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The James and von Neumann-Jordan type constants and uniform normal structure in Banach spaces

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Abstract

Recently, Takahashi has introduced the James and von Neumann-Jordan type constants. In this paper, we present some sufficient conditions for uniform normal structure and therefore the fixed point property of a Banach space in terms of the James and von Neumann-Jordan type constants and the Ptolemy constant. Our main results of the paper significantly generalize and improve many known results in the recent literature.

Keywords: James type constant; von Neumann-Jordan type constant; Ptolemy constant; fixed point property; uniform normal structure.
2010 MSC: Primary 46B20; Secondary 47H10, 37C25, 54H25.

1. Introduction

The concepts of normal structure and uniform normal structure play an important role in metric fixed point theory for nonexpansive mappings (see [12]). Whether or not a Banach space has normal structure depends on the geometry of its unit ball and its unit sphere. It is not always easy to check whether a given Banach space has normal structure, and it is even less easy to check whether it has the fixed point property. Many mathematicians have established that, under various geometric properties of a Banach space often measured by different geometric constants, normal structure of the space is guaranteed. Recently, many geometric constants for a Banach space have been investigated. It has been shown that these constants are very useful in the description of various geometric structures of Banach spaces, which enable us to classify several important concepts of Banach spaces such as uniformly nonsquareness and uniform normal structure. Therefore, many recent studies have focused on these constants. The readers interested in this topic are referred to [1, 2, 5, 6, 7, 8, 9, 10, 11, 14, 19, 22, 23, 26] and the references mentioned therein.

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Definition 1.1. ([12]) A Banach space X is said to have the fixed point property if whenever C is a nonempty bounded closed convex subset of X it follows that every nonexpansive mapping $T : C \to C$, that is, $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, always has a fixed point. If the "bounded closed" condition of the set C above is replaced by "weakly compact", then we say that X has the weak fixed point property.

For a reflexive Banach space, fixed point property and weak fixed point property are the same. Recently, Lin [18] successfully constructed a nonreflexive space that has the fixed point property.

It is well known that a contractive mapping has unique one fixed point on Banach space X. However, a nonexpansive mapping may have no fixed point if X is anyone Banach space. One of the remaining unsolved questions is whether each nonexpansive mapping on a closed bounded convex subset in reflexive Banach spaces has a fixed point. The question for nonreflexive Banach space is false in general. In the last fifty years the question of whether a Banach space X has, or has not, the weak fixed point property has been intensively studied and although the list of sufficient conditions for the weak fixed point property is still increasing, perhaps the most celebrated result in this direction is the one given by Kirk [17], which states that every Banach space with weak normal structure has the weak fixed point property. Since then, many geometric properties guaranteeing (weak) normal structure and the (weak) fixed point property have been widely investigated.

The main aim of the present paper is to investigate some sufficient conditions for a Banach space to have uniform normal structure and therefore the fixed point property in terms of several geometric parameters and constants. The obtained results generalize and improve many comparable results in the existing literature.

2. Preliminaries

We start by reviewing some notions, definitions and results which will be needed in the sequel.

We shall assume throughout this paper that X stands for real nontrivial Banach space, that is, $\dim X \geq 2$ with the dual space X^*. Let $S_X = \{x \in X \,:\, \|x\| = 1\}$ and $B_X = \{x \in X \,:\, \|x\| \leq 1\}$ be the unit sphere and the closed unit ball of X, respectively.

Brodskĭi and Mil'man [3] introduced the following geometric concepts in 1948 to study the fixed point properties under isometry which maps a weakly compact set to itself.

