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Global existence, stability results and compact invariant sets for a quasilinear nonlocal wave equation on $\mathbb{R}^N$

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Abstract

We discuss the asymptotic behaviour of solutions for the nonlocal quasilinear hyperbolic problem of Kirchhoff Type

$$u_{tt} - \phi(x)||\nabla u(t)||^2 \Delta u + \delta u_t = |u|^a u,\ x \in \mathbb{R}^N,\ t \geq 0\ ,$$

with initial conditions $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$, in the case where $N \geq 3$, $\delta \geq 0$ and $(\phi(x))^{-1} = g(x)$ is a positive function lying in $L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. It is proved that, when the initial energy $E(u_0, u_1)$, which corresponds to the problem, is non-negative and small, there exists a unique global solution in time in the space $\mathcal{X}_0 =: D(A) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$. When the initial energy $E(u_0, u_1)$ is negative, the solution blows-up in finite time. For the proofs, a combination of the modified potential well method and the concavity method is used. Also, the existence of an absorbing set in the space $\mathcal{X}_1 =: \mathcal{D}^{1,2}(\mathbb{R}^N) \times L^2_g(\mathbb{R}^N)$ is proved and that the dynamical system generated by the problem possess an invariant compact set $\mathcal{A}$ in the same space.

Finally, for the generalized dissipative Kirchhoff's String problem

$$u_{tt} = -||A^{1/2}u||^2_H Au - \delta A u_t + f(u),\ \ x \in \mathbb{R}^N,\ \ t \geq 0\ ,$$

with the same hypotheses as above, we study the stability of the trivial solution $u \equiv 0$. It is proved that if $f'(0) > 0$, then the solution is unstable for the initial Kirchhoff's system, while if

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$f'(0) < 0$ the solution is asymptotically stable. In the critical case, where $f'(0) = 0$, the stability is studied by means of the central manifold theory. To do this study we go through a transformation of variables similar to the one introduced by R. Pego.

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1. Introduction and preliminaries

We study the following quasilinear hyperbolic initial value problem

$$u_{tt} - \phi(x)||\nabla u(t)||^2 \Delta u + \delta u_t - |u|^a u = 0, \quad x \in \mathbb{R}^N, \quad t \geq 0\,, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

with initial conditions u_0, u_1 in appropriate function spaces, $N \geq 3$, and $\delta \geq 0$. Throughout the paper we assume that the functions $\phi, g : \mathbb{R}^N \to \mathbb{R}$ satisfy the following condition:

$(\mathcal{G})$ $\phi(x) > 0$, for all $x \in \mathbb{R}^N$ and $(\phi(x))^{-1} =: g(x) \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

This class will include functions of the form $\phi(x) \sim c_0 + \varepsilon |x|^\alpha$, for some $\varepsilon > 0$ and $\alpha > 0$, resembling phenomena of *slowly varying wave speed* around the speed c_0.

G.Kirchhoff in 1883 proposed the so called *Kirchhoff string* model in the study of oscillations of stretched strings and plates

$$ph\frac{\vartheta^2 u}{\vartheta t} = \left\{ p_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\vartheta u}{\vartheta x}\right)^2 dx \right\} \frac{\vartheta^2 u}{\vartheta x^2} + f, \quad 0 < x < L, \quad t \geq 0\,,$$

where $u = u(x,t)$ is the lateral displacement at the space coordinate x and the time t, E the Young modules, p the mass density, h the cross-section area, L the length, p_0 the initial axial tension and f the external force (see [10]). When $p_0 = 0$ the equation is considered to be of *degenerate type*, otherwise it is of *nondegenerate type*.

In the case of *bounded domain*, T. Kobayashi [11] constructed a unique weak solution by a Faedo-Galerkin method for a quasilinear wave equation with strong dissipation (see also [1, 13]). K. Nishihara [14] has derived a decay estimate from below of the potential of solutions. Also R. Ikehata [4] has shown that for sufficiently small initial data, global existence can be obtained, even when the influence of the source terms is stronger than that of the damping terms. Finally K. Ono [15] for $\delta \geq 0$ has proved global existence and blow up results for a degenerate non-linear wave equation of Kirchhoff type with strong dissipation.

