Int. J. Nonlinear Anal. Appl. 5 (2014) No. 2, 1-6
ISSN: 2008-6822 (electronic)
http://www.ijnaa.semnan.ac.ir

Some fixed point theorems and common fixed point theorem in log-convex structure

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(Communicated by A. Ebadian)

Abstract

Some fixed point theorems and common fixed point theorem in Logarithmic convex structure are proved.

Keywords: Convex metric space; common fixed point; logarithmic convex structure.
2010 MSC: 47H09, 47H10, 47H19.

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1. Introduction

In 1970, W. Takahashi [5] introduced the notion of convexity in metric spaces and studies some fixed point theorems for non expansive mapping in such spaces. He proved that all norm spaces and their convex subsets are convex metric spaces. Takahashi also gave many examples of convex metric spaces which are not embeded in any normed Banach space. Subsequently, M. D. Guay, K. L. Singh and J. H. M. Whitfield [2], Machado [3], Tallman [6], Shimizu [7] and Ciric [1] were among others who obtained results in this setting. Recently, the auther and et al. [4] define the notion of the logarithmic convex structure. In this paper, some fixed point theorems and a common fixed point theorem is proved.

2. Definitions and propositions

For a metric space (X, d), a continuous mapping, $W : X \times X \times [0, 1] \longrightarrow X$ is said to be a convex structure on X if for all $x, y \in X$ and $\lambda \in [0, 1]$

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y) \tag{2.1}$$

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Received: January 2013 *Revised:* May 2013.

holds for all $u \in X$. The metric space (X,d) with a convex structure is called a convex metric space. A Banach space and each of its convex subsets are convex metric spaces. But a Frechet space is not necessary a convex metric space [5].
Recently, the author and et al. define the logaritmic convex structure as the following:[4]

Definition 2.1. *Let X be a set and $D : X \times X \to [1,\infty)$ be a mapping satisfying the following conditions:*

(i) For all $x, y \in X$, $D(x,y) \geq 1$ and $D(x,y) = 1$ if and only if $x = y$.

(ii) For all $x, y \in X, D(x,y) = D(y,x)$.

(iii) $\forall x, y, z \in X; D(x,y) \leq D(x,z)D(z,y)$.

(iv) For all $x, y, z \in X$, $z \neq x, y$ and $\lambda \in (0,1)$;
$D\Big(z, W(x,y,\lambda)\Big) \leq D^{\lambda}(x,z)D^{1-\lambda}(y,z)$ and

$$D(x,y) = D\Big(x, W(x,y,\lambda)\Big)D\Big(y, W(x,y,\lambda)\Big) \tag{2.2}$$

where, $W : X \times X \times [0,1] \to X$ is a continuous mapping. We name this a logarithmic convex structure.

A subset K of a logarithmic convex structure is said to be log-convex if $W(x,y,\lambda) \in K$ for all $x, y \in K$ and $\lambda(0 \leq \lambda \leq 1)$.

Remark 2.2. *Note that for every convex metric space (X,d), by defining $D(x,y) = e^{d(x,y)}$, we obtain a log-convex structure. In the following, we make a log–convex structure which is not the above form.*

Let $D(x,y) = 1 + e^{-|x-y|}$ for $x \neq y$ and one for the case $x = y$, then $(\mathbb{R}, D, W)$ is a log-convex structure, where $W(x,y,\lambda) = \lambda x + (1-\lambda)y$. For inequality in (iv), one may apply the following inequality:

$$1 + a^{\lambda}b^{1-\lambda} \leq (1+a)^{\lambda}(1+b)^{1-\lambda} \qquad (0 < a, b \leq 1).$$

Since, the function $f(a,b) = (1+a)^{\lambda}(1+b)^{1-\lambda} - a^{\lambda}b^{1-\lambda}$ attains its minimum at points (a,a), so the above inequality is hold.

Proposition 2.3. *The open balls $B_r(x)$ and the closed balls $\overline{B_r(x)}$ in X are log-convex subsets of X.*

Proposition 2.4. *Let $\{K_\alpha : \quad \alpha \in I\}$ be a family of log-convex subsets of X, then $\bigcap_{\alpha \in I} K_\alpha$ is a log-convex subset of X.*

Definition 2.5. *For a subset A of a log- convex structure X, we denote the log-convex hull of A as $Lco(A)$ and define as the intersection of all log-convx sets containing A.*

