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On the s^{th} derivative of a polynomial

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Abstract

For every $1 \leq s < n$, the s^{th} derivative of a polynomial $P(z)$ of degree n is a polynomial $P^{(s)}(z)$ whose degree is $(n-s)$. This paper presents a result which gives generalizations of some inequalities regarding the s^{th} derivative of a polynomial having zeros outside a circle. Besides, our result gives interesting refinements of some well-known results.

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1. Introduction and preliminaries

If $P(z)$ is a polynomial of degree n, then concerning the estimate of $|P'(z)|$ on the unit disk $|z| = 1$, we have

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

The above inequality is an immediate consequence of Bernstein's inequality [3] on the derivative of a Trigonometric polynomial and is best possible with equality holding for the polynomial $P(z) = \lambda z^n, \lambda$ being a complex number.

If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then the above inequality can be sharpened. In fact, Erdös conjectured and later Lax [8] proved that if $P(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

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The above inequality is best possible and equality holds for all polynomials having their zeros on $|z| = 1$.

As an extension of (1.2), Malik [9] proved that if $P(z) \neq 0$ in $|z| < k, k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{1.3}$$

Govil and Rahman [6] extended inequality (1.3) to the s^{th} derivative of a polynomial and proved under the same hypothesis for $1 \leq s < n$, that

$$\max_{|z|=1} \left|P^{(s)}(z)\right| \leq \frac{n(n-1)\cdots(n-s+1)}{1+k^s} \max_{|z|=1} |P(z)|. \tag{1.4}$$

Inequality (1.4) was further refined by Govil [5] who under the same hypothesis proved for $1 \leq s < n$, that

$$\max_{|z|=1} \left|P^{(s)}(z)\right| \leq \frac{n(n-1)\cdots(n-s+1)}{1+k^s} \left(\max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)| \right). \tag{1.5}$$

Inequality (1.4) was also refined by Aziz and Rather [2] by involving the binomial coefficient and coefficients of the polynomial $P(z)$. In fact, they proved that, if $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \neq 0$ in $|z| < k,\ k \geq 1$, then for $1 \leq s < n$,

$$\max_{|z|=1} \left|P^{(s)}(z)\right| \leq \frac{n(n-1)\cdots(n-s+1)}{1+\delta_{k,s}} \max_{|z|=1} |P(z)|, \tag{1.6}$$

where

$$\delta_{k,s} = k^{s+1} \left\{ \frac{1 + \frac{1}{C(n,s)} \left|\frac{a_s}{a_0}\right| k^{s-1}}{1 + \frac{1}{C(n,s)} \left|\frac{a_s}{a_0}\right| k^{s+1}} \right\}. \tag{1.7}$$

In this paper, we shall prove the following result which refines the inequalities (1.5) and (1.6). Besides this, many other results can be also easily deduced.

Theorem 1.1. If $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n which does not vanish in $|z| < k\ k \geq 1$, then for $1 \leq s < n$,

$$\max_{|z|=1} \left|P^{(s)}(z)\right| \leq \frac{n(n-1)\cdots(n-s+1)}{\phi_{k,s}+1} \left(\max_{|z|=1} |P(z)| - m \right), \tag{1.8}$$

where

$$\phi_{k,s} = k^{s+1} \left\{ \frac{1 + \frac{1}{C(n,s)} \frac{|a_s|}{|a_0|-m} k^{s-1}}{1 + \frac{1}{C(n,s)} \frac{|a_s|}{|a_0|-m} k^{s+1}} \right\}, \quad m = \min_{|z|=k} |P(z)|. \tag{1.9}$$

Remark 1.2. For $m = 0$, inequality (1.8) reduces to (1.6). Also, for $s = 1$ and $m = 0$, inequality (1.8) reduces to a result of Govil et.al. [7] and for $k = s = 1$, (1.8) gives a result of Aziz and Dawood [1]. In general, Theorem 1.1 sharpens results of Malik [9], Govil and Rahman [6], Govil [5] and Aziz and Rather [2].

We need the following lemmas for the proof of Theorem 1.1.