In the case of *unbounded domain*, P. D'Ancona and S. Spagnolo [2] have shown the global existence of a unique C^∞ solution for the non-degenerate type with small C_0^∞ data. N. Karahalios and N. Stavrakakis [5]–[9], have proved global existence and blow-up results for some semilinear wave equations with variable wave speed on all $\mathbb{R}^N$. T Mizumachi (see [12]) studied the asymptotic behavior of solutions to the Kirchhoff equation with a viscous damping term with no external force. In our previous work (see [18]), we prove global existence and blow-up results of an equation of Kirchhoff type in all of $\mathbb{R}^N$. Finally, in [19] we study the stability of the trivial solution $u = 0$ for the generalized Kirchhoff's string equation, using the central manifold theory.

As we will see the space setting for the initial conditions and the solutions of our problem is the product space $\mathcal{X}_0 =: D(A) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$. By $\mathcal{D}^{1,2}(\mathbb{R}^N)$ we define the closure of the $C_0^\infty(\mathbb{R}^N)$ functions with respect to the energy norm $||u||_{\mathcal{D}^{1,2}} =: \int_{\mathbb{R}^N} |\nabla u|^2 dx$. It is known that $\mathcal{D}^{1,2}(\mathbb{R}^N) = \left\{u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N) : \nabla u \in (L^2(\mathbb{R}^N))^N\right\}$. The weighted Lebesque space $L_g^2(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ functions with respect to the inner product $(u,v)_{L_g^2} =: \int_{\mathbb{R}^N} g\ u\ v\ dx$ (see [3]). We also have that the operator $A = -\phi\Delta$ is self-adjoint and therefore graph-closed. Its domain $D(A)$, is a Hilbert space with respect to the norm

$$||Au||_{L_g^2} = \left\{ \int_{\mathbb{R}^N} \phi|\Delta u|^2 dx \right\}^{1/2} . \tag{1.3}$$

So, we construct the following *evolution quartet*, with compact and dense embeddings:

$$D(A) \subset \mathcal{D}^{1,2}(\mathbb{R}^N) \subset L_g^2(\mathbb{R}^N) \subset \mathcal{D}^{-1,2}(\mathbb{R}^N) .$$

For the positive selfadjoint operator $A = -\varphi\Delta$, we may define the fractional powers in the following way. For every $s > 0$, A^s is an unbounded selfadjoint operator in $L_g^2(\mathbb{R}^N)$ with its domain $D(A^s)$ to be a dense subset in $L_g^2(\mathbb{R}^N)$. The operator A^s is strictly positive and injective. Also $D(A^s)$, endowned with the scalar product

$$(u,v)_{D(A^s)} = (u,v)_{L_g^2} + (A^s u, A^s v)_{L_g^2} ,$$

becomes a Hilbert space. We write as usual $V_{2s} = D(A^s)$ and we have the following identifications

$$D(A^{-1/2}) = \mathcal{D}^{-1,2}(\mathbb{R}^N), \quad D(A^0) = L_g^2, \quad D(A^{1/2}) = \mathcal{D}^{1,2}(\mathbb{R}^N) .$$

Moreover, the mapping

$$A^{s/2} : V_x \longrightarrow V_{x-s}$$

is an isomorphism. Furthermore, we have that the injection $D(A^{s_1}) \subset D(A^{s_2})$ is compact and dense, for every $s_1, s_2 \in R,\ \ s_1 > s_2$.

In order to clarify the kind of solutions we are going to obtain for our problem, we give the definition of the *weak solution* for the problem.

