Int. J. Nonlinear Anal. Appl. 8 (2017) No. 1, 165-176
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2017.2498

Existence results for equilibrium problems under strong sign property

Somaye Jafari, Ali Farajzadeh*

Department of Mathematics, Razi University, Kermanshah 67149, Iran

(Communicated by M. Eshaghi)

Abstract

This paper concerns equilibrium problems in real metric linear spaces. Considering a modified notion of upper sign property for bifunctions, we obtain the relationship between the solution sets of the local Minty equilibrium problem and the equilibrium problem, where the technical conditions on f used in the literature are relaxed. The KKM technique is used to generalize and unify some existence results for the relaxed μ-quasimonotone equilibrium problems in the literature.

Keywords: metric linear space; equilibrium problem; Minty equilibrium problem; strong upper sign property.
2010 MSC: Primary 26A25; Secondary 39B62.

1. Introduction

Equilibrium problem theory is a unified and general model to investigate a wide class of problems arising in finance, economics, transportation, and optimization. In 1994, Blum and Oettli [8] introduced the concept of equilibrium problems in order to unify some known problems in nonlinear analysis such as variational inequalities, fixed point, Nash equilibrium, game theory and etc.

Let K be a nonempty convex subset of a real Hausdorff topological vector space X and $f : K \times K \to \mathbb{R}$ be a given bifunction. The definition of the equilibrium problem (in brief EP) is to find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \geq 0 \quad \text{for all} \quad y \in K.$$

*Corresponding author
Email addresses: s.jafari.math@gmail.com (Somaye Jafari), farajzadehali@gmail.com (Ali Farajzadeh)

Received: July 2015 *Revised:* November 2015

To the best of our knowledge, this general model is closely related to the minimax inequalities which studied by Fan [11], but after the paper of Blum and Oettli, a large number of publications have been appeared for this developing area of analysis ([4, 5, 6, 9, 10, 17, 18] and the references therein). Equilibrium problems were also studied from the perspective of convergence of numerical algorithms and related topics, see more details and recent developments in the survey paper [7].

By some topological and algebraic assumptions on a bifunction f, Fan [11] obtained the existence of solutions for minimax inequality which is closely related to equilibrium problems. For this aim, the upper semicontinuity of $f(.,y)$ for all $y \in K$, the quasiconvexity of $f(x,.)$ for all $x \in K$ together with the compactness of K was assumed. In this regard, using the generalized monotonicity, the weak concepts of the continuity and some coercivity conditions, many researchers obtained existence results for equilibria [4, 5, 15].

The relationship between Minty and Stampacchia variational inequalities is a key tool to investigate the existence results for variational inequalities [20]. Moreover, the existence results for equilibrium problems are usually taken from the notions and tools introduced in the framework of variational inequalities. Therefore inspired and motivated by the theory of the variational inequality, establishing a link between the solution set of equilibrium problem and Minty equilibrium problem (in brief MEP) are widely studied [4, 10, 13, 14]. The Minty equilibrium problem is as follows

$$\text{find} \quad \bar{x} \in K \quad \text{such that} \quad f(y, \bar{x}) \leq 0 \quad \text{for all} \quad y \in K.$$

We designate by $S(f,K)$ the solution set of EP and by $M(f,K)$ the solution set of MEP.

Since, the solution set $M(f,K)$ may be empty, Bianchi and Pini [4] used a relationship between the solution set $S(f,K)$ and a greater solution set of local Minty equilibrium problem which defined as follows:

If $\bar{x} \in K$ and there exists an open neighborhood U of $\bar{x}$ such that

$$f(y, \bar{x}) \leq 0 \quad \text{for all} \quad y \in K \cap U,$$

then $\bar{x}$ is a solution of the local Minty equilibrium problem and the solution set of it will be denoted by $M_L(f,K)$.

For presenting the above definition, the authors followed the idea and steps of variational inequality theory [2]. The assumptions in Bianchi and Pini's paper [4] which guaranteed, the solution set $S(f,K)$ involves the solution set $M_L(f,K)$, also obtained from the adaptation of a notion in the variational inequality context to the equilibrium problem setting. Indeed, investigating continuity and maximality properties of pseudomonotone operators, Hadjisavvas [16] defined the notion of the sign continuity of an operator in the case of the variational inequalities. Bianchi and Pini [4] generalized the concept of the upper sign continuity to bifunctions. Let us say the definition of upper sign continuity for set-valued mappings and bifunctions.

Consider the set-valued map $T : K \rightrightarrows X^*$ and bifunction $f : K \times K \to \mathbb{R}$, T and f are respectively called upper sign continuous at $x \in K$, if for every $y \in K$, the following implications respectively hold

$$\Big(\inf_{z_t^* \in T(z_t)} \langle z_t^*, y - x \rangle \geq 0, \qquad \forall t \in (0,1)\Big) \Rightarrow \qquad \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \geq 0,$$

and

$$f(z_t, y) \geq 0, \quad \forall t \in (0,1) \Rightarrow \qquad f(x,y) \geq 0,$$

where $z_t = (1-t)x + ty$.