Definition 2.1. ([3]) A nonempty bounded convex subset K of a Banach space X is said to have normal structure if for every convex subset E of K that contains more than one point, there exists a point $x_0 \in E$ such that

$$\sup\left\{\|x_0 - y\| \,:\, y \in E\right\} < \mathrm{diam}(E),$$

where $\mathrm{diam}(E) = \sup\{\|x - y\| \,:\, x, y \in E\}$ denotes the diameter of E. A Banach space X is said to have normal structure if every bounded convex subset of X has normal structure. A Banach space X is said to have weak normal structure if for each weakly compact convex set K of X that contains more than one point has normal structure. X is said to have uniform normal structure if there exists $0 < c < 1$ such that for any closed bounded convex subset K of X that contains more than one point, there exists $x_0 \in K$ such that

$$\sup\left\{\|x_0 - y\| \,:\, y \in K\right\} < c\,\mathrm{diam}(K).$$

It is clear that for a reflexive Banach space, normal structure and weak normal structure coincide. Moreover, if a space has uniform normal structure, then it is reflexive.

It is worth mentioning that if X fails to have weak normal structure, then there exist a weakly compact convex subset $C \subset X$ and a sequence $\{x_n\} \subset C$ such that $\operatorname{dist}(x_{n+1}, \operatorname{co}\{x_k\}_{k=1}^n) \to \operatorname{diam}(C) = 1$ (see [12]).

Before going to the results, let us recall some basic facts about ultrapowers of Banach spaces and state a lemma which we will use in the proof of our main results.

Let $\mathcal{F}$ be a filter on $\mathbb{N}$ and let X be a Banach space. A sequence $\{x_n\}$ in X converges to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in \mathbb{N} \ : \ x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on $\mathbb{N}$ is called an ultrafilter if it is maximal with respect to set inclusion. An ultrafilter is called trivial if it is of the form $\{A \subset \mathbb{N} \ : \ i_0 \in A\}$ for some fixed $i_0 \in \mathbb{N}$, otherwise, it is called nontrivial. Let $\ell_\infty(X)$ denotes the subspace of the product space $\prod_{n\in\mathbb{N}} X$ equipped with the norm

$$\|(x_n)\| := \sup_{n\in\mathbb{N}} \|x_n\| < \infty.$$

Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and let

$$N_{\mathcal{U}} = \left\{(x_n) \in \ell_\infty(X) \ : \ \lim_{\mathcal{U}} \|x_n\| = 0\right\}.$$

The ultrapower of X, denoted by $\widetilde{X}$, is the quotient space $\frac{\ell_\infty(X)}{N_{\mathcal{U}}}$ equipped with the quotient norm. Write $(x_n)_{\mathcal{U}}$ to denote the elements of the ultrapower. It follows from the definition of the quotient norm that

$$\left\|(x_n)_{\mathcal{U}}\right\| = \lim_{\mathcal{U}} \|x_n\|.$$

Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically. For a widespread discussion on the Banach space ultrapower construction, we refer the reader to [12, 15, 16, 24]. We also note that if X is super-reflexive, that is $\widetilde{X^*} = (\widetilde{X})^*$, then X has uniform normal structure if and only if $\widetilde{X}$ has normal structure (see [15]).

Lemma 2.2. ([23]) If X is a super-reflexive Banach space and fails to have normal structure, then for $r \in (0, 1]$ there are $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3 \in S_{\widetilde{X}}$ and $\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3 \in S_{(\widetilde{X})^*}$ such that

(1) $\|\widetilde{x}_i - \widetilde{x}_j\| = 1$ and $\widetilde{f}_i(\widetilde{x}_j) = 0$ for all $i \neq j$;

(2) $\widetilde{f}_i(\widetilde{x}_i) = 1$ for $i = 1, 2, 3$;

(3) $\|\widetilde{x}_3 - (\widetilde{x}_2 + r\widetilde{x}_1)\| \geq \|\widetilde{x}_2 + r\widetilde{x}_1\|$.

3. The James type constant

The aim of this section is to present a sufficient condition for uniform normal structure by considering the James type constant.

A Banach space X is called uniformly nonsquare, in the sense of James, if there exists a positive number $\delta < 2$ such that $\min(\|x + y\|, \|x - y\|) \leq \delta$ for any $x, y \in S_X$. The nonsquare or James constant (see [10]), defined as $J(X) = \sup\{\min(\|x+y\|, \|x-y\|) \ : \ x, y \in S_X\}$, is introduced to characterize such a concept. Obviously, X is uniformly nonsquare in the sense of James if and only if $J(X) < 2$ (see [14]). It was proved in [13] that uniformly nonsquare Banach spaces are reflexive, indeed super-reflexive.

