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Hermite-Hadamard inequalities for $\mathbb{B}$-convex and $\mathbb{B}^{-1}$-convex functions

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Abstract

Hermite-Hadamard inequality is one of the fundamental applications of convex functions in Theory of Inequality. In this paper, Hermite-Hadamard inequalities for $\mathbb{B}$-convex and $\mathbb{B}^{-1}$-convex functions are proven.

Keywords: Hermite-Hadamard Inequality; $\mathbb{B}$-convex functions; $\mathbb{B}^{-1}$-convex functions; abstract convexity.
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1. Introduction

Theory of Inequality is one of the most important application fields of convex analysis. A great number of inequalities can be obtained by taking advantage of convexity concept. Hermite-Hadamard inequality is one of the most important applications within these inequalities. Firstly, for convex functions Hermite-Hadamard inequality was proven by Hermite in [11] and then, ten years later, Hadamard rediscovered its left-hand side in [10] (see also [8] for the historical considerations), then examined in numerous article, like [8, 14]. Moreover, Hermite-Hadamard inequalities for different types of abstract convex functions were studied in [1, 2, 3, 6, 8, 12, 15, 16, 18, 20].

In this article, Hermite-Hadamard inequalities for $\mathbb{B}$-convex and $\mathbb{B}^{-1}$-convex functions which are new kinds of abstract convex functions are proven.

In section of preliminaries, we mention some definitions and theorems of $\mathbb{B}$-convexity and $\mathbb{B}^{-1}$-convexity which will be necessary in the sequel (Section 2.1 and Section 2.2), also recall Hermite-Hadamard inequalities of some types of abstract convex functions (Section 2.3). In Section 3 and

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Section 4, we prove Hermite-Hadamard inequalities for $\mathbb{B}$-convex functions and $\mathbb{B}^{-1}$-convex functions, respectively.

In this paper, we will use the following notations:

$\mathbb{Z}^-$	is the set of negative integers;
$\mathbb{R}_*$	is $\mathbb{R}\setminus\{0\}$;
$\mathbb{R}^n$	is the n-dimensional vector space;
$\mathbb{R}^n_+$	$\{(x_1,\ldots,x_n)\in\mathbb{R}^n : x_i\geq 0,\ i=1,2,\ldots,n\}$;
$\mathbb{R}^n_{++}$	$\{(x_1,\ldots,x_n)\in\mathbb{R}^n : x_i> 0,\ i=1,2,\ldots,n\}$;
$Co^r(A)$	is the r-convex hull of A;
$Co^\infty(A)$	is the $\mathbb{B}$-polytope of A;
$Co^{-\infty}(A)$	is the $\mathbb{B}^{-1}$-polytope of A;
$epi(f)$	$\{(x,\mu)\,\lvert x\in U,\ \mu\in\mathbb{R},\ \mu\geq f(x)\}$;
$epi^*(f)$	$\{(x,\mu)\,\lvert x\in U,\ \mu\in\mathbb{R}_*,\ \mu\geq f(x)\}$;
$\overset{m}{\underset{i=1}{\vee}} x^{(i)}$	$\overset{m}{\underset{i=1}{\vee}} x^{(i)} = \left(\max\left\{x_1^{(1)},x_1^{(2)},\ldots,x_1^{(m)}\right\},\ldots,\max\left\{x_n^{(1)},x_n^{(2)},\ldots,x_n^{(m)}\right\}\right)$;
$\overset{m}{\underset{i=1}{\wedge}} x^{(i)}$	$\overset{m}{\underset{i=1}{\wedge}} x^{(i)} = \left(\min\left\{x_1^{(1)},x_1^{(2)},\ldots,x_1^{(m)}\right\},\ldots,\min\left\{x_n^{(1)},x_n^{(2)},\ldots,x_n^{(m)}\right\}\right)$.

2. Preliminaries

2.1. $\mathbb{B}$-convexity

Let $r\in\mathbb{N}$, $\varphi_r:\mathbb{R}\to\mathbb{R}$, $\varphi_r(x)=x^{2r+1}$ and $\Phi_r:\mathbb{R}^n\to\mathbb{R}^n$, $\Phi_r(x)=\Phi_r(x_1,x_2,\ldots,x_n)=(\varphi_r(x_1),\varphi_r(x_2),\ldots,\varphi_r(x_n))$. For a finite nonempty set $A=\left\{x^{(1)},x^{(2)},\ldots,x^{(m)}\right\}\subset\mathbb{R}^n$, the r-convex hull of A, denoted as $Co^r(A)$, is given by

$$Co^r(A)=\left\{\Phi_r^{-1}\left(\sum_{i=1}^m t_i\Phi_r(x^{(i)})\right) : t_i\geq 0, \sum_{i=1}^m t_i=1\right\}.$$

Definition 2.1. [5] The Kuratowski-Painleve upper limit of the sequence of sets $(Co^r(A))_{r\in\mathbb{N}}$, denoted by $Co^\infty(A)$ where A is a finite subset of $\mathbb{R}^n$, is called $\mathbb{B}$-polytope of A.