The approach of [4] has been pursued by Farajzadeh and Zafarani [13] who provided necessary and sufficient conditions for the non-emptiness of $M_L(f,K)$. It must be noted that if the set-valued mapping $T : K \rightrightarrows X^*$ has w^*-compact and nonempty values (which is always assumed), then the associated bifunction to T and K is denoted by $f_T : K \times K \to \mathbb{R}$ and defined by

$$f_T(x,y) = \sup_{x^* \in T(x)} \langle x^*, y - x \rangle.$$

Unfortunately, we cannot deduce upper sign continuity of T by the upper sign continuity of the bifunction f_T. Hence, a natural question is how the definition of upper sign continuity can be modified such that the upper sign continuity f_T implies the upper sign continuity of T. Recently, Castellani and Giuli [10] answered to the question by proposing a new definition named (local) upper sign property. We say that the bifunction $f : K \times K \to \mathbb{R}$ has the (local) upper sign property (with respect to the first variable) at $x \in K$, if there exists an open neighborhood U of x such that for any $y \in K \cap U$, the following implication holds:

$$f(z_t, x) \leq 0, \quad \forall t \in (0,1) \ \Rightarrow \quad f(x,y) \geq 0,$$

where $z_t = (1-t)x + ty$.

The aim of this paper is to weaken the assumptions of Castellani and Giuli [10] in such a way that we deduce the important relationship $M_L(f,K) \subseteq S(f,K)$ by using a modified notion of the upper sign property which is named, strong upper sign property. We also introduced the notion of the strong μ-upper sign property in order to obtain existence results for μ-quasimonotone equilibrium problems.

The remainder of the paper is organized as follows: in Section 2 we fix the notations and recall some definitions and discuss about different technical conditions on the bifunctions for deriving the important inclusion $M_L(f,K) \subseteq S(f,K)$, Section 3 is devoted to the notion of the upper sign property and its generalizations, while Section 4 deals with a modified notion of the upper sign property in order to relaxed the conditions on $f(x,x)$ for every $x \in K$ and the technical condition for deriving $M_L(f,K) \subseteq S(f,K)$ and in Section 5 we introduce the notion of the strong μ-upper sign property in order to obtain the relationship between $S(f,K)$ and $M_L^{\mu}(f,K)$. Using the KKM technique, we end in Section 6 with establishing existence results for $EP(f,K)$ in real metric linear spaces.

2. Preliminaries

A real linear topological space E is called a metric real linear space if its topology is given by a meter. Throughout the paper, unless otherwise stated, we assume that $\mathbb{X}$ is a metric real linear space and $\mathbb{X}^*$ is its dual.

It is well known that every normed space is a metric linear space, but the converse may fail. For instance, Fréchet spaces are locally convex spaces that are complete with respect to a translation invariant metric and the metric need not arise from a norm.

The symbol $\langle .,. \rangle$ signifies the duality pairing between $\mathbb{X}$ and $\mathbb{X}^*$. Let A be a nonempty subset of $\mathbb{X}$, $conv(A)$ denotes the convex hull of A and the neighborhood with center $x \in \mathbb{X}$ and radius $r > 0$ is denoted by $B_r(x)$.

In recent years, some applications of the generalized convexity and the generalized monotonicity to obtain results for equilibrium problems were investigated [4, 10, 13, 17].

Let $f : K \to \mathbb{R}$ be a function, f is called convex, if for every $x, y \in K$ and $t \in [0,1]$

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Let $f : K\times K \to \mathbb{R}$ and $y \in K$. We say that $f(.,y)$ is upper hemicontinuous in the first argument at $x \in K$, if the restriction of f on all lines is upper semicontinuous at $x \in K$, i.e.,

$$f(x,y) \geq \limsup_{t\downarrow 0} f((1-t)x + tz, y), \quad \forall z \in K.$$

Several definitions of monotonicity for bifunctions have been introduced. In the following for some $\mu \geq 0$, the definitions of parametric generalized monotonicity for bifunctions are presented [10].

Definition 2.1. [10] Given a bifunction $f : K \times K \to \mathbb{R}$ and fixed $\mu \geq 0$. f is called

- relaxed μ-monotone if for all $x, y \in K$, we have
$$f(x,y) + f(y,x) \leq \mu(d(x,y))^2;$$
- relaxed μ-pseudomonotone if for all $x, y \in K$, we have
$$f(x,y) \geq 0 \quad \Rightarrow \quad f(y,x) \leq \mu(d(x,y))^2;$$
- relaxed μ-quasimonotone if for all $x, y \in K$, we have
$$f(x,y) > 0 \quad \Rightarrow \quad f(y,x) \leq \mu(d(x,y))^2;$$
- properly relaxed μ-quasimonotone if for all $n \in \mathbb{N}$, $x_1,\ldots,x_n \in K$ and $\bar{x} \in conv\{x_1,\ldots,x_n\}$, there exists $0 \leq k \leq n$ such that
$$f(x_k,\bar{x}) \leq \mu(d(x_k,\bar{x}))^2.$$

If $\mu = 0$, the above definitions reduce to the usual definitions of monotonicity, pseudomonotonicity, quasimonotonicity, and properly quasimonotonicity of f. Clearly, if f is relaxed μ-monotone, then it is relaxed μ-pseudomonotone, which implies relaxed μ-quasimonotonicity of f.