Recently, Takahashi [25] has introduced the James type constant

$$J_{X,t}(\tau) = \sup\left\{\mu_t\big(\|x+\tau y\|, \|x-\tau y\|\big) \ : \ x, y \in S_X\right\},$$

where $\tau \geq 0$, $-\infty \leq t < +\infty$. Here, we denote

$$\mu_t(a,b) = \left(\frac{a^t + b^t}{2}\right)^{\frac{1}{t}} \ (t \neq 0) \qquad \text{and} \qquad \mu_0(a,b) = \lim_{t\to 0} \mu_t(a,b) = \sqrt{ab}$$

for two positive numbers a and b. It is well known that $\mu_t(a,b)$ is nondecreasing and

$$\mu_{-\infty}(a,b) = \lim_{t\to-\infty} \mu_t(a,b) = \min(a,b) \qquad \text{and} \qquad \mu_{+\infty}(a,b) = \lim_{t\to+\infty} \mu_t(a,b) = \max(a,b).$$

Therefore, $J(X) = J_{X,-\infty}(1)$. It is obvious that the James type constant includes some known constants, such as Alonso-Llorens-Fuster's constant $T(X)$ (see [1]), Baronti-Casini-Papini's constant $A_2(X)$ (see [2]), Gao's constant $E_\tau(X)$ (see [9]), Yang-Wang's modulus $\gamma_X(\tau)$ (see [26]) and the modulus of smoothness $\rho_X(\tau)$ (see [12]). These constants are defined by $T(X) = J_{X,0}(1)$, $A_2(X) = J_{X,1}(1)$, $E_\tau(X) = 2J^2_{X,2}(\tau)$, $\gamma_X(\tau) = J^2_{X,2}(\tau)$ and $\rho_X(\tau) = J_{X,1}(\tau) - 1$. It is interesting to remark at this point that $A_2(X) = A_2(X^*)$.

Proposition 3.1. ([27]) If $t \geq 1$, then for any Banach space X, the following conditions are equivalent:

(1) X is uniformly nonsquare.

(2) $J_{X,t}(\tau) < 1 + \tau$ for all $\tau > 0$.

(3) $J_{X,t}(\tau_0) < 1 + \tau_0$ for some $\tau_0 > 0$.

Proposition 3.2. ([27]) For any Banach space X,

$$\begin{aligned} J_{X,t}(\tau) &= \sup\left\{\left(\frac{\|x+\tau y\|^t + \|x-\tau y\|^t}{2}\right)^{\frac{1}{t}} \ : \ x \in S_X,\ y \in B_X\right\} \\ &= \sup\left\{\left(\frac{\|x+\tau y\|^t + \|x-\tau y\|^t}{2}\right)^{\frac{1}{t}} \ : \ x, y \in B_X\right\}, \end{aligned}$$

where $t \geq 1$.

Now, we are in a position to prove our first result concerning the James type constant.

Theorem 3.3. *Let X be a Banach space such that*

$$J_{X,t}(\tau) < \frac{\tau + \sqrt{4+\tau^2}}{2}$$

for some $\tau \in (0,1]$. Then X has uniform normal structure.

Proof . By virtue of Proposition 3.1 and our hypothesis, X is uniformly nonsquare, and consequently, X is super-reflexive. It suffices to prove only that X has normal structure if $J_{X,t}(\tau) < \frac{\tau+\sqrt{4+\tau^2}}{2}$. Suppose on the contrary that X fails to have normal structure. Then there are elements $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3 \in S_{\widetilde{X}}$ and $\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3 \in S_{(\widetilde{X})^*}$ satisfying all the conditions in Lemma 2.2. Let $\alpha(\tau) = \frac{2-\tau+\sqrt{4+\tau^2}}{2}$ and consider, without loss of generality, the following cases:

Case 1: $\|\widetilde{x}_2 + \widetilde{x}_1\| \le \alpha(\tau)$.