Definition 1.1. *A* weak solution *of the problem is a function u such that*
(i) $u \in L^2[0,T; D(A)],\ \ u_t \in L^2[0,T; \mathcal{D}^{1,2}(\mathbb{R}^N)],\ \ u_{tt} \in L^2[0,T; L_g^2(\mathbb{R}^N)]$,
(ii) for all $v \in C_0^\infty([0,T] \times (\mathbb{R}^N))$, satisfies the generalized formula

$$\int_0^T (u_{tt}(\tau), v(\tau))_{L_g^2}\, d\tau \;+\; \int_0^T \left(||\nabla u(t)||^2 \int_{\mathbb{R}^N} \nabla u(\tau)\nabla v(\tau) dx d\tau \right)$$
$$+\delta \int_0^T (u_t(\tau), v(\tau))_{L_g^2}\, d\tau \;-\; \int_0^T (f(u(\tau)), v(\tau))_{L_g^2}\, d\tau = 0 , \tag{1.4}$$

where $f(s) = |s|^a s$, and

(iii) satisfies the initial conditions

$$u(x,0) = u_0(x) \in D(A) ,\ \ u_t(x,0) = u_1(x) \in \mathcal{D}^{1,2}(\mathbb{R}^N) .$$

In the following section we briefly discuss the results concerning the asymptotic behaviour of solutions for the problem (1.1)-(1.2). Among the global existence and blow-up results we also prove existence of a compact functional invariant set. We would like to mention that up to our knowledge, this is the first result concerning existence of functional invariant sets for mathematical models of Kirchoff's strings type.

2. Global Existence, Blow-up Results and Invariant Sets

In this section we give global existence and blow-up results for the problem (1.1)-(1.2) in the space $\mathcal{X}_0$. We also prove existence of an *attractor like* set. For the proofs we refer on [18].

In order to obtain a local existence result for the problem (1.1)-(1.2), we need information concerning the solvability of the corresponding nonhomogeneous linearized problem around the function v, where $(v, v_t) \in C\left(0, T; D(A) \times \mathcal{D}^{1,2}\right)$ is given, restricted in the sphere B_R.

$$\begin{array}{ll} u_{tt} - \phi(x)||\nabla v(t)||^2 \Delta u + \delta u_t = |v|^a v, & (x,t) \in B_R \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in B_R, \\ u(x,t) = 0, & (x,t) \in \partial B_R \times (0,T), \end{array} \tag{2.1}$$

where $v \in C\left(0, T; D(A)\right), \quad v_t \in C\left(0, T; \mathcal{D}^{1,2}\right)$.

Proposition 2.1. *Assume that* $u_0 \in D(A), \quad u_1 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ *and* $0 \le a \le 4/(N-2)$, *then the linear wave Eq.* (2.1) *has a unique solution such that*

$$u \in C\left(0, T; D(A)\right) \quad and \quad u_t \in C\left(0, T; \mathcal{D}^{1,2}\right).$$

Proof . The proof follows the lines of [6, Proposition 3.1]. The Galerkin method is used, based on the information taken from the associated eigenvalue problem.
Next, we have the following theorem (for the proof we refer to [18]). □

Theorem 2.2. If $(u_0, u_1) \in D(A) \times \mathcal{D}^{1,2}$ and satisfy the nondegenerate condition

$$||\nabla u_0|| > 0 \ ,$$

then there exists $T > 0$, such that the problem (1.1)-(1.2) admits a unique local weak solution u satisfying

$$u \in C(0, T; D(A)), \quad u_t \in C(0, T; \mathcal{D}^{1,2}) \ .$$

Moreover, at least one of the following statements holds true, either

(i) $T = +\infty$, or

(ii) $e(u(t)) =: ||u_t||^2_{\mathcal{D}^{1,2}} + ||u||^2_{D(A)} \to \infty$, as $t \to T_-$.