Let X and Y be Hausdorff topological vector spaces, A set-valued mapping $F : X \rightrightarrows Y$ is called upper semicontinuous at $x \in X$ if, for any neighborhood G of the set $F(x)$, there is a neighborhood of x, say $O(x)$, such that $F(y) \subseteq G$ for all $y \in O(x)$. F is upper hemicontinuous at $x \in X$ if the restriction of F on all lines is upper semicontinuous at $x \in X$.

Let us compare different technical conditions which frequently used in [4, 10, 13] to derive the important relationship $M_L(f,K) \subseteq S(f,K)$.

Suppose that $f : K \times K \to \mathbb{R}$ be a bifunction:

(F1) For every $x, y_1, y_2 \in K$
$$f(x,y_1) \leq 0 \text{ and } f(x,y_2) < 0 \Rightarrow f(x,z_t) < 0 \qquad \forall t \in (0,1),$$
where $z_t = (1-t)y_1 + ty_2$;

(F2) For every $x, y_1, y_2 \in K$
$$f(x,y_1) = 0 \text{ and } f(x,y_2) < 0 \Rightarrow f(x,z_t) < 0 \qquad \forall t \in (0,1),$$
where $z_t = (1-t)y_1 + ty_2$;

(F3) For every $x, y \in K$

$$f(x,y) < 0 \ \Rightarrow f(x,z_t) < 0, \quad \forall t \in (0,1),$$

where $z_t = (1-t)x + ty$.

It is obvious that condition (F1) implies condition (F2) and condition (F2) implies condition (F3) provided that $f(x,x) = 0$ for every $x \in K$. In order to show the important relationship $M_L(f,K) \subseteq S(f,K)$, we use the following condition for f which is weaker than conditions (F1), (F2) and (F3) provided that $f(x,x) = 0$ for every $x \in K$.

(F4) For every $x, y \in K$

$$f(x,y) < 0 \ \Rightarrow \ \exists (t_n)_{n\in\mathbb{N}} \subseteq (0,1) \ : \ t_n \to 0 \ \text{ and } \ f(x,z_{t_n}) < 0 \quad \forall n \in \mathbb{N}, \tag{2.1}$$

where $z_{t_n} = (1-t_n)x + t_n y$.

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(x,y) = \begin{cases} -1, & \text{if } x \text{ and } y \text{ are rational and } x \neq y \\ 0, & \text{if } x \text{ or } y \text{ is irrational or } x = y. \end{cases}$$

One can shows that the bifunction f satisfies (F4), but it doesn't satisfy conditions (F1), (F2) and (F3).

3. Upper sign property

Motivated by the definition of the upper sign continuity for an operator T, Bianchi and Pini [4] introduced the notion of the upper sign continuity for a bifunction and applied it to establish a link between $S(f,K)$ and $M_L(f,K)$.

If the set-valued mapping $T : K \rightrightarrows X^*$ has w^*-compact and convex values, then the associated bifunction to T and K is defined by

$$f_T(x,y) = \sup_{x^* \in T(x)} \langle x^*, y - x \rangle.$$

In this case the upper sign continuity of T doesn't imply the upper sign continuity of f_T. Hence a natural question is how the definition of upper sign continuity of f_T can be modified such that the mentioned notions imply each other. Recently, Castellani and Giuli in [10] answered to the question by proposing a new definition named *upper sign property*. In the next definition, we restate the definition in Hausdorff real topological vector spaces.

Definition 3.1. Let K be a convex subset of a Hausdorff real topological linear space X. We say that the bifunction $f : K \times K \to \mathbb{R}$ has the local upper sign property (with respect to the first variable) at $x \in K$ if there exists an open neighborhood U of x such that for any $y \in K \cap U$, the following implication holds:

$$f(z_t, x) \leq 0 \quad \text{for all} \quad t \in (0,1) \Longrightarrow f(x,y) \geq 0, \tag{3.1}$$

where $z_t = (1-t)x + ty$. We say that f has the upper sign property (with respect to the first variable) at $x \in K$ if for every $y \in K$, the implication 3.1 is satisfied.

Recently, Aussel et al.[1] showed the upper sign property and its local counterpart coincide under condition (F1). In the next lemma, the same was proved under condition (F4).