In this case, let us put $\widetilde{u} = \widetilde{x}_2 - \widetilde{x}_1$ and $\widetilde{v} = \frac{\widetilde{x}_2+\widetilde{x}_1}{\alpha(\tau)}$. It follows that $\widetilde{u}, \widetilde{v} \in B_{\widetilde{X}}$, and

$$\begin{aligned}\|\widetilde{u} + \tau\widetilde{v}\| &= \left\|\left(1+\frac{\tau}{\alpha(\tau)}\right)\widetilde{x}_2 - \left(1-\frac{\tau}{\alpha(\tau)}\right)\widetilde{x}_1\right\| \\ &\ge \left(1+\frac{\tau}{\alpha(\tau)}\right)\widetilde{f}_2(\widetilde{x}_2) - \left(1-\frac{\tau}{\alpha(\tau)}\right)\widetilde{f}_2(\widetilde{x}_1) \\ &= 1+\frac{\tau}{\alpha(\tau)},\end{aligned}$$

$$\begin{aligned}\|\widetilde{u} - \tau\widetilde{v}\| &= \left\|\left(1+\frac{\tau}{\alpha(\tau)}\right)\widetilde{x}_1 - \left(1-\frac{\tau}{\alpha(\tau)}\right)\widetilde{x}_2\right\| \\ &\ge \left(1+\frac{\tau}{\alpha(\tau)}\right)\widetilde{f}_1(\widetilde{x}_1) - \left(1-\frac{\tau}{\alpha(\tau)}\right)\widetilde{f}_1(\widetilde{x}_2) \\ &= 1+\frac{\tau}{\alpha(\tau)}.\end{aligned}$$

Case 2: $\|\widetilde{x}_2 + \widetilde{x}_1\| > \alpha(\tau)$.

We subdivide Case 2 into Case 2.1 $\big(\|\widetilde{x}_3 - \widetilde{x}_2 + \widetilde{x}_1\| \le \alpha(\tau)\big)$ and Case 2.2 $\big(\|\widetilde{x}_3 - \widetilde{x}_2 + \widetilde{x}_1\| > \alpha(\tau)\big)$.

Case 2.1. $\|\widetilde{x}_3 - \widetilde{x}_2 + \widetilde{x}_1\| \le \alpha(\tau)$.

In this case, let us put $\widetilde{u} = \widetilde{x}_3 - \widetilde{x}_1$ and $\widetilde{v} = \frac{\widetilde{x}_3-\widetilde{x}_2+\widetilde{x}_1}{\alpha(\tau)}$. It follows that $\widetilde{u}, \widetilde{v} \in B_{\widetilde{X}}$, and

$$\begin{aligned}\|\widetilde{u} + \tau\widetilde{v}\| &= \left\|\left(1+\frac{\tau}{\alpha(\tau)}\right)\widetilde{x}_3 - \left(1-\frac{\tau}{\alpha(\tau)}\right)\widetilde{x}_1 - \frac{\tau}{\alpha(\tau)}\widetilde{x}_2\right\| \\ &\ge \left(1+\frac{\tau}{\alpha(\tau)}\right)\widetilde{f}_3(\widetilde{x}_3) - \left(1-\frac{\tau}{\alpha(\tau)}\right)\widetilde{f}_3(\widetilde{x}_1) - \frac{\tau}{\alpha(\tau)}\widetilde{f}_3(\widetilde{x}_2) \\ &= 1+\frac{\tau}{\alpha(\tau)},\end{aligned}$$

$$\begin{aligned}\|\widetilde{u} - \tau\widetilde{v}\| &= \left\|\left(1+\frac{\tau}{\alpha(\tau)}\right)\widetilde{x}_1 - \left(1-\frac{\tau}{\alpha(\tau)}\right)\widetilde{x}_3 - \frac{\tau}{\alpha(\tau)}\widetilde{x}_2\right\| \\ &\ge \left(1+\frac{\tau}{\alpha(\tau)}\right)\widetilde{f}_1(\widetilde{x}_1) - \left(1-\frac{\tau}{\alpha(\tau)}\right)\widetilde{f}_1(\widetilde{x}_3) - \frac{\tau}{\alpha(\tau)}\widetilde{f}_1(\widetilde{x}_2) \\ &= 1+\frac{\tau}{\alpha(\tau)}.\end{aligned}$$