The next theorem deals with the global existence, blow-up results and the energy decay property of the problem. The proofs of the results are in [18].
First we define as the energy of the problem (1.1)-(1.2) the quantity

$$E(t) =: E(u(t),\ u_t(t)) =: ||u_t(t)||^2_{L^2_g} + \frac{1}{2}||u(t)||^4_{\mathcal{D}^{1,2}} - \frac{2}{a+2}||u(t)||^{a+2}_{L^{a+2}_g} . \tag{2.2}$$

Also we introduce the potential of the problem (1.1)-(1.2), as

$$\mathcal{J}(u) =: \frac{1}{2}||u(t)||^4_{\mathcal{D}^{1,2}} - \frac{2}{a+2}||u(t)||^{a+2}_{L^{a+2}_g} . \tag{2.3}$$

So, we get the following relation

$$E(t) = ||u_t(t)||^2_{L^2_g} + \mathcal{J}(u) . \tag{2.4}$$

Finally, we introduce a modified version of the modified potential well used in [6] (see also [13]), by

$$\mathcal{W} =: \left\{ u \in D(A);\ \mathcal{K}(u) = ||u(t)||^4_{\mathcal{D}^{1,2}} - ||u(t)||^{a+2}_{L^{a+2}_g} > 0 \right\} \cup \{0\} . \tag{2.5}$$

Theorem 2.3. Assume that $N = 3$, $8/3 < a < 4$, $u_0 \in \mathcal{W}\,(\subset D(A))$ and $u_1 \in \mathcal{D}^{1,2}$. Also suppose that the following inequality holds

$$E(u_0, u_1) \leq \left(\frac{1}{C_0 \mu_0^{p_1}}\right)^{1/p_2}, \quad \text{if} \quad \frac{8}{3} < a < 4 \quad \text{and} \quad p_2 > 0 . \tag{2.6}$$

Then $a)$ for $p_1 =: \frac{2(a+2)-3a}{2}$ and $p_2 =: \frac{3a-8}{8}$ there exists a unique global solution $u \in \mathcal{W}$ of the problem (1.1)-(1.2) satisfying

$$u \in C\left([0,+\infty);\ D(A)\right) \quad \text{and} \quad u_t \in C\left([0,+\infty);\ \mathcal{D}^{1,2}(\mathbb{R}^n)\right) .$$

$b)$ Moreover, this solution obeys the following energy estimates

$$||u_t||^2_{L^2_g} + d_*^{-1}||\nabla u||^4 \leq E(u,\ u_t) \leq \left\{E(u_0, u_1)^{-1/2} + d_0^{-1}[t-1]^+\right\}^{-2}, \tag{2.7}$$

where $d_* = \frac{2(a+2)}{a-2}$ and $d_0 \geq 1$, that is,

$$||\nabla u||^4 \leq C_*(1+t)^{-1}, \tag{2.8}$$

where C_* is some constant depending on $||u_0||^4_{\mathcal{D}^{1,2}}$ and $||u_1||_{L^2_g}$.
$c)$ Suppose that $a \geq 2$, $N \geq 3$ and the initial energy $E(u_0, u_1)$ is negative. Then there exists a time T, where

$$\begin{aligned} 0 < T \leq \quad & a^{-2}\left(-E(u_0,\ u_1)\right)^{-1}\Big[\Big\{(2\delta||u_0||^2_{L^2_g} - a(u_0,\ u_1)_{L^2_g})^2 \\ & + a^2\left(-E(u_0,\ u_1)\right)||u_0||^2_{L^2_g}\Big\}^{1/2} + 2\delta||u_0||^2_{L^2_g} - a(u_0,\ u_1)_{L^2_g}\Big], \end{aligned} \tag{2.9}$$

such that the (unique) solution of the problem (1.1)-(1.2) blows-up at T, i.e.,

$$\lim_{t\to T-} ||u(t)||^2_{L^2_g} = +\infty . \tag{2.10}$$

The existence of an absorbing set in $\mathcal{X}_0$ is given below.