Lemma 3.2. *Let K be a convex subset of $\mathbb{X}$ and $f : K \times K \to \mathbb{R}$ be a given bifunction. Suppose that (F4) is also satisfied. Then f has the local upper sign property on K if and only if f has the upper sign property on K.*

Proof . *It is obvious that f with the upper sign property on K has also the local upper sign property on K. Conversely, assume that f has the local upper sign property at $x \in K$. Let $y \in K$ and*

$$f(z_t, x) \leq 0, \qquad \forall t \in (0,1),$$

where $z_t = (1-t)x + ty$. From the local upper sign property of f at $x \in K$, there exists $r > 0$ such that $f(x, y) \geq 0$ provided that $y \in B_r(x) \cap K$.

Take $y' \in K$. Arguing by a contradiction assume that $f(x, y') < 0$. Then by using (F4) there exists $(t_n)_{n\in\mathbb{N}}$ in $(0,1)$ which converges to zero and $f(x, z_{t_n}) < 0$ for all $n \in \mathbb{N}$. Moreover we can choose $n_0 \in \mathbb{N}$ such that $z_{t_{n_0}} \in B_r(x) \cap K$ ($z_{t_{n_0}} = (1-t_{n_0})x + t_{n_0}y'$). Therefore it follows by the local upper sign property that $f(x, z_{t_{n_0}}) \geq 0$ and this is a contradiction. Hence, $f(x, y') \geq 0$ and this completes the proof. $\square$

Existence results for equilibrium problems can be obtained by establishing a link between $M_L(f, K)$ and $S(f, K)$ and showing the non-emptiness of the solution set of local Minty equilibrium problem. The following theorem extends Lemma 2.1 in [4], Lemma 2.1 in [13] and consequently Theorem 1 in [10].

Theorem 3.3. *Let $f : K \times K \to \mathbb{R}$ be a bifunction with the upper sign property. If f satisfies condition (F4), then $M_L(f, K) \subseteq S(f, K)$.*

Proof . *Let $\bar{x}$ be an element of $M_L(f, K)$. So, there exists $r > 0$ such that*

$$f(y, \bar{x}) \leq 0, \quad \forall y \in B_r(x) \cap K.$$

Take $y' \in K$. If $f(x, y') < 0$, then by condition (F4), there exists $(t_n)_{n\in\mathbb{N}}$ in $(0,1)$ which converges to zero and $f(x, z_{t_n}) < 0$ for all $n \in \mathbb{N}$. Moreover, for any sequence $(t_n)_{n\in\mathbb{N}}$ in $(0,1)$ which converges to zero, we can find $n_0 \in \mathbb{N}$ such that $z_{t_{n_0}} \in B_r(x) \cap K$, where $z_{t_{n_0}} = (1-t_{n_0})x + t_{n_0}y'$. Thus

$$f(z_t, \bar{x}) \leq 0, \quad \forall t \in (0,1),$$

where $z_t = (1-t)\bar{x} + tz_{t_{n_0}}$. Now from the upper sign property of f we deduce that $f(x, z_{t_{n_0}}) \geq 0$ and this is a contradiction. Hence $f(x, y') \geq 0$ and this completes the proof. $\square$

Remark 3.4. By using of above theorem we can derive $M_L(f, \mathbb{R}) \subseteq S(f, \mathbb{R})$ for the bifunction $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which defined as follows

$$f(x, y) = \begin{cases} x + y, & \text{if } x < 0,\ y > 0,\ x, y \in \mathbb{Q} \\ 0, & \text{if } x = y \\ 1, & \text{otherwise.} \end{cases}$$

Notice that by the results in literature, we cannot obtain this important inclusion for f.

4. Strong upper sign property

In this subsection, we introduce the notions of the strong upper sign property and the strong upper sign continuity. The notion of the strong sign property is useful due to it directly implies the link between $M_L(f,K)$ and $S(f,K)$.

Definition 4.1. Let $f : K \times K \to \mathbb{R}$ be a given bifunction. f is said to have the strong upper sign property (with respect to the first variable) at $x \in K$ if for every $y \in K$ the following implication holds

$$\exists \delta \in (0,1] \ : \ f(z_t, x) \leq 0, \qquad \forall t \in (0,\delta) \quad \Rightarrow \quad f(x,y) \geq 0,$$

where $z_t = (1-t)x + ty$. We say that f has the strong upper sign property, if f has the property at every $x \in K$.

Notice that if for every $y \in K$ we have $\delta = 1$, then the notions of the strong upper sign property and the upper sign property are the same.

Definition 4.2. Let $f : K \times K \to \mathbb{R}$ be a bifunction. We say that f is strong upper sign continuous (with respect to the first variable) at $x \in K$ if for every $y \in K$ the following implication holds

$$\exists \delta \in (0,1] \ : \ f(z_t, y) \geq 0, \qquad \forall t \in (0,\delta) \quad \Rightarrow \quad f(x,y) \geq 0,$$

where $z_t = (1-t)x + ty$. We say that f is strong upper sign continuous, if f is strong upper sign continuous at every $x \in K$.