Case 2.2. $\|\widetilde{x}_3 - \widetilde{x}_2 + \widetilde{x}_1\| > \alpha(\tau)$.

In this case, let us put $\widetilde{u} = \widetilde{x}_3 - \widetilde{x}_2$ and $\widetilde{v} = \widetilde{x}_1$. It follows that $\widetilde{u}, \widetilde{v} \in S_{\widetilde{X}}$, and

$$\begin{aligned}
\|\widetilde{u} + \tau\widetilde{v}\| &= \|\widetilde{x}_3 - \tau\widetilde{x}_2 + \widetilde{x}_1\| \\
&\geq \|\widetilde{x}_3 - \widetilde{x}_2 + \widetilde{x}_1\| - (1-\tau) \\
&\geq \alpha(\tau) + \tau - 1, \\
\|\widetilde{u} - \tau\widetilde{v}\| &= \|\widetilde{x}_3 - (\tau\widetilde{x}_2 + \widetilde{x}_1)\| \\
&\geq \|\widetilde{x}_3 - (\widetilde{x}_2 + \widetilde{x}_1)\| - (1-\tau) \\
&\geq \alpha(\tau) + \tau - 1.
\end{aligned}$$

Therefore, from the equivalent definition of the James type constant (see Proposition 3.2) and the fact that $J_{X,t}(\tau) = J_{\widetilde{X},t}(\tau)$, we obtain

$$J_{X,t}(\tau) \geq \min\left\{1 + \frac{\tau}{\alpha(\tau)}, \alpha(\tau) + \tau - 1\right\} = \frac{\tau + \sqrt{4+\tau^2}}{2},$$

which is a contradiction. This completes the proof. □

According to Theorem 3.3 and the inequalities

$$J(X) \leq T(X) \leq A_2(X) \tag{3.1}$$

(see [1]), we get the following results.

Corollary 3.4. If $J(X) < \frac{1+\sqrt{5}}{2}$, then X has uniform normal structure.

Corollary 3.5. If $T(X) < \frac{1+\sqrt{5}}{2}$, then X has uniform normal structure.

Corollary 3.6. If $A_2(X) < \frac{1+\sqrt{5}}{2}$, then X and its dual X^* have uniform normal structure.

Corollary 3.7. If $J(X) < 1 + \frac{1}{T(X)}$, then X has uniform normal structure.

Corollary 3.8. If $J(X) < 1 + \frac{1}{A_2(X)}$, then X has uniform normal structure.

Corollary 3.9. If $T(X) < 1 + \frac{1}{J(X)}$, then X has uniform normal structure.

Corollary 3.10. If $T(X) < 1 + \frac{1}{A_2(X)}$, then X has uniform normal structure.

Corollary 3.11. If $A_2(X) < 1 + \frac{1}{J(X)}$, then X has uniform normal structure.

Corollary 3.12. If $A_2(X) < 1 + \frac{1}{T(X)}$, then X has uniform normal structure.

Corollary 3.13. If X is a Banach space with

$$\rho_X(\tau) < \frac{\tau - 2 + \sqrt{4+\tau^2}}{2}$$

for some $\tau \in (0,1]$, then X has uniform normal structure.

Remark 3.14. In [21] it was proved that $\rho_X(\tau) < \frac{\tau}{2}$ for some $\tau > 0$ implies that X has uniform normal structure. So our Corollary 3.13 is an improvement of such a result.