Lemma 2.4. *Assume that* $\rho_1 > 4\alpha^{-1/2}R^2c_3^2, \ \ 0 \leq a < 2/(N-2), \ \ N \geq 3$ *and* $||\nabla u_0|| > 0$. *Then the unique local solution defined by Proposition* 2.1 *exists globally in time.*

Proof . Given the constants $T > 0, \ R > 0$, we introduce the two parameter space of solutions

$$\begin{aligned} X_{T,R} \ &=: \ \left\{ w \in C\left(0,T; D(A)\right) : w_t \in C\left(0,T; \mathcal{D}^{1,2}\right), w(0) = u_0, \right. \\ & \qquad \left. w_t(0) = u_1, \ \ e(w) \leq R^2, \ \ t \in [0,T] \right\}, \end{aligned}$$

where $e(w) =: ||w_t||^2_{\mathcal{D}^{1,2}} + ||w||^2_{D(A)}$. Also u_0 satisfies the nondegenerate condition

$$||\nabla u_0||^2 > 0 \ . \tag{2.11}$$

It is easy to see that the set $X_{T,R}$ is a complete metric space under the distance $d(u, \ v) =: \sup_{0\leq t\leq T} e(u(t) - v(t))$. We may introduce the notation

$$M_0 =: \frac{1}{2}||\nabla u_0||^2, \ T_0 =: \sup\left\{t \in [0, \ \infty) : \ ||\nabla w(s)||^2 > M_0, \ 0 \leq s \leq t\right\}.$$

By condition Eq. (2.11), we may see that $M_0 > 0$, $T_0 > 0$ and $||\nabla w(t)||^2 > M_0$, for all $t \in [0, T_0]$. Multiplying Eq. (2.1) by

$$gAv = g(-\varphi\Delta)v = -\Delta v = -\Delta(u_t + \varepsilon u)$$

where $v = u_t + \varepsilon u$ and integrating over $\mathbb{R}^n$, we obtain the following inequality

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\{||w||^2_{\mathcal{D}^{1,2}}||u||^2_{D(A)}\right. \ &+ \ \left. ||v||^2_{\mathcal{D}^{1,2}} + \frac{\varepsilon(\delta-\varepsilon)}{2}||u||^2_{\mathcal{D}^{1,2}} \ \right\} \\ &+ \ (\delta-\varepsilon)||v||^2_{\mathcal{D}^{1,2}} + \varepsilon||w||^2_{\mathcal{D}^{1,2}}||u||^2_{D(A)} \\ &+ \ \varepsilon^2(\delta-\varepsilon)||u||^2_{\mathcal{D}^{1,2}} \\ \leq & \quad \left|\left(\frac{d}{dt}||w||^2_{\mathcal{D}^{1,2}}\right)||u||^2_{D(A)}\right| \\ &+ \ k_2||w||^a_{L^{Na}}||\nabla w||_{L^{\frac{2N}{N-2}}}||\nabla v|| \ . \end{aligned} \tag{2.12}$$

We observe that

$$\begin{aligned} \theta(t) \ &=: \ ||w||^2_{\mathcal{D}^{1,2}}||u||^2_{D(A)} + ||v||^2_{\mathcal{D}^{1,2}} + \frac{\varepsilon(\delta-\varepsilon)}{2}||u||^2_{\mathcal{D}^{1,2}} \\ &\geq \ ||w||^2_{\mathcal{D}^{1,2}}||u||^2_{D(A)} + ||v||^2_{\mathcal{D}^{1,2}} \\ &\geq \ M_0||u||^2_{D(A)} + ||u_t||^2_{\mathcal{D}^{1,2}} \geq c_3^{-2}e(u) \ , \end{aligned} \tag{2.13}$$

with $c_3 =: \left(\max\left\{1, \ M_0^{-1}\right\}\right)^{1/2}$. We also have that

$$\begin{aligned} \left|\left(\frac{d}{dt}||w||^2_{\mathcal{D}^{1,2}}\right)||u||^2_{D(A)}\right| \ &= \ \left|\left(2\int_{\mathbb{R}^n}\Delta w w_t \varphi g dx\right)||u||^2_{D(A)}\right| \\ &\leq \ 2\left(||w||^2_{D(A)}\right)^{1/2}\left(||w_t||^2_{L^2_g}\right)^{1/2}||u||^2_{D(A)} \\ &\leq \ 2\alpha^{-1/2}R||w_t||_{\mathcal{D}^{1,2}}||u||^2_{D(A)} \\ &\leq \ 2\alpha^{-1/2}R^2e(u) \leq 2\alpha^{-1/2}R^2c_3^2 \ \theta(t) \ . \end{aligned} \tag{2.14}$$