Lemma 1.3. If $P(z)=\sum\limits_{v=0}^{n} a_v z^v$ is a polynomial of degree n which does not vanish in $|z|<k, k\geq 1$, then for $1\leq s<n$ and $|z|=1$,

$$\delta_{k,s}\left|P^{(s)}(z)\right|\leq\left|Q^{(s)}(z)\right| \tag{1.10}$$

and

$$\frac{1}{C(n,s)}\left|\frac{a_s}{a_0}\right|k^s\leq 1, \tag{1.11}$$

where here and throughout this paper $Q(z)=z^n\overline{P(\frac{1}{\bar{z}})}$ and $\delta_{k,s}$ is defined by (1.7).

The above lemma is due to Aziz and Rather [2].

Lemma 1.4. If $P(z)=\sum\limits_{v=0}^{n} a_v z^v$ is a polynomial of degree n with $P(z)\neq 0$ for $|z|<k, k\geq 1$, then $|P(z)|>m$ for $|z|<k$, and in particular $|a_0|>m$, where $m=\min_{|z|=k}|P(z)|$. The above lemma is due to Gardner, Govil and Musukula [4].

Lemma 1.5. *The function*

$$T(x)=k^{s+1}\left\{\frac{1+\frac{1}{C(n,s)}\left(\frac{|a_s|}{x}\right)k^{s-1}}{1+\frac{1}{C(n,s)}\left(\frac{|a_s|}{x}\right)k^{s+1}}\right\}$$

is an increasing function of x.

Proof . The proof follows by considering the first derivative test of $T(x)$. □

The following two lemmas are due to Govil [5].

Lemma 1.6. If $P(z)$ is a polynomial of degree n having no zeros in $|z|<k, k\geq 1$, then for $|z|\geq\frac{1}{k}$,

$$\left|Q^{(s)}(z)\right|\geq mn(n-1)\ldots(n-s+1)|z|^{n-s}, \tag{1.12}$$

where $m=\min_{|z|=k}|P(z)|$.

Lemma 1.7. If $P(z)$ is a polynomial of degree n such that $P(z)\neq 0$ in $|z|<k,\ k\geq 1$, then

$$k^s\left|P^{(s)}(z)\right|\leq\left|Q^{(s)}(z)\right| \quad for\ |z|=1. \tag{1.13}$$

2. Proof of Theorem

Proof of Theorem 1.1**.** Since $P(z)$ has all its zeros in $|z|\geq k\geq 1$ and $m=\min_{|z|=k}|P(z)|$, therefore,

$$m\leq|P(z)|\ for\ |z|=k.$$

Hence it follows by Rouche's theorem that for $m > 0$ and for every real or complex number λ with $|\lambda| < 1$, the polynomial $P(z) - \lambda m$ does not vanish in $|z| < k, k \geq 1$. Applying inequality (1.10) of Lemma 1.3 to the polynomial $P(z) - \lambda m$, we get for $|z| = 1$ that

$$k^{s+1}\left\{\frac{1+\frac{1}{C(n,s)}\left(\frac{|a_s|}{|a_0-\lambda m|}\right)k^{s-1}}{1+\frac{1}{C(n,s)}\left(\frac{|a_s|}{|a_0-\lambda m|}\right)k^{s+1}}\right\}\left|P^{(s)}(z)\right| \leq \left|Q^{(s)}(z) - \overline{\lambda} mn(n-1)\ldots(n-s+1)z^{n-s}\right|. \tag{2.1}$$

Since for every λ with $|\lambda| < 1$, we have

$$|a_0 - \lambda m| \geq |a_0| - |\lambda| m \geq |a_0| - m, \tag{2.2}$$

and $|a_0| > m$ by Lemma 1.4, we get on combining (2.1), (2.2) and Lemma 1.5 that for every λ with $|\lambda| < 1$,

$$k^{s+1}\left\{\frac{1+\frac{1}{C(n,s)}\left(\frac{|a_s|}{|a_0|-m}\right)k^{s-1}}{1+\frac{1}{C(n,s)}\left(\frac{|a_s|}{|a_0|-m}\right)k^{s+1}}\right\}\left|P^{(s)}(z)\right| \leq \left|Q^{(s)}(z) - \overline{\lambda} mn(n-1)\ldots(n-s+1)z^{n-s}\right|, \quad for \quad |z| = 1. \tag{2.3}$$