Corollary 3.15. If X is a Banach space with

$$E_\tau(X) < 2 + \tau^2 + \tau\sqrt{4+\tau^2}$$

for some $\tau \in (0,1]$, then X has uniform normal structure.

Remark 3.16. In [9] it was proved that $E_\tau(X) < 1 + (1+\tau)^2$ for some $(0,1]$ implies that X has uniform normal structure. So our Corollary 3.15 is an improvement of such a result.

Corollary 3.17. If X is a Banach space with

$$\gamma_X(\tau) < \frac{2+\tau^2+\tau\sqrt{4+\tau^2}}{2}$$

for some $\tau \in (0,1]$, then X has uniform normal structure.

Remark 3.18. In [26] it was proved that $\gamma_X(\tau) < \frac{1+(1+\tau)^2}{2}$ for some $(0,1]$ implies that X has uniform normal structure. So our Corollary 3.17 is an improvement of such a result.

4. The von Neumann-Jordan type and Ptolemy constants

In this section, we are going to present some sufficient conditions for uniform normal structure in terms of the von Neumann-Jordan type and Ptolemy constants.

The von Neumann-Jordan type constant was defined by Takahashi [25] as

$$C_t(X) = \sup\left\{ \frac{J^2_{X,t}(\tau)}{1+\tau^2} \ : \ 0 \le \tau \le 1 \right\},$$

where $J_{X,t}(\tau)$ is James type constant. In particular, if $\tau = 1$ or $\tau = 1$, $t = -\infty$, then we get $C'_t(X)$ and $C_{-\infty}(X)$, respectively. It is obvious that $C'_t(X) \le C_t(X)$. Note that the von Neumann-Jordan type constant includes some known constants, such as the von Neumann-Jordan constant $C_{NJ}(X)$ (see [4]) and the Zbăganu constant $C_Z(X)$ (see [28]). These constants are defined by $C_{NJ}(X) = C_2(X)$ and $C_Z(X) = C_0(X)$. It is worthwhile to mention that

$$\frac{1}{2}\big(J(X)\big)^2 \le C'_t(X) \le C_t(X). \tag{4.1}$$

Now, we are ready to state and prove our main results concerning the von Neumann-Jordan type and Ptolemy constants.

Theorem 4.1. *If X is a Banach space such that $C'_t(X) < \frac{1+\sqrt{3}}{2}$, then X has uniform normal structure.*

Proof . Since $C'_t(X) < 2$, it then follows that X is uniformly nonsquare, and consequently, X is super-reflexive. It suffices to prove only that X has normal structure if $C'_t(X) < \frac{1+\sqrt{3}}{2}$. Suppose on the contrary that X fails to have normal structure. Then there are elements $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3 \in S_{\widetilde{X}}$ and $\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3 \in S_{(\widetilde{X})^*}$ satisfying all the conditions in Lemma 2.2. Let $\alpha^2 = 1+\sqrt{3}$ and consider, without loss of generality, the following cases:

Case 1: $\|\widetilde{x}_1 + \widetilde{x}_2\| \le \alpha$.

In this case, let us put $\widetilde{u} = \widetilde{x}_1 - \widetilde{x}_2$ and $\widetilde{v} = \frac{\widetilde{x}_1+\widetilde{x}_2}{\alpha}$. It follows that $\widetilde{u}, \widetilde{v} \in B_{\widetilde{X}}$, and