The strong sign continuity is a very weak form of continuity. In fact, we may derive the strong upper sign continuity of f at $x \in K$ by upper hemicontinuity of f with respect to the first variable at $x \in K$. It must be mentioned that the strong upper sign continuity of f at $x \in K$ implies the upper sign continuity of f at $x \in K$ in the sense of Bianchi and Pini [4].

The following proposition shows that the concept of the strong upper sign property is weaker than the strong upper sign continuity under some mild assumptions which is also used in [10].

Proposition 4.3. *Let $f : K \times K \to \mathbb{R}$ be a bifunction satisfying $f(x,x) = 0$ for every $x \in K$. If f is strong upper sign continuous and condition (F1) is satisfied, then f has the strong upper sign property.*

Proof . Let f is strong upper sign continuous at $x \in K$ and let for some $y \in K$ there exists $\delta \in (0,1]$ such that

$$f(z_t, x) \leq 0 \qquad \forall t \in (0,\delta). \tag{4.1}$$

On the contradiction, let $f(x,y) < 0$. By the strong upper sign continuity of f, there exists $t_\delta \in (0,\delta)$ such that $f(z_{t_\delta}, y) < 0$, where $z_{t_\delta} = (1-t_\delta)x + t_\delta y$. Moreover by (4.1) $f(z_{t_\delta}, x) \leq 0$. Now, using condition (F1) we get that

$$f(z_{t_\delta}, z_t) < 0 \qquad \forall t \in (0,1),$$

where $z_t = (1-t)x + ty$. This contradicts to $f(z_{t_\delta}, z_{t_\delta}) = 0$. □

Notice that although the strong upper sign property is a stronger notion than the upper sign property, but the strong upper sign property could be more useful.

Theorem 4.4. *If the bifunction $f : K \times K \to \mathbb{R}$ has the strong upper sign property. Then $M_L(f,K) \subseteq S(f,K)$.*

Proof . Take $x \in M_L(f,K)$. Then there exists $r > 0$ such that

$$f(y', x) \leq 0, \quad \forall y' \in K \cap B_r(x).$$

Let $y \in K$ and consider the open line segment between x and y, namely all $z_t = (1-t)x + ty$ such that $t \in (0,1)$. We can choose $\delta \in (0,1]$ such that $z_t \in K \cap B_r(x)$ for all $t \in (0,\delta)$. Thus, $f(z_t, x) \leq 0$ for all $t \in (0,\delta)$. Now the strong upper sign property of f implies that $f(x,y) \geq 0$. This means that $x \in S(f,K)$. This completes the proof. □

Notice that we relaxed the condition $f(x,x) = 0$ for every $x \in K$ which is frequently supposed in literature [4, 10] for deriving the relationship between $M_L(f,K)$ and $S(f,K)$.

In the following proposition we prove that the strong upper sign property is a weaker assumption to derive $M_L(f,K) \subseteq S(f,K)$ than the assumptions in literature. Hence, Theorem 4.4 is a real generalization of Lemma 2.1 in [4], Lemma 2.1 in [13] and Theorem 1 in [10]. Moreover, one can consider Theorem 3.3 as a special case of it.

Proposition 4.5. *Let $f : K \times K \to \mathbb{R}$ be a bifunction with the upper sign property. If f fulfills the condition (F4), then it has the strong upper sign property.*

Proof . Assume that $x \in K$ and f satisfies the upper sign property at $x \in K$. Let $y \in K$ and there exists $\delta \in (0,1]$ such that

$$f(x_t, x) \leq 0, \quad \forall t \in (0,\delta),$$

where $x_t = (1-t)x + ty$. If $f(x,y) < 0$ then there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in $(0,1)$ which converges to zero such that $f(x, z_{t_n}) < 0$ for all $n \in \mathbb{N}$. Moreover we can choose $n_0 \in \mathbb{N}$ such that $t_n \in (0,\delta)$ for all $n \geq n_0$. Therefore

$$f(z_s, x) \leq 0, \quad \forall s \in (0,1),$$

where $z_s = (1-s)x + s z_{t_{n_0}}$. It follows from the upper sign property of f that $f(x, z_{t_{n_0}}) \geq 0$ which is a contradiction. This completes the proof. □

The next example shows the importance of Theorem 4.4.

Example 4.6. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} y, & x > 0, y < 0 \\ 0, & x = y \\ xy, & x < 0, y > 0 \\ 1, & \text{otherwise.} \end{cases}$$

Let $x \in \mathbb{R}$. For every $y \in \mathbb{R}$, there is no $\delta \in (0,1)$ such that $f(z_t, x) \leq 0$ for all $t \in (0,\delta)$ where $z_t = (1-t)x + ty$. Therefore f has the strong sign property at x. Thus by Theorem 4.4, we have $M_L(f,\mathbb{R}) \subseteq S(f,\mathbb{R})$.