Definition 2.2. A subset U of $\mathbb{R}^n$ is $\mathbb{B}$-convex if for all finite subset $A\subset U$ the $\mathbb{B}$-polytope $Co^\infty(A)$ is contained in U.

In $\mathbb{R}^n_+$, $\mathbb{B}$-convex set is defined in a different way [5]:

A subset U of $\mathbb{R}^n_+$ is $\mathbb{B}$-convex if and only if for all $x^{(1)},x^{(2)}\in U$ and all $\lambda\in[0,1]$ one has $\lambda x^{(1)}\vee x^{(2)}\in U$.

Here, we denote the least upper bound with respect to the coordinate-wise order relation of $x^{(1)},x^{(2)},\ldots,x^{(m)}\in\mathbb{R}^n$ by $\overset{m}{\underset{i=1}{\vee}} x^{(i)}$, that is:

$$\overset{m}{\underset{i=1}{\vee}} x^{(i)} = \left(\max\left\{x_1^{(1)},x_1^{(2)},\ldots,x_1^{(m)}\right\},\ldots,\max\left\{x_n^{(1)},x_n^{(2)},\ldots,x_n^{(m)}\right\}\right)$$

where, $x_j^{(i)}$ denotes jth coordinate of the point $x^{(i)}$.

Remark 2.3. In $\mathbb{R}_+$, $\mathbb{B}$-convex sets are intervals because of definition.

Furthermore, in [5, 13], the definition of $\mathbb{B}$-convex functions is given as follows:

Definition 2.4. Let $U \subset \mathbb{R}^n$. A function $f : U \to \mathbb{R} \bigcup \{\pm\infty\}$ is called a $\mathbb{B}$-convex function if $epi\,(f) = \{(x, \mu)\,|x \in U, \mu \in \mathbb{R}, \mu \geq f\,(x)\}$ is a $\mathbb{B}$-convex set.

The following theorem provides a sufficient and necessary condition for $\mathbb{B}$-convex functions in $\mathbb{R}^n_+$ [5, 13].

Theorem 2.5. Let $U \subset \mathbb{R}^n_+, f : U \to \mathbb{R}_+ \cup \{+\infty\}$. The function f is $\mathbb{B}$-convex if and only if U is a $\mathbb{B}$-convex set and for all $x, y \in U$ and all $\lambda \in [0, 1]$ the following inequality holds:

$$f\,(\lambda x \vee y) \leq \lambda f\,(x) \vee f\,(y)\ . \tag{2.1}$$

2.2. $\mathbb{B}^{-1}$-convexity

For $r \in \mathbb{Z}^-$, the map $x \to \varphi_r(x) = x^{2r+1}$ is a homeomorphism from $\mathbb{R}_* = \mathbb{R} \setminus \{0\}$ to itself; $x = (x_1, x_2, \ldots, x_n) \to \Phi_r(x) = (\varphi_r(x_1), \varphi_r(x_2), \ldots, \varphi_r(x_n))$ is homeomorphism from $\mathbb{R}^n_*$ to itself.

For a finite nonempty set $A = \{x^{(1)}, x^{(2)}, \ldots, x^{(m)}\} \subset \mathbb{R}^n_*$ the Φ_r-convex hull (shortly r-convex hull) of A, which we denote $Co^r(A)$ is given by

$$Co^r(A) = \left\{ \Phi_r^{-1}\left(\sum_{i=1}^{m} t_i \Phi_r(x^{(i)})\right) : t_i \geq 0, \sum_{i=1}^{m} t_i = 1 \right\}\ .$$

We denote by $\overset{m}{\underset{i=1}{\wedge}} x^{(i)}$ the greatest lower bound with respect to the coordinate-wise order relation of $x^{(1)}, x^{(2)}, \ldots, x^{(m)} \in \mathbb{R}^n$, that is:

$$\overset{m}{\underset{i=1}{\wedge}} x^{(i)} = \left(\min\left\{x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(m)}\right\}, \ldots, \min\left\{x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(m)}\right\}\right)$$

where, $x_j^{(i)}$ denotes jth coordinate of the point $x^{(i)}$.

Thus, we can define $\mathbb{B}^{-1}$-polytopes as follows:

Definition 2.6. [4] The Kuratowski-Painleve upper limit of