$$\begin{aligned}
\frac{\left(\frac{\|\widetilde{u}+\widetilde{v}\|^t+\|\widetilde{u}-\widetilde{v}\|^t}{2}\right)^{\frac{2}{t}}}{2} &\ge \frac{\left(\frac{\left\|\left(1+\frac{1}{\alpha}\right)\widetilde{x}_1-\left(1-\frac{1}{\alpha}\right)\widetilde{x}_2\right\|^t+\left\|\left(1+\frac{1}{\alpha}\right)\widetilde{x}_2-\left(1-\frac{1}{\alpha}\right)\widetilde{x}_1\right\|^t}{2}\right)^{\frac{2}{t}}}{2}\\
&\ge \frac{\left(\frac{\left(\left(1+\frac{1}{\alpha}\right)\widetilde{f}_1(\widetilde{x}_1)-\left(1-\frac{1}{\alpha}\right)\widetilde{f}_1(\widetilde{x}_2)\right)^t+\left(\left(1+\frac{1}{\alpha}\right)\widetilde{f}_2(\widetilde{x}_2)-\left(1-\frac{1}{\alpha}\right)\widetilde{f}_2(\widetilde{x}_1)\right)^t}{2}\right)^{\frac{2}{t}}}{2}\\
&\ge \frac{\left(1+\frac{1}{\alpha}\right)^2}{2}.
\end{aligned}$$

Case 2: $\|\widetilde{x}_1 + \widetilde{x}_2\| > \alpha$.

We subdivide Case 2 into Case 2.1 ($\|\widetilde{x}_3 + \widetilde{x}_2 - \widetilde{x}_1\| \le \alpha$) and Case 2.2 ($\|\widetilde{x}_3 + \widetilde{x}_2 - \widetilde{x}_1\| > \alpha$).

Case 2.1. $\|\widetilde{x}_3 + \widetilde{x}_2 - \widetilde{x}_1\| \le \alpha$.

In this case, let us put $\widetilde{u} = \widetilde{x}_2 - \widetilde{x}_3$ and $\widetilde{v} = \frac{\widetilde{x}_3+\widetilde{x}_2+\widetilde{x}_1}{\alpha}$. It follows that $\widetilde{u}, \widetilde{v} \in B_{\widetilde{X}}$, and

$$\begin{aligned}
\frac{\left(\frac{\|\widetilde{u}+\widetilde{v}\|^t+\|\widetilde{u}-\widetilde{v}\|^t}{2}\right)^{\frac{2}{t}}}{2} &\ge \frac{\left(\frac{\left\|\left(1+\frac{1}{\alpha}\right)\widetilde{x}_2-\left(1-\frac{1}{\alpha}\right)\widetilde{x}_3-\frac{1}{\alpha}\widetilde{x}_1\right\|^t+\left\|\left(1+\frac{1}{\alpha}\right)\widetilde{x}_3-\left(1-\frac{1}{\alpha}\right)\widetilde{x}_2-\frac{1}{\alpha}\widetilde{x}_1\right\|^t}{2}\right)^{\frac{2}{t}}}{2}\\
&\ge \frac{\left(\frac{\left(\left(1+\frac{1}{\alpha}\right)\widetilde{f}_2(\widetilde{x}_2)-\left(1-\frac{1}{\alpha}\right)\widetilde{f}_2(\widetilde{x}_3)-\frac{1}{\alpha}\widetilde{f}_2(\widetilde{x}_1)\right)^t+\left(\left(1+\frac{1}{\alpha}\right)\widetilde{f}_3(\widetilde{x}_3)-\left(1-\frac{1}{\alpha}\right)\widetilde{f}_3(\widetilde{x}_2)-\frac{1}{\alpha}\widetilde{f}_3(\widetilde{x}_1)\right)^t}{2}\right)^{\frac{2}{t}}}{2}\\
&\ge \frac{\left(1+\frac{1}{\alpha}\right)^2}{2}.
\end{aligned}$$

Case 2.2. $\|\widetilde{x}_3 + \widetilde{x}_2 - \widetilde{x}_1\| > \alpha$.