Note that f doesn't satisfy condition (F4) and consequently (F1), (F2) and (F3). For example if you take $x = 1$ and $y = -1$, then $f(1,-1) = -1 < 0$. But for every sequence $(t_n)_{n\in\mathbb{N}}$ which converges to zero, there exists $n_0 \in \mathbb{N}$ such that

$$f(1, (1-t_n)1 + t_n(-1)) = 1 > 0, \quad \forall n \geq n_0.$$

Therefore by Theorem 3.3, Lemma 2.1 in [4], Lemma 2.1 in [13] and Theorem 1 in [10], we cannot obtain $M_L(f,\mathbb{R}) \subseteq S(f,\mathbb{R})$.

5. Strong μ-upper sign property

The notion of the relaxed μ-quasimonotonicity in the setting of set-valued mappings has been considered by Bai and Hadjisavvas [3]. They established a link between the solution set of the variational inequality and the solution set of a suitable μ-Minty problem. Recently, this approach has been investigated also for equilibrium problems.

Definition 5.1. [10] Let $f : K \times K \to \mathbb{R}$ be a given bifunction and $\mu \geq 0$ be fixed. A local μ-Minty solution is a point $x \in K$ for which there exists $r > 0$ such that

$$f(y,x) \leq \mu(d(x,y))^2 \quad \text{for all} \quad y \in K \cap B_r(x).$$

The solution set of the all local μ-Minty solutions is denoted by $M_L^{\mu}(f,K)$.

Notice that if $\mu = 0$, then $M_L^0(f,K) = M_L(f,K)$ and if $\mu < \mu'$, then $M_L^{\mu}(f,K) \subseteq M_L^{\mu'}(f,K)$.

Definition 5.2. [10] Let $f : K \times K \to \mathbb{R}$ be a given bifunction and $\mu \geq 0$ be fixed. We say that f has the μ-upper sign property (with respect to the first variable) at $x \in K$ if there exists $r > 0$ such that for every $y \in K \cap B_r(x)$ the following implication holds

$$f(z_t,x) \leq \mu(d(z_t,x))^2, \quad \forall t \in (0,1) \quad \Rightarrow \quad f(x,y) \geq 0$$

where $z_t = (1-t)x + ty$.

In order to show that $M_L^{\mu}(f,K) \subseteq S(f,K)$, in the following, we introduce the strong μ-upper sign property.

Definition 5.3. Let $f : K \times K \to \mathbb{R}$ be a given bifunction and $\mu \geq 0$ be fixed. We say that f has the strong μ-upper sign property (with respect to the first variable) at $x \in K$ if for all $y \in K$ the following implication holds

$$\exists \delta \in (0,1] \ : \ f(z_t,x) \leq \mu(d(z_t,x))^2, \quad \forall t \in (0,\delta) \quad \Rightarrow \quad f(x,y) \geq 0$$

where $z_t = (1-t)x + ty$.

Remark 5.4. 1) It must be mentioned that if for all $y \in K$, there exists $\delta = 1$ satisfying $f(z_t,x) \leq \mu(d(z_t,x))^2$ for all $t \in (0,\delta)$, then f satisfies the μ-upper sign property if and only if f fulfills the μ-upper sign property.
2) If $\mu > \mu'$, every bifunction with the strong μ-upper sign property has the strong μ'-upper sign property and the strong upper sign property coincides with the strong 0-upper sign property.
3) As it was shown in Lemma 4 of [10], under a mild assumption of convexity, the μ-upper sign property is a kind of weak continuity. It is easy to check that under some assumptions of convexity for the bifunction f, the strong μ-upper sign property is also a kind of weak continuity.

By the same lines given in Theorem 4.4 and Proposition 4.5, we can get the following results.

Theorem 5.5. Let $\mu \geq 0$ be fixed and $f : K \times K \to \mathbb{R}$ be a given bifunction with the strong μ-upper sign property. Then $M_L^{\mu}(f,K) \subseteq S(f,K)$.

Proposition 5.6. Let $\mu \geq 0$ be fixed and $f : K \times K \to \mathbb{R}$ be a bifunction with the μ-upper sign property. If condition (F4) holds, then f has the strong μ-upper sign property.

Corollary 5.7. Let $\mu \geq 0$ be fixed and $f : K \times K \to \mathbb{R}$ be a given bifunction with the μ-upper sign property. If f satisfies condition (F4), then $M_L^{\mu}(f,K) \subseteq S(f,K)$.

6. Existence results

In this section, we use the KKM theory to prove the non-emptiness of $M_L^\mu(f,K)$ for a suitable $\mu \geq 0$ and then we apply Theorem 5.5 to obtain the non-emptiness of $S(f,K)$. For every $x \in K$ consider the set-valued map $F_\mu : K \rightrightarrows K$ as following

$$F_\mu(x) = \{y \in K \ : \ f(x,y) \leq \mu(d(x,y))^2\}.$$

Theorem 6.1. *Assume that $\mu \geq 0$ is fixed and $f : K \times K \to \mathbb{R}$ is a relaxed μ-quasimonotone bifunction which is not properly relaxed μ-quasimonotone. If the set $F_\mu(x)$ is closed and convex for every $x \in K$, then $M_L^\mu(f,K)$ is nonempty.*