Now choosing the argument of λ on the right hand side of (2.3) so that on $|z| = 1$,

$$\left|Q^{(s)}(z) - \overline{\lambda} mn(n-1)\ldots(n-s+1)z^{n-s}\right| = \left|Q^{(s)}(z)\right| - |\lambda| mn(n-1)\ldots(n-s+1),$$

which is possible by inequality (1.12) of Lemma 1.6. Hence we conclude from (2.3) that on $|z| = 1$,

$$k^{s+1}\left\{\frac{1+\frac{1}{C(n,s)}\left(\frac{|a_s|}{|a_0|-m}\right)k^{s-1}}{1+\frac{1}{C(n,s)}\left(\frac{|a_s|}{|a_0|-m}\right)k^{s+1}}\right\}\left|P^{(s)}(z)\right| \leq \left|Q^{(s)}(z)\right| - |\lambda| mn(n-1)\ldots(n-s+1). \tag{2.4}$$

Letting $|\lambda| \to 1$ in (2.4), we obtain

$$\phi_{k,s}\left|P^{(s)}(z)\right| \leq \left|Q^{(s)}(z)\right| - mn(n-1)\ldots(n-s+1). \tag{2.5}$$

Now, if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then $g(z) = z^n\overline{p(\frac{1}{\overline{z}})}$ has no zero in $|z| < 1$. Hence by inequality (2.4) of Lemma 1.7 with $k = 1$, we have for $|z| = 1$,

$$\left|g^{(s)}(z)\right| \leq \left|p^{(s)}(z)\right|. \tag{2.6}$$

Let $M = \max_{|z|=1}|P(z)|$, then for every γ with $|\gamma| > 1$, it follows by Rouche's theorem that the polynomial $T(z) = P(z) - \gamma M z^n$ has all zeros in $|z| < 1$. Taking $S(z) = z^n\overline{T(\frac{1}{\overline{z}})} = Q(z) - \overline{\gamma} M$ and apply inequality (2.6) to $T(z)$, we get for $1 \leq s < n$ and for $|z| = 1$,

$$\left|S^{(s)}(z)\right| \leq \left|T^{(s)}(z)\right|,$$

which implies

$$\left|Q^{(s)}(z)\right| \leq \left|P^{(s)}(z) - \gamma Mn(n-1)\cdots(n-s+1)z^{n-s}\right| \quad for\ |z| = 1. \tag{2.7}$$

Since $P(z)$ is of degree n, it follows for every $1 \leq s < n$ that the polynomial $P^{(s)}(z)$ is of degree $(n-s)$. By the repeated application of inequality (1.1), we obtain for $|z| = 1$,

$$\left|P^{(s)}(z)\right| \leq n(n-1)\cdots(n-s+1)M. \tag{2.8}$$

Choose argument of γ suitably and note inequality (2.8), we obtain from inequality (2.7) for $|z| = 1$,

$$\left|Q^{(s)}(z)\right| \leq Mn(n-1)\cdots(n-s+1) - \left|P^{(s)}(z)\right|.$$

That is, for $|z| = 1$

$$\left|P^{(s)}(z)\right| + \left|Q^{(s)}(z)\right| \leq Mn(n-1)\cdots(n-s+1). \tag{2.9}$$

Combining inequalities (2.5) and (2.9), we have for $|z| = 1$,

$$\begin{aligned}\left(1+\phi_{k,s}\right)\left|P^{(s)}(z)\right| &\leq \left|P^{(s)}(z)\right| + \left|Q^{(s)}(z)\right| - mn(n-1)\ldots(n-s+1)\\ &\leq Mn(n-1)\cdots(n-s+1) - mn(n-1)\ldots(n-s+1)\\ &= n(n-1)\ldots(n-s+1)(M-m),\end{aligned}$$

which is equivalent to the desired result.

Remark 2.1. As is seen in the proof of Theorem 1.1 that the polynomial $P(z) - \lambda m$ doses not vanish in $|z| < k,\ k \geq 1$ for every λ with $|\lambda| < 1$, it follows by applying inequality (1.11) of Lemma 1.3 to $P(z) - \lambda m$, that

$$C(n,s)|a_0 - \lambda m| \geq |a_s|k^s.$$

Choosing argument of λ suitably and noting Lemma 1.4, we get

$$C(n,s)\big(|a_0| - |\lambda|m\big) \geq |a_s|k^s.$$

Letting $|\lambda| \to 1$, we get

$$C(n,s)(|a_0| - m) \geq |a_s|k^s,$$

which in turn implies $\phi_{k,s} \geq k^s$ for $1 \leq s < n$. From this, it follows that inequality (1.8) is a refinement of inequality (1.5).

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