In this case, let us put $\widetilde{u} = \widetilde{x}_3 - \widetilde{x}_1$ and $\widetilde{v} = \widetilde{x}_2$. It follows that $\widetilde{u}, \widetilde{v} \in S_{\widetilde{X}}$, and

$$\begin{aligned}
\frac{\left(\frac{\|\widetilde{u}+\widetilde{v}\|^t+\|\widetilde{u}-\widetilde{v}\|^t}{2}\right)^{\frac{2}{t}}}{2} &\ge \frac{\left(\frac{\|\widetilde{x}_3+\widetilde{x}_2-\widetilde{x}_1\|^t+\|\widetilde{x}_3-(\widetilde{x}_1+\widetilde{x}_2)\|^p}{2}\right)^{\frac{2}{t}}}{2}\\
&\ge \frac{\left(\frac{\alpha^t+\|\widetilde{x}_1+\widetilde{x}_2\|^t}{2}\right)^{\frac{2}{t}}}{2}\\
&\ge \frac{\alpha^2}{2}.
\end{aligned}$$

Therefore, by using definition of $C'_t(X)$ and the fact that $C'_t(X) = C'_t(\widetilde{X})$, we obtain

$$C'_t(X) \ge \min\left\{\frac{\left(1+\frac{1}{\alpha}\right)^2}{2}, \frac{\alpha^2}{2}\right\} = \frac{1+\sqrt{3}}{2},$$

which is a contradiction. This finishes the proof. □

By virtue of Theorem 4.1 and bearing in mind inequalities (3.1) and (4.1), we obtain the following results.

Corollary 4.2. If $C'_t(X) < 1 + \frac{1}{(A_2(X))^2}$, then X has uniform normal structure.

Corollary 4.3. If $C'_t(X) < 1 + \frac{1}{(T(X))^2}$, then X has uniform normal structure.

Corollary 4.4. If $C'_t(X) < 1 + \frac{1}{(J(X))^2}$, then X has uniform normal structure.

Remark 4.5. Theorem 4.1 and Corollary 4.4 are improvements of the following results: A Banach space X has uniform normal structure if one of the following conditions is satisfied

(1) $C_{NJ}(X) < \frac{1+\sqrt{3}}{2}$ [5, Theorem 3.16] and [22, Theorem 2];

(2) $C_{NJ}(X) < 1 + \frac{1}{(J(X))^2}$ [5, Corollary 3.17];

(3) $C_Z(X) < \frac{1+\sqrt{3}}{2}$ [11, Corollary 8] and [19, Theorem 5];

(4) $C_Z(X) < 1 + \frac{1}{(J(X))^2}$ [11, Corollary 7].

According to Theorem 4.1 and Corollaries 4.2, 4.3 and 4.4 and the fact that $C'_t(X) \leq C_t(X)$, we immediately obtain the following corollaries.

Corollary 4.6. If $C_t(X) < \frac{1+\sqrt{3}}{2}$, then X has uniform normal structure.

Corollary 4.7. If $C_t(X) < 1 + \frac{1}{(A_2(X))^2}$, then X has uniform normal structure.

Corollary 4.8. If $C_t(X) < 1 + \frac{1}{(T(X))^2}$, then X has uniform normal structure.

Corollary 4.9. If $C_t(X) < 1 + \frac{1}{(J(X))^2}$, then X has uniform normal structure.

Recall that for a normed space X, the real number

$$C_p(X) = \sup\left\{\frac{\|x-y\|\,\|z\|}{\|x-z\|\,\|y\| + \|z-y\|\,\|x\|}\ :\ x, y, z \in X\backslash\{0\},\ x \neq y \neq z \neq x\right\}$$

is called the Ptolemy constant of X. The notion of the Ptolemy constant of Banach spaces was introduced in [20] and recently it has been studied by Llorens-Fuster et al. in [19].

By applying Corollaries 4.6, 4.7, 4.8 and 4.9 and the fact that $C_Z(X) \leq C_p(X)$ (see [19]), we immediately obtain the following results.

Corollary 4.10. If $C_p(X) < \frac{1+\sqrt{3}}{2}$, then X has uniform normal structure.

Corollary 4.11. If $C_p(X) < 1 + \frac{1}{(A_2(X))^2}$, then X has uniform normal structure.

Corollary 4.12. If $C_p(X) < 1 + \frac{1}{(T(X))^2}$, then X has uniform normal structure.

Corollary 4.13. If $C_p(X) < 1 + \frac{1}{(J(X))^2}$, then X has uniform normal structure.

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