Suppose that $A_n = W^n(A) = W(W...(W(A)...)$ then, $\{A_n\}_{n=1}^{\infty}$ is an increasing sequence of log-convex subsets of X. One may show that

$$\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} An = Lco(A).$$

Definition 2.6. *Let X be a log-convex structure and A be a nonempty log-convex bounded set in X. For $x \in X$, let us set*

$$r_x(A) = \sup_{y\in A} d(x,y),$$

and

$$r(A) = \inf_{x\in A} r_x(A).$$

We thus define $A_c = \{x \in A \quad : \quad r_x(A) = r(A)\}$ to be the center of A. We denote the diameter of a subset A of X by

$$\delta(A) = \sup\{d(x,y) \quad : \quad x, y \in A\}.$$

A point $x \in A$ is called a diametral point of A iff

$$\sup_{y\in A} d(x,y) = \delta(A).$$

A log-convex structure X will be said to have Property (LC) *if every bounded decreasing net of nonempty closed log-convex subsets of X has a nonempty intersection.*

Proposition 2.7. *If X has* Property(LC)*, then A_c is nonempty, closed and log-convex.*

Proposition 2.8. *Let M be nonempty compact subset of X and let K be the least closed log-convex set containing M. If the diameter $\delta(M)$ is positive, then there exists an element $u \in K$ such that* $\sup\{d(x,u) : x \in M\} < \delta(M)$.

Definition 2.9. *A log-convex structure X is said to have normal structure if for each closed bounded log-convex subset A of X which contains at least two points, there exists $x \in A$ which is not a diametral point of A.*

Let K be a nonempty subset of a log-convex structure X. A selfmapping $T : K \longrightarrow K$ has property (G) if

$$\exists\lambda \in [0,1]\forall x, y \in K; d(Tx,Ty) \leq d^{\lambda}(x,Tx)d^{1-\lambda}(y,Ty)$$

T is said to have property (B) on K, if for every closed and log-convex subset F of K, which has nonzero diameter and is invariant under T, there exists some $x \in F$ such that

$$d(x,Tx) < \sup_{y\in F} d(y,Ty)$$

The set S in a log-convex structure X is said to be log-star shaped if there exists $x_0 \in S$ such that $W(x, x_0, \lambda) \in S$ for all $x \in S$ and $\lambda \in [0,1]$.

3. main result

Theorem 3.1. *Let T be a mapping of a non-empty bounded closed log-star shaped subset S of a log-convex structure X, into itself which has property (G). Suppose that for every closed T-invariant log-star shaped subset C of S we have*

$$\sup d(y, Ty) < \sqrt{\delta(C)},$$

with $\delta(C) > 0$ which $\delta(C)$ is diameter of C. Then, T has a unique fixed point in S if S posses a minimal closed T-invariant log-star shaped subset S^* of S.

Proof . If $\delta(S^*) = 0$ then the point in S^* is fixed point. By contrary, assume that $\delta(S^*) > 0$. So, there is $x_0 \in S^*$ such that for every $x \in S^*$, $\lambda \in [0,1]$, $W(x, x_0, \lambda) \in s^*$. For each $y \in S^*$ and for some λ_0 we have

$$\begin{aligned} d(Ty, Tx_0) &\leq d^{\lambda_0}(y, Ty) d^{1-\lambda_0}(x_0, Tx_0) \\ &\leq \sup_{z \in S^*} d(z, Tz) \doteq r. \end{aligned}$$

So, $T(S^*) \subseteq B_r(Tx_0)$. Since, $S^* \cap B_r(Tx_0)$ is T-invariant log-star shaped, by minimality of S^* it follows that $S^* \subseteq B_r(Tx_0)$. Hence,

$$\sup_{z \in S^*} d(z, Tx_0) \leq \sup_{z \in S^*} d(z, Tz).$$

Let $S' = \{y \in S^* : \sup_{z \in S^*} d(z, y) \leq r^2\}$. Now, for $z \in S^*$,

$$d(x_0, z) \leq d(x_0, Tx_0) \times d(Tx_0, z) \leq r^2.$$

This implies that $x_0 \in S'$.
For $y \in S'$ we have

$$d(Ty, z) \leq d(Ty, Tx_0) \times d(Tx_0, z) \leq (\sup_{z \in S^*} d(z, Tz))^2,$$

and so, S' is T-invariant.
For $x \in S'$, $y \in S^*$ and $\lambda \in [0,1]$ we have

$$d(W(x, x_0, \lambda), y) \leq d^{\lambda}(x, y) d^{1-\lambda}(x_0, z) \leq r^2.$$

therefore, $\sup_{y \in X^*} d(W(x, x_0, \lambda) \leq r^2$. It follows that $W(x, x_0, \lambda) \in S'$ for all $x \in S'$ and $\lambda \in [0,1]$. Hence, S' is log-star shaped.
Suppose that $y \in \overline{S'}$, the closure of S'. Then, there is a sequence $\{y_n\}$ converging to y and

$$\sup_{z \in S^*} d(y_n, z) \leq (\sup_{z \in S^*} d(z, Tz))^2.$$

By tending n to infinity, we have $\sup_{z \in S^*} d(y, z) \leq r^2$, and so $y \in S'$. On the other hand,

$$\delta(S') \leq r^2 < \delta(S^*).$$

Therefore, S' is a proper closed T–invariant log-star shaped subset of S^*, a contradition. Verification the uniqueness of the fixed point is easy. □

Lemma 3.2. *Let M be nonempty compact subset of X and let K be the least closed log-convex set containing M. If the diameter $\delta(M)$ is positive, then there exists an element $u \in K$ such that* $\sup\{d(x,u) : x \in M\} < \delta(M)$.