By relations Eqs. (2.13)-(2.14) the inequality (2.12) becomes

$$\begin{aligned}\frac{d}{dt}\theta(t) &+ (\delta-\varepsilon)||v||^2_{\mathcal{D}^{1,2}} + \varepsilon||w||^2_{\mathcal{D}^{1,2}}||u||^2_{D(A)} + \frac{2\varepsilon\varepsilon(\delta-\varepsilon)}{2}||u||^2_{D(A)} \\ &\leq 2\alpha^{-1/2}R^2c_3^2\,\theta(t) + k_2||w||^a_{L^{Na}}||\nabla w||_{L^{\frac{2N}{N-2}}}||\nabla v||\,. \end{aligned} \tag{2.15}$$

We also have that

$$||w||^a_{L^{Na}} \leq R^a \quad \text{and} \quad ||\nabla w||_{L^{\frac{2N}{N-2}}} \leq ||w||_{D(A)} \leq R. \tag{2.16}$$

Applying Young's inequality for $\varepsilon = \delta/2$, in the last term of Eq. (2.15) we obtain

$$\frac{d}{dt}\theta(t) + \frac{\rho_1}{2}\theta(t) - 2\alpha^{-1/2}R^2c_3^2\,\theta(t) \leq \frac{C(R)}{\delta}\,, \tag{2.17}$$

where $\rho_1 = \min(\frac{\delta}{2}-\varepsilon,\ \varepsilon,\ 2\varepsilon)$ and $C(R) = k_2R^{2(a+1)}$. So

$$\frac{d}{dt}\theta(t) + C_*\theta(t) \leq \frac{C(R)}{\delta}\,, \tag{2.18}$$

where $C_* = \frac{1}{2}\left(\rho_1 - 4\alpha^{-1/2}R^2c_3^2\right) > 0$. Applying Gronwall's Lemma in Eq. (2.18) we get

$$\theta(t) \leq \theta(0)\,e^{-C_*t} + \frac{1-e^{-C_*t}}{C_*}\frac{C(R)}{\delta}\,. \tag{2.19}$$

By using the nondegenerate condition $||\nabla u_0||^2 > 0$ we may assume that $||\nabla w(s)|| > M_0, \quad 0 \leq s \leq t, \quad t \in [0,\infty)$, (see [16] and [17, Theorem 1.1]). Letting $t \to \infty$, in relation Eq. (2.19) we conclude that

$$\lim_{t\to\infty}\sup\theta(t) \leq \frac{C(R)}{\delta C_*} =: R_*^2\,. \tag{2.20}$$

From inequality (2.20) and following the arguments of Theorem 3.1 (see [18]) we conclude that the solution of Eq. (2.1) exists globally in time. □

Remark 2.5. (Global Solutions) From the last Lemma 2.4 we may observe that solutions of the problem (1.1)-(1.2), (given by Theorem 2.2), belong to the space $C_b(\mathbb{R}_+,\ \mathcal{X}_0)$, i.e., we have achieved global solutions for the given problem. Let us remark that, in the Theorem 2.3, using a modified potential well technique, we have proved global existence results under the conditions $N = 3,\ \ 8/3 < \alpha < 4$ and the initial energy $E(0)$ been non-negative and small. On the other hand, in Lemma 2.4, we could achieve global results for different type of nonlinearities, i.e., $\alpha \in (0, 2/(N-2))$, but for any $N \geq 3$ and independently of the sign of the initial energy $E(0)$.