Proof . Let $\mu > 0$. Since f is not properly relaxed μ-quasimonotone, there exist $x_1, \dots, x_n \in K$ and $\bar{x} \in conv\{x_1, \dots, x_n\}$ such that for all $i \in \{1, \dots, n\}$

$$f(x_i, \bar{x}) > \mu(d(x_i, \bar{x}))^2.$$

From the the fact that $F_\mu(x_i)$ is closed for every $i \in \{1, \dots, n\}$, the set $\{y \in K \ : \ f(x_i, y) > \mu(d(x_i, y))^2\}$ which includes $\bar{x}$ is open. So there exists $r > 0$ such that for every fixed $y \in K \cap B_r(\bar{x})$ we have

$$f(x_i, y) > \mu(d(x_i, y))^2 > 0, \quad i \in \{1, \dots, n\}.$$

It follows from the relaxed μ-quasimonotonicity of f that

$$f(y, x_i) \leq \mu(d(y, x_i))^2, \quad i \in \{1, \dots, n\}.$$

For every $y \in B_r(\bar{x}) \cap K$, the set $F_\mu(y)$ is convex, thus

$$f(y, \bar{x}) \leq \mu(d(y, \bar{x}))^2.$$

So $\bar{x} \in M_L^\mu(f,K)$.

Now, let $\mu = 0$. In this case, since $F_0(x) = lev(f,x)$ ($lev(f,x) = \{y \in K \ : \ f(x,y) \leq 0\}$ for every $x \in K$), the proof follows from Theorem 3 in [10]. □

Remark 6.2. 1) Notice that in [10], the same result was obtained under the convexity of $lev(f,x)$ for every $x \in K$, instead of using the convexity of $F_\mu(x)$ for every $x \in K$.
2) The result of Theorem 5 in [10] was obtained in the framework of reflexive Banach space where we have to assume that $F_\mu(x)$ is weakly closed for every $x \in K$ (however it was not mentioned in Theorem 5 of [10]). According to the fact that every closed and convex set is weakly closed, the assumption of convexity of $F_\mu(x)$ for every $x \in K$ is more useful in Theorem 5 of [10].

Corollary 6.3. Assume that $\mu \geq 0$ is fixed and $f : K \times K \to \mathbb{R}$ is a relaxed μ-quasimonotone bifunction which is not properly relaxed μ-quasimonotone. Suppose that f has the strong μ-upper sign property and the set $F_\mu(x)$ is closed for every $x \in K$. Then $S(f,K)$ is nonempty.

It is well known that Fan [11] in 1961 extended the famous Knaster–Kuratowski–Mazurkiewicz Theorem [19] (known also as the KKM Theorem, or the three Polish Lemma). The KKM theory is the wide area of the nonlinear analysis that could provide key tools and techniques for the study of equilibrium problems. In the sequel, we use the following lemma by Fan [12].

Definition 6.4. Let K be a subset of a Hausdorff topological vector space X. A set-valued map $\Gamma : K \rightrightarrows X$ is called a KKM map if for any $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in K$

$$conv\{x_1, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} \Gamma(x_i).$$

Lemma 6.5. (Fan-KKM lemma)[12] Let K be a nonempty subset of a Hausdorff topological vector space X and $\Gamma : K \rightrightarrows X$ be a KKM mapping with closed values. Assume that there exists a nonempty compact convex subset B of K such that $\cap_{x \in B} \Gamma(x)$ is compact. Then

$$\cap_{x \in K} \Gamma(x) \neq \emptyset.$$

Note that f is properly relaxed μ-quasimonotone if and only if F_μ is a KKM-map.

Corollary 6.6. *Let $\mu \geq 0$ be fixed. Assume that K is compact and $f : K \times K \to \mathbb{R}$ is properly relaxed μ-quasimonotone. If $F_\mu(x)$ is closed for every $x \in K$, then there exists $\bar{x} \in K$ such that*

$$f(y, \bar{x}) \leq \mu(d(y, \bar{x}))^2, \quad \forall y \in K.$$

Namely, the set of μ-Miny solutions ($M^\mu(f, K)$) is non-empty.

Proof . It is easy to verify that $\cap_{x \in K} F_\mu(x) \subseteq M^\mu(f, K)$. So by Fan-KKM lemma we get the result. □

The next theorem is the main result of the paper.

Theorem 6.7. *Let $\mu \geq 0$ be fixed. Assume that $f : K \times K \to \mathbb{R}$ be a bifunction satisfing the following conditions:*

(i) f is relaxed μ-quasimonotone.

(ii) f has the strong μ-upper sign property.

(iii) $F_\mu(x)$ is closed for every $x \in K$.

(iv) $F_\mu(x)$ is convex for every $x \in K$.

Moreover suppose that the following coercivity condition holds:
(C) there exist a nonempty compact subset $D \subseteq K$ and a non-empty convex compact subset $B \subseteq K$ such that for each $x \in K \backslash D$, there exists $y \in B$ such that $f(x, y) > \mu(d(x, y))^2$.
Then $S(f, K)$ is non-empty.