Theorem 3.3. *Let K be a compact log-convex metric space. If Υ is a family of nonexpansive mappings with invariant property in K, then the family Υ has a common fixed point.*

Proof . Setting Φ as family of nonempty convex compact subset of K such that are an invariant under each $T \in \Upsilon$. It is follow from Zorns lemma, Φ has a minimal member, that is A. If A is a singleton subset of K, then theorem is proved. Otherwise, there exists a compact subset M of A such that $M = T(M) = \{T(x) : x \in M\}$ for each $T \in \Upsilon$. If M contains more than one point, by lemma 3.2 there exists an element u in the least convex set M such that

$$\alpha = \sup\{d(u,x) : x \in M\} < \delta(M)$$

where $\delta(M)$ is diameter of M. We setting

$$A_0 = \bigcap_{x\in M} \{y \in A : d(x,y) \leq \alpha\},$$

then A_0 is the nonempty closed convex proper subset of A invariant under each $T \in \Upsilon$. This is a contradiction to the minimality of A. □

Theorem 3.4. *Let X be log-convex metric space X having property (LC) and S be a non-empty bounded closed and log-convex subset of X. If $T : S \to S$ is a continuous selfmapping which has properties (G) and (B) then, T has a unique fixed point in S.*

Proof . Let Γ be the family of all bounded closed and log-convex subsets of S, mapped into itself by T. Also, assume that M be the minimal element of Γ with respect to being nonempty bounded closed and log-convex and invariant under T. If $\delta(M) = 0$, each member of M is a fixed point. If $\delta(M) > 0$, by property (B), there is $x \in M$ such that

$$d(x,Tx) < \sup_{y\in M} d(y,Ty) \doteq r.$$

Let $N = \{x \in M : d(x.Tx) \leq r\}$. If $x \in N$, then

$$d(Tx,T^2x) \leq d^{\lambda_0}(x,Tx)d^{1-\lambda_0}(Tx,T^2x) \leq r.$$

this implies that $Tx \in N$ for all $x \in N$. So, $TN \subseteq N$. Let $\overline{Lco}(T(N)$ be the closed log-convex hull of $T(N)$. If $z \in \overline{Lco}(T(N)$, then one of the following cases may arise:
(i) $z \in T(N)$ and since $T(N) \subseteq N$, hence, $Tz \in T(N) \subseteq \overline{Lco}(T(N)$
(ii) $z \in Lco(T(N)) = \bigcup_{n=1}^{\infty} An$ where, $A_n = W^n(A)$. it follows that there exists n so that, $z \in A_n$. Assume that $n = 1$, then there are x_1 and y_1 in A and $\lambda_1 \in [0,1]$ such that $z = W(x_1,y_1,\lambda_1)$. hence, $d(z,Tz) \leq d^{\lambda_1}(x_1,Tz)d^{1-\lambda_1}(y_1,Tz)$. Note that $d(x_1,Tz) \leq d^{\lambda_0}(x_1,Tx_1)d^{1-\lambda_0}(Tx_1,Tz) \leq r^{\lambda_0}\Big(d^{\lambda_0}(x_1,Tx_1)dd^{1-\lambda_0}(z,Tz)\Big)^{1-\lambda_0} \leq r$. By applying the priciple of mathematical induction, for the case $z \in A_n$ one may prove that $d(z,Tz) \leq r$. This implies that $z \in N$ and so, $Tz \in T(N) \subseteq \overline{Lco}(T(N)$.
(iii) $z \in \overline{Lco}(T(N)) - T(N)$, then there is a sequence $\{z_n\}$ in $Lco(T(N))$ such that $z_n \to z$. By continuity of T, the sequence $\{T(z_n)\}$ tends to $T(z)$ and

$$d(z,T(z)) = \lim_{n\to\infty} d(z_n,T(z_n)) \leq r.$$

Hence, $z \in N$ and $T(z) \in T(N) \subseteq \overline{Lco}(T(N)$. Thus, $\overline{Lco}(T(N)$ is a closed and log-convex subset of M which is invariant under T and $d(z,T(z)) \leq r$ for all $z \in \overline{Lco}(T(N)$. this implies that $\overline{Lco}(T(N)$ is a proper subset of M, a contradiction. □

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