Lemma 2.4 has an immediate consequence:

Remark 2.6. A nonlinear semigroup $S(t) : \mathcal{X}_0 \to \mathcal{X}_0, \quad t \geq 0$, may be associated to the problem (1.1)-(1.2) such that for $\psi = \{u_0, u_1\} \in \mathcal{X}_0, \quad S(t)\psi = \{u(t), u_t(t)\}$ is the weak solution of the problem (1.1)-(1.2). Moreover the ball $B_0 =: B_{\mathcal{X}_0}(0, \bar{R}_*)$ for any $\bar{R}_* > R_*$, where R_* is defined by Lemma 2.4, is an absorbing set for the semigroup $S(t)$ in the energy space $\mathcal{X}_0 \subset \mathcal{X}_1$, compactly.

In the rest of the paper we show that the ω-limit set of the absorbing set B_0 is a compact invariant set. To this end, we need to decompose the semigroup $S(t)$, in the form $S(t) = S_1(t) + S_2(t)$, where for a suitable bounded set $\mathcal{B} \subset \mathcal{X}_0$, the semigroups $S_1(t),\ S_2(t)$ satisfy the following properties:

($\mathcal{S}_1$) $S_1(t)$ is uniformly compact for t large, i.e., $\cup_{t\geq t_0} S_1(t)\mathcal{B}$ is relatively compact in $\mathcal{X}_1$.
($\mathcal{S}_2$) $\sup_{k\in\mathcal{B}} ||S_2(t)k||_{\mathcal{X}_1} \to 0, \quad \text{as} \quad t \to \infty.$
As a consequence of the above properties we have the following result.

Theorem 2.7. Let ϕ satisfy hypothesis $(\mathcal{G})$. Then the semigroup $S(t)$ associated with the problem (1.1)-(1.2) possesses a functional invariant set $\mathcal{A} = \omega(B_0)$, which is compact in the weak topology of X_1.

Remark 2.8. We have that $\mathcal{X}_0$ is compactly embedded in $\mathcal{X}_1$, so the set $\overline{\cup_{t\geq t_0} S_1(t)\mathcal{B}}$ is compact with respect to the strong topology in $\mathcal{X}_1$. For the functional invariant compact set $\mathcal{A} = \omega(B_0)$, we observe that $(u_0, u_1) \in \mathcal{A}$, if $|\nabla u_0| > 0$. So, $\mathcal{A}$ is an attractor like set.

Finally, in the following section we study the stability of the initial solution $u = 0$ for the generalized Kirchhoff equation.

3. Stability Results

We consider the generalized quasilinear dissipative Kirchhoff's String problem

$$u_{tt} = -||A^{1/2}u||_H^2 Au - \delta A u_t + f(u), \quad x \in \mathbb{R}^n, \quad t \geq 0\ ,$$

under the same initial conditions as above and H is a Hilbert space. First, we prove existence of solution for our problem, under small initial data (for the proof we refer to [19]).

Theorem 3.1. (Local Existence) Let $f(u)$ a C^1-function such that $|f(u)| \leq k_1|u|^{a+1}$, $|f'(u)| \leq k_2|u|^a$, $0 \leq a \leq 4/(N-2)$ and $N \geq 3$. Consider that $(u_0,\ u_1) \in D(A) \times V$ and satisfy the nondegenerate condition

$$||A^{1/2}u_0|| > 0\ . \tag{3.1}$$

Then there exists $T_0 > 0$ such that the problem (1.1)-(1.2) admits a unique local weak solution u satisfying

$$u \in C(0,T;D(A)) \ \ and \ \ u_t \in C(0,T;V)\ .$$

The linearized equation of the system around solution $u = 0$ is

$$\bar{u}_t + \hat{A}\bar{u} = 0\ , \tag{3.2}$$

where

$$\bar{u}_t = (w,\ v)^T \quad \text{and} \quad \hat{A} = \begin{bmatrix} \delta A & -f'(0) \\ -1 & 0 \end{bmatrix}. \tag{3.3}$$