Proof . We consider two cases:
a) f is relaxed μ-quasimonotone bifunction which is not properly relaxed μ-quasimonotone. In this case by Theorem 6.1 and Theorem 5.5 we derive that $S(f, K)$ is non-empty.
b) f is properly relaxed μ-quasimonotone. So F_μ is a KKM map. The coercivity condition (C) implies that

$$\cap_{x \in B} F_\mu(x) \subseteq D \cap K = D,$$

which implies that $\cap_{x \in B} F_\mu(x)$ is compact. Thus by Fan-KKM Lemma, we conclude that $\cap_{x \in K} F_\mu(x) \neq \emptyset$. But, considering the fact that $\cap_{x \in B} F_\mu(x) \subseteq M^\mu(f, K)$, it follows that $M^\mu(f, K) \neq \emptyset$. Now, the strong μ-upper sign property of f implies that $S(f, K) \neq \emptyset$. □

Remark 6.8. 1) We didn't assume any condition on $f(x,x)$ where $x \in K$. Indeed, we relaxed the condition $f(x,x) = 0$ for every $x \in K$ which is used in [4, 10] and the condition $f(x,x) \geq 0$ which is used in [13].
2) No technical condition was supposed on f. In fact, the conditions (F1)-(F3) which are frequently used in literature were relaxed.
3) Existence results in this paper was obtained in a metric linear space, where we use a stronger coercivity condition (C) than the one used in [10]. We know that every reflexive Banach space is a metric linear space, but the converse is not true. For instance, a Fréchet space is a metric linear space which is not a normed space.

References

[1] D. Aussel, J. Cotrina and A. Iusem, *Existence results for quasi-equilibrium problems*, Preprint, serie A 751 (2014), IMPA Instituto de Matemática Pura e Aplicada.
[2] D. Aussel and N. Hadjisavvas, *On quasimonotone variational inequalities*, J. Optim. Theory Appl. 121 (2004) 445–450.
[3] M.R. Bai and N. Hadjisavvas, *Relaxed quasimonotone operators and relaxed quasiconvex functions*, J. Optim. Theory Appl. 138 (2008) 329–339.
[4] M. Bianchi and R. Pini, *Coercivity conditions for equilibrium problems*, J. Optim. Theory Appl. 124 (2005) 79–92.
[5] M. Bianchi and S. Schaible, *Generalized monotone bifunctions and equilibrium problems*, J. Optim. Theory Appl. 90 (1996) 31–43.
[6] G. Bigi, M. Castellani and G. Kassay, *A dual view of equilibrium problems*, J. Math. Anal. Appl. 342 (2008) 17–26.
[7] G. Bigi, M. Castellani, M. Pappalardo and M. Passacantando, *Existence and solution methods for equilibria*, Eur. J. Oper. Res. 227 (2013) 1–11.
[8] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student 63 (1994) 123–145.
[9] M. Castellani and M. Giuli, *On equivalent equilibrium problems*, J. Optim. Theory Appl. 147 (2010) 157–168.
[10] M. Castellani and M. Giuli, *Refinements of existence results for relaxed quasimonotone equilibrium problem*, J. Glob. Optim. 57 (2013) 1213–1227.
[11] K. Fan, *A minimax inequality and applications*, In: Shisha, O. (ed.) Inequalities III. Academic Press, New York (1972) 103–113.
[12] K. Fan, *Some properties of convex sets related to fixed point theorems*, Math. Ann. 266 (1984) 519–537.
[13] A.P. Farajzadeh and J. Zafarani, *Equilibrium problems and variational inequalities in topological vector spaces*, Optimization 59 (2010) 485–499.
[14] A.P. Farajzadeh, S. Jafari and C-T. Pang, *On η-upper sign property and upper sign continuity and their applications in equilibrium-like problems*, Abstr. Appl. Anal. 2014 (2014) 1–6.
[15] F. Flores-Bazán, *Existence theorems for generalized noncoercive equilibrium problems: the quasiconvex case*, SIAM J. Optim. 11 (2000) 675–690.
[16] N. Hadjisavvas, *Continuity and maximality properties of pseudomonotone operators*, J. Conv. Anal. 10 (2003) 459–469.
[17] A.N. Iusem, G. Kassay and W. Sosa, *On certain conditions for the existence of solutions of equilibrium problems*, Math. Program. 116 (2009) 259–273.
[18] A.N. Iusem and W. Sosa, *New existence results for equilibrium problems*, Nonlinear Anal. 52 (2003) 621–635.
[19] B. Knaster, C. Kuratowski and S. Mazurkiewicz, *Ein Beweies des Fixpunktsatzes fr N Dimensionale Simplexe*, Fund. Math. 14 (1929) 132–137.
[20] G.J. Minty, *On the generalization of a direct method of the calculus of variations*, Bull. Am. Math. Soc. 73 (1967) 315–321.