1}^{\infty} f(m) > \int_{1}^{\infty} f(x)dx$$

$$= \int_{-\alpha}^{\infty} f(x)dx - \int_{-\alpha}^{1} f_1(x)dx = \frac{rs}{\lambda} - \frac{s}{\lambda}(n+\alpha)^{-\frac{\lambda}{s}}(1+\alpha)^{\frac{\lambda}{s}},$$

then we have the left-hand side inequality of (2.2). □

Lemma 2.3. *As the assumption of Lemma 1, define the weight coefficients* $\omega_i(m_i)=\omega_\lambda(m_i,r_i;r_1,\cdots,r_{i-1},r_{i+1},\cdots,r_n)$ *as*

$$\omega_i(m_i) \;:=\; \frac{1}{(m_i+\alpha)^{\lambda/r_i}}\sum_{m_n=1}^{\infty}\cdots\sum_{m_{i+1}=1}^{\infty}\sum_{m_{i-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty} \times(\min_{1\leq k\leq n}\{m_k\}+\alpha)^{\lambda}\prod_{j=1(j\neq i)}^{n}\frac{1}{(m_j+\alpha)^{1+(\lambda/r_j)}} \tag{2.3}$$

$(\ i=1,\cdots,n,\ )$*, then we have*

$$\frac{1}{\lambda^{n-1}}\prod_{j=1}^{n} r_j\left[1-O(\frac{1}{(m_n+\alpha)^{\lambda/r_n}})\right] < \omega_n(m_n) = \frac{1}{(m_n+\alpha)^{\lambda/r_n}} \times\sum_{m_{n-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{(\min_{1\leq k\leq n}\{m_k\}+\alpha)^{\lambda}}{\prod_{j=1}^{n-1}(m_j+\alpha)^{1+(\lambda/r_j)}} < \frac{1}{\lambda^{n-1}}\prod_{j=1}^{n} r_j. \tag{2.4}$$

Moreover, it follows

$$\omega_i(m_i) < \frac{1}{\lambda^{n-1}}\prod_{j=1}^{n} r_j (i=1,\cdots,n). \tag{2.5}$$

Proof . Proof. *We prove (2.4) by mathematical induction. For* $n=2$*, by (2.2), setting* $n=m_2, m=m_1,\ r=r_1, s=r_2$*, we have (2.4). Assuming that for* $n(\geq 2)$*, (2.4) are valid, then for* $n+1$*, setting* $m_{j_0}+\alpha=\min_{2\leq k\leq n+1}\{m_k\}+\alpha$*,* $s_1:=(1-\frac{1}{r_1})^{-1}$*, then by (2.2), we have the following:*

$$\omega_{n+1}(m_{n+1}) = \frac{1}{(m_{n+1}+\alpha)^{\lambda/r_{n+1}}}\sum_{m_n=1}^{\infty}\cdots \cdots\sum_{m_2=1}^{\infty}\frac{(m_{j_0}+\alpha)^{\lambda/s_1}}{\prod_{j=2}^{n}(m_j+\alpha)^{1+(\lambda/r_j)}}\left[\sum_{m_1=1}^{\infty}\frac{(\min\{m_{j_0},m_1\}+\alpha)^{\lambda}}{(m_{j_0}+\alpha)^{\lambda/s_1}(m_1+\alpha)^{1+(\lambda/r_1)}}\right]$$

$$< \frac{r_1 s_1}{\lambda(m_{n+1}+\alpha)^{\lambda/r_{n+1}}}\sum_{m_n=1}^{\infty}\cdots\sum_{m_2=1}^{\infty}\frac{(\min_{2\leq k\leq n+1}\{m_k\}+\alpha)^{\lambda/s_1}}{\prod_{j=2}^{n}(m_j+\alpha)^{1+(\lambda/r_j)}}. \tag{2.6}$$

Setting $\widetilde{\lambda} = \frac{\lambda}{s_1}, \widetilde{r}_j = \frac{r_j}{s_1}$ *in (2.6),since* $\sum_{j=2}^{n+1} \widetilde{r}_j^{-1} = 1, 0 < \widetilde{\lambda} \leq 1$, *by the assumption of induction, it follows that*

$$\omega_{n+1}(m_{n+1}) < \frac{r_1 s_1}{\lambda} \frac{1}{\widetilde{\lambda}^{n-1}} \prod_{j=2}^{n+1} \widetilde{r}_j = \frac{1}{\lambda^n} \prod_{j=1}^{n+1} r_j. \tag{2.7}$$

By (2.2) and the assumption of induction, we still have

$$\begin{aligned} & \omega_{n+1}(m_{n+1}) > \frac{r_1 s_1}{\lambda} \frac{1}{(m_{n+1}+\alpha)^{\lambda/r_{n+1}}} \\ & \sum_{m_n=1}^{\infty} \cdots \sum_{m_2=1}^{\infty} \frac{(\min_{2\leq k\leq n+1}\{m_k\}+\alpha)^{\lambda/s_1}}{\prod_{j=2}^{n}(m_j+\alpha)^{1+(\lambda/r_j)}} \left[1 - \frac{1}{r_1}(\frac{1+\alpha}{m_{j_0}+\alpha})^{\frac{\lambda}{s_1}}\right] \\ = & \frac{r_1 s_1}{\lambda (m_{n+1}+\alpha)^{\widetilde{\lambda}/\widetilde{r}_{n+1}}} \left[\sum_{m_n=1}^{\infty} \cdots \sum_{m_2=1}^{\infty} \frac{(\min_{2\leq k\leq n+1}\{m_k\}+\alpha)^{\widetilde{\lambda}}}{\prod_{j=2}^{n}(m_j+\alpha)^{1+(\widetilde{\lambda}/\widetilde{r}_j)}} - \beta\right] \\ > & \frac{r_1 s_1}{\lambda} \left[\frac{1}{\widetilde{\lambda}^{n-1}} \prod_{j=2}^{n+1} \widetilde{r}_j \left(1 - \widetilde{O}(\frac{1}{(m_{n+1}+\alpha)^{\widetilde{\lambda}/\widetilde{r}_{n+1}}})\right) - \frac{\beta}{(m_{n+1}+\alpha)^{\widetilde{\lambda}/\widetilde{r}_{n+1}}}\right] \\ = & \frac{1}{\lambda^n} \prod_{j=1}^{n+1} r_j \left[1 - O\left(\frac{1}{(m_{n+1}+\alpha)^{\lambda/r_{n+1}}}\right)\right], \end{aligned} \tag{2.8}$$

where $\beta = \frac{(1+\alpha)^{\lambda/s_1}}{r_1} \prod_{j=2}^{n} \sum_{m_j=1}^{\infty} \frac{1}{(m_j+\alpha)^{1+(\lambda/r_j)}} \in \mathbf{R}$. *By (2.7) and (2.8), (2.4) are valid for* $n+1$. *By mathematical induction, (2.4) are valid for* $n \in \mathbf{N}\backslash\{1\}$. □

Setting $\widetilde{m}_n = m_i, \widetilde{r}_n = r_i, \widetilde{m}_j = m_{j+1}, \widetilde{r}_j = r_{j+1} (j = i, \cdots, n-1)$, $\widetilde{m}_j = m_j$, $\widetilde{r}_j = r_j (j = 1, \cdots, i-1)$, *then we have the following:*

$$\omega_i(m_i) = \omega_\lambda(\widetilde{m}_n, \widetilde{r}_n; \widetilde{r}_1, \cdots, \widetilde{r}_{n-1}) < \frac{1}{\lambda^{n-1}} \prod_{i=1}^{n} \widetilde{r}_i = \frac{1}{\lambda^{n-1}} \prod_{i=1}^{n} r_i.$$

Hence (2.5) is valid. □

3. Main Results

Theorem 3.1. *Suppose that* $n \in \mathbf{N}\backslash\{1\}, p_i, r_i > 1$ $(i = 1, \cdots, n), \sum_{i=1}^{n} \frac{1}{p_i} = 1$, $\sum_{i=1}^{n} \frac{1}{r_i} = 1, \frac{1}{q_n} = 1 - \frac{1}{p_n}$, $0 < \lambda \leq 1, \alpha \geq \frac{\sqrt{21}}{12} - \frac{3}{4}$. *If* $a_{m_i}^{(i)} \geq 0, 0 < \sum_{m_i=1}^{\infty} (m_i+\alpha)^{p_i(1+\frac{\lambda}{r_i})-1} (a_{m_i}^{(i)})^{p_i} < \infty (i = 1, \cdots, n)$, *then we have the following equivalent inequalities:*

$$\begin{aligned} I \ : & = \sum_{m_n=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} (\min_{1\leq k\leq n}\{m_k\}+\alpha)^{\lambda} \prod_{i=1}^{n} a_{m_i}^{(i)} \\ & < \frac{1}{\lambda^{n-1}} \prod_{i=1}^{n} r_i \left\{\sum_{m_i=1}^{\infty} (m_i+\alpha)^{p_i(1+\frac{\lambda}{r_i})-1} (a_{m_i}^{(i)})^{p_i}\right\}^{\frac{1}{p_i}}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} J \ : & = \left\{\sum_{m_n=1}^{\infty} \frac{1}{(m_n+\alpha)^{1+(q_n\lambda/r_n)}} \left[\sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} (\min_{1\leq k\leq n}\{m_k\}+\alpha)^{\lambda} \right.\right. \\ & \left.\left. \times \prod_{i=1}^{n-1} a_{m_i}^{(i)}\right]^{q_n}\right\}^{\frac{1}{q_n}} < \frac{r_n}{\lambda^{n-1}} \prod_{i=1}^{n-1} r_i \left\{\sum_{m_i=1}^{\infty} (m_i+\alpha)^{p_i(1+\frac{\lambda}{r_i})-1} (a_{m_i}^{(i)})^{p_i}\right\}^{\frac{1}{p_i}}. \end{aligned} \tag{3.2}$$

Lemma 3.2. Proof . *We have proven the theorem for $n = 2$ (cf. [7]). In the following, we prove the theorem for $n \geq 3$.* □

Since $\frac{1}{p_n} + \frac{1}{q_n} = 1$, by (2.1), (2.4) and Hölder's inequality (cf. [4]), we find

$$\begin{aligned}
&\left[\sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} (\min_{1\leq k\leq n}\{m_k\} + \alpha)^{\lambda} \prod_{i=1}^{n-1} a_{m_i}^{(i)}\right]^{q_n} \\
= &\left\{\sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} (\min_{1\leq k\leq n}\{m_k\} + \alpha)^{\lambda} \left[\prod_{j=1}^{n-1} \frac{(m_n+\alpha)^{(\frac{\lambda}{r_n}+1)(p_n-1)}}{(m_j+\alpha)^{1+(\lambda/r_j)}}\right]^{\frac{1}{p_n}}\right. \\
&\left.\times \prod_{i=1}^{n-1}\left[(m_i+\alpha)^{(\frac{\lambda}{r_i}+1)(p_i-1)} \prod_{j=1(j\neq i)}^{n} \frac{1}{(m_j+\alpha)^{1+(\lambda/r_j)}}\right]^{\frac{1}{p_i}} a_{m_i}^{(i)}\right\}^{q_n} \\
\leq &\left\{\omega_n(m_n)(m_n+\alpha)^{p_n(\frac{\lambda}{r_n}+1)-1}\right\}^{\frac{q_n}{p_n}} \sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} (\min_{1\leq k\leq n}\{m_k\} + \alpha)^{\lambda} \\
&\times \prod_{i=1}^{n-1}\left[(m_i+\alpha)^{(\frac{\lambda}{r_i}+1)(p_i-1)} \prod_{j=1(j\neq i)}^{n} \frac{1}{(m_j+\alpha)^{1+(\lambda/r_j)}}\right]^{\frac{q_n}{p_i}} (a_{m_i}^{(i)})^{q_n} \\
\leq &\left(\frac{1}{\lambda^{n-1}}\prod_{i=1}^{n} r_i\right)^{q_n-1} (m_n+\alpha)^{1+\frac{q_n\lambda}{r_n}} \sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} (\min_{1\leq k\leq n}\{m_k\} + \alpha)^{\lambda} \\
&\times \prod_{i=1}^{n-1}\left[(m_i+\alpha)^{(\frac{\lambda}{r_i}+1)(p_i-1)} \prod_{j=1(j\neq i)}^{n} \frac{1}{(m_j+\alpha)^{1+(\lambda/r_j)}}\right]^{\frac{q_n}{p_i}} (a_{m_i}^{(i)})^{q_n},
\end{aligned}$$

$$\begin{aligned}
J \leq &\left(\frac{1}{\lambda^{n-1}}\prod_{i=1}^{n} r_i\right)^{\frac{1}{p_n}} \left\{\sum_{m_n=1}^{\infty}\sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} (\min_{1\leq k\leq n}\{m_k\} + \alpha)^{\lambda}\right. \\
&\left.\times \prod_{i=1}^{n-1}\left[(m_i+\alpha)^{(\frac{\lambda}{r_i}+1)(p_i-1)} \prod_{j=1(j\neq i)}^{n} \frac{1}{(m_j+\alpha)^{1+(\lambda/r_j)}}\right]^{\frac{q_n}{p_i}} (a_{m_i}^{(i)})^{q_n}\right\}^{\frac{1}{q_n}} \\
= &\left(\frac{1}{\lambda^{n-1}}\prod_{i=1}^{n} r_i\right)^{\frac{1}{p_n}} \left\{\sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \left[\sum_{m_n=1}^{\infty} \frac{(\min_{1\leq k\leq n}\{m_k\}+\alpha)^{\lambda}}{(m_n+\alpha)^{1+(\lambda/r_n)}}\right]\right. \\
&\left.\times \prod_{i=1}^{n-1}\left[\frac{(m_i+\alpha)^{p_i(1+\frac{\lambda}{r_i})-1}}{(m_i+\alpha)^{\frac{\lambda}{r_i}}} \prod_{j=1(j\neq i)}^{n-1} \frac{1}{(m_j+\alpha)^{1+\frac{\lambda}{r_j}}}\right]^{\frac{q_n}{p_i}} (a_{m_i}^{(i)})^{q_n}\right\}^{\frac{1}{q_n}}. \qquad (3.3)
\end{aligned}$$

For $n \geq 3$, since $\sum_{i=1}^{n-1} \frac{q_n}{p_i} = 1$, by Hölder's inequality again in (3.3), we find

$$J \leq \left(\frac{1}{\lambda^{n-1}}\prod_{i=1}^{n} r_i\right)^{\frac{1}{p_n}} \prod_{i=1}^{n-1}\left\{\sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty}\sum_{m_n=1}^{\infty} \frac{(\min_{1\leq k\leq n}\{m_k\}+\alpha)^{\lambda}}{(m_n+\alpha)^{1+(\lambda/r_n)}}\right.$$

$$\times\left[\frac{(m_i+\alpha)^{p_i(1+\frac{\lambda}{r_i})-1}}{(m_i+\alpha)^{\frac{\lambda}{r_i}}}\prod_{j=1(j\neq i)}^{n-1}\frac{1}{(m_j+\alpha)^{1+(\lambda/r_j)}}\right](a_{m_i}^{(i)})^{p_i}\Bigg\}^{\frac{1}{p_i}}$$

$$=\left(\frac{1}{\lambda^{n-1}}\prod_{i=1}^{n}r_i\right)^{\frac{1}{p_n}}\prod_{i=1}^{n-1}\left\{\sum_{m_i=1}^{\infty}\omega_i(m_i)(m_i+\alpha)^{p_i(1+\frac{\lambda}{r_i})-1}(a_{m_i}^{(i)})^{p_i}\right\}^{\frac{1}{p_i}}. \quad (3.4)$$

Then by (2.4), we have (3.2). Since $\frac{1}{q_n}+\frac{1}{p_n}=1$, *by Hölder's inequality, we have*

$$I=\sum_{m_n=1}^{\infty}\left[\frac{1}{(m_n+\alpha)^{\frac{1}{q_n}+\frac{\lambda}{r_n}}}\sum_{m_{n-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}(\min_{1\le k\le n}\{m_k\}+\alpha)^{\lambda}\prod_{i=1}^{n-1}a_{m_i}^{(i)}\right]$$

$$\times\left[(m_n+\alpha)^{\frac{1}{q_n}+\frac{\lambda}{r_n}}a_{m_n}^{(n)}\right]\le J\left\{\sum_{m_n=1}^{\infty}(m_n+\alpha)^{p_n(1+\frac{\lambda}{r_n})-1}(a_{m_n}^{(n)})^{p_n}\right\}^{\frac{1}{p_n}}. \quad (3.5)$$

Then by (3.2), we have (3.1). Assuming that (3.1) is valid, setting

$$a_{m_n}^{(n)}:=\frac{1}{(m_n+\alpha)^{1+(q_n\lambda/r_n)}}\left[\sum_{m_{n-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}(\min_{1\le k\le n}\{m_k\}+\alpha)^{\lambda}\prod_{i=1}^{n-1}a_{m_i}^{(i)}\right]^{q_n-1},$$

then by (3.4), it follows $J=\left\{\sum_{m_n=1}^{\infty}(m_n+\alpha)^{p_n(1+\frac{\lambda}{r_n})-1}(a_{m_n}^{(n)})^{p_n}\right\}^{\frac{1}{q_n}}<\infty$. *If* $J=0$, *then (3.2) is naturally valid. Suppose* $0<J<\infty$. *By (3.1), we find*

$$\begin{aligned}0 &< \sum_{m_n=1}^{\infty}(m_n+\alpha)^{p_n(1+\frac{\lambda}{r_n})-1}(a_{m_n}^{(n)})^{p_n}=J^{q_n}=I\\ &< \frac{1}{\lambda^{n-1}}\prod_{i=1}^{n}r_i\left\{\sum_{m_i=1}^{\infty}(m_i+\alpha)^{p_i(1+\frac{\lambda}{r_i})-1}(a_{m_i}^{(i)})^{p_i}\right\}^{\frac{1}{p_i}}<\infty,\\ &\left\{\sum_{m_n=1}^{\infty}(m_n+\alpha)^{p_n(1+\frac{\lambda}{r_n})-1}(a_{m_n}^{(n)})^{p_n}\right\}^{\frac{1}{q_n}}\\ &= J<\frac{r_n}{\lambda^{n-1}}\prod_{i=1}^{n-1}r_i\left\{\sum_{m_i=1}^{\infty}(m_i+\alpha)^{p_i(1+\frac{\lambda}{r_i})-1}(a_{m_i}^{(i)})^{p_i}\right\}^{\frac{1}{p_i}}.\end{aligned}$$

Then (3.2) is valid, which is equivalent to (3.1). □

Theorem 3.3. *As the assumption of Theorem 1, the same constant factor* $\frac{1}{\lambda^{n-1}}\prod_{i=1}^{n}r_i$ *in (3.1) and (3.2) is the best possible.*

Proof . *For* $0<\varepsilon<\frac{q_n\lambda}{r_n}$, *setting* $\widetilde{r}_i=(\frac{1}{r_i}+\frac{\varepsilon}{p_i\lambda})^{-1}$, $\widetilde{a}_{m_i}^{(i)}=(m_i+\alpha)^{-\frac{\lambda}{\widetilde{r}_i}-1}$ $(i=1,\cdots,n-1)$, $\widetilde{r}_n=(\frac{1}{r_n}-\frac{\varepsilon}{q_n\lambda})^{-1}$, $\widetilde{a}_{m_n}^{(n)}=(m_n+\alpha)^{-\frac{\lambda}{\widetilde{r}_n}-1-\varepsilon}$, *we have* $\widetilde{r}_i>1(i=1,\cdots,n)$, $\sum_{i=1}^{n}\frac{1}{\widetilde{r}_i}=1$. *Then by (2.4), we find*

$$\widetilde{I}\ :=\sum_{m_n=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}(\min_{1\le k\le n}\{m_k\}+\alpha)^{\lambda}\prod_{i=1}^{n}\widetilde{a}_{m_i}^{(i)}=\sum_{m_n=1}^{\infty}\frac{1}{(m_n+\alpha)^{\varepsilon+1}}$$

$$\times\left[\frac{1}{(m_n+\alpha)^{\lambda/\widetilde{r}_n}}\sum_{m_{n-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}(\min_{1\le k\le n}\{m_k\}+\alpha)^{\lambda}\prod_{i=1}^{n-1}\frac{1}{(m_i+\alpha)^{1+(\lambda/\widetilde{r}_i)}}\right]$$

$$\begin{aligned}
&= \sum_{m_n=1}^{\infty} \frac{1}{(m_n+\alpha)^{\varepsilon+1}} \cdot \omega_\lambda(m_n, \widetilde{r}_n; \widetilde{r}_1, \cdots, \widetilde{r}_{n-1}) \\
&> \frac{1}{\lambda^{n-1}} \prod_{i=1}^{n} \widetilde{r}_i \sum_{m_n=1}^{\infty} \frac{1}{(m_n+\alpha)^{\varepsilon+1}} \left[1 - O(\frac{1}{(m_n+\alpha)^{\lambda/\widetilde{r}_n}})\right] \\
&= \frac{1}{\lambda^{n-1}} \prod_{i=1}^{n} \widetilde{r}_i \sum_{m_n=1}^{\infty} \frac{1}{(m_n+\alpha)^{\varepsilon+1}} \\
&\quad \times \left\{1 - \left[\sum_{m_n=1}^{\infty} \frac{1}{(m_n+\alpha)^{\varepsilon+1}}\right]^{-1} \sum_{m_n=1}^{\infty} O(\frac{1}{(m_n+\alpha)_n^{\varepsilon+1+(\lambda/\widetilde{r}_n)}})\right\}. \qquad (3.6)
\end{aligned}$$

□

If there exists a constant $k(\leq \frac{1}{\lambda^{n-1}} \prod_{i=1}^{n} r_i)$*, such that (3.1) is still valid as we replace* $\frac{1}{\lambda^{n-1}} \prod_{i=1}^{n} r_i$ *by* k*, then in particular, we have*

$$\widetilde{I} < k \prod_{i=1}^{n} \left\{ \sum_{m_i=1}^{\infty} (m_i+\alpha)^{p_i(1+\frac{\lambda}{r_i})-1} (\widetilde{a}_{m_i}^{(i)})^{p_i} \right\}^{\frac{1}{p_i}} = k \sum_{m_n=1}^{\infty} \frac{1}{(m_n+\alpha)^{\varepsilon+1}}. \qquad (3.7)$$

In virtue of (3.6) and (3.7), it follows,

$$\frac{\prod_{i=1}^{n} \widetilde{r}_i}{\lambda^{n-1}} \left\{1 - \left[\sum_{m_n=1}^{\infty} \frac{1}{(m_n+\alpha)^{\varepsilon+1}}\right]^{-1} \sum_{m_n=1}^{\infty} O\left(\frac{1}{(m_n+\alpha)^{\varepsilon+1+(\lambda/\widetilde{r}_n)}}\right)\right\} < k,$$

and then $\frac{1}{\lambda^{n-1}} \prod_{i=1}^{n} r_i \leq k$*, for* $\varepsilon \to 0^+$*. Hence* $k = \frac{1}{\lambda^{n-1}} \prod_{i=1}^{n} r_i$ *is the best value of (3.1). We conform that the constant factor* $\frac{1}{\lambda^{n-1}} \prod_{i=1}^{n} r_i$ *in (3.2) is the best possible, otherwise we can get a contradiction by (3.5) that the constant factor in (3.1) is not the best possible.* □

Remark 3.4. *For* $\alpha = 0$ *in (3.1), we have a multiple best extension of (1.2) as*

$$\begin{aligned}
&\sum_{m_n=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \left(\min_{1\leq k\leq n} \{m_k\}\right)^{\lambda} \prod_{i=1}^{n} a_{m_i}^{(i)} \\
&< \frac{1}{\lambda^{n-1}} \prod_{i=1}^{n} r_i \left\{ \sum_{m_i=1}^{\infty} {m_i}^{p_i(1+\frac{\lambda}{r_i})-1} (a_{m_i}^{(i)})^{p_i} \right\}^{\frac{1}{p_i}}. \qquad (3.8)
\end{aligned}$$

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