So, in order to study the stability of the solution, we study the spectrum of the operator $\hat{A}$. The characteristic polynomial of $\hat{A}$ is

$$\begin{vmatrix} -\delta\lambda_j + \mu_j & f'(0) \\ 1 & \mu_j \end{vmatrix} = 0\ ,$$

or equivalently

$$\mu_j^2 - \delta\lambda_j\mu_j - f'(0) = 0 \ .$$

Let, $\Delta = \delta^2\lambda_j^2 + 4f'(0)$. Then according to the sign of $f'(0)$, we have the following cases:
I) Let $f'(0) > 0$, then we have that 0 is unstable for the initial Kirchhoff's system.
II) Let $f'(0) < 0$. This implies that the operator $\hat{A}$ admits two real eigenvalues which are both positive. Thus we obtain that the solution $u = 0$ is asymptotically stable for the initial Kirchhoff's system.
III) Let $f'(0) = 0$. In this case we use the central manifold theory in order to study the stability of the initial solution $u = 0$. Making use of the change of variables similar to what is found by Pego (see [21]), namely

$$\begin{cases} p(x,t) &= A^{-1/2}u_t, \\ q(x,t) &= -\delta A^{1/2}u - p \ , \end{cases} \tag{3.4}$$

we can rewrite Eqs. (3.2)-(3.3) in the form of a *reaction-diffusion system*:

$$\begin{cases} p_t(x,t) &= -\delta Ap + (\frac{1}{\delta^3}||p+q||_H^2)(p+q) + A^{-1/2}f(u), \\ q_t(x,t) &= -(\frac{1}{\delta^3}||p+q||_H^2)(p+q) - A^{-1/2}f(u), \\ p(x,t) &= 0, \qquad t > 0, \\ p(x,0) &= p_0(x), \qquad q(x,0) = q_0(x) \ , \end{cases} \tag{3.5}$$

where $p + q = -\delta A^{1/2}u$.

In order to prove the existence of a local central manifold we need the following result (for the proof see [21])

Proposition 3.2. For some neighborhood U of 0 in $X^{1/2} =: V \times H$, system (3.5) has a local center manifold defined by

$$W_{loc}^c(0) = \{\xi + \eta \ |\xi = h^c(\eta), \ \xi \in X_+^{1/2} \cap U, \ \eta \in X_0 \cap U\} \ ,$$

where we have that $h^c(0) = Dh^c(0) = 0$.

We get that the center manifold is approximated in the following form

$$h^c(q) = \frac{1}{\delta^4}||q||_H^2 A^{-1}q + \frac{2A^{-3/2}f(u)}{\delta} + O(||q||_H^4) \ . \tag{3.6}$$

Solutions on the center manifold satisfy

$$\begin{aligned} p(t) &= h^c(q(t)), \\ q_t(t) &= -\frac{1}{\delta^3}||h^c(q) + q||_H^2(h^c(q) + q) \ . \end{aligned} \tag{3.7}$$

From system (3.7) we obtain that the stability of the solution $u = 0$ depends on f. Thus we have the following cases

(i) if $f(u_0) < 0$, then we get that $(p,q) = (0,0)$ is unstable, so $u = 0$ is also unstable for the initial Kirchhoff's system,

(ii) if $f(u_0) > 0$, then $(p,q) = (0,0)$ is asymptotically stable, so $u = 0$ is also asymptotically stable for the initial system,

(iii) if $f(u_0) = 0$, we have that solutions on the center manifold satisfy the following system

$$\begin{aligned} p(t) &= h^c(q(t)) \, , \\ q_t(t) &= -\frac{1}{\delta^3}||q||_H^2 q + O(||q||_H^5) \, . \end{aligned}$$

So, we obtain that $(p,q) = (0,0)$ is stable, that is, $u = 0$ is stable for the initial Kirchhoff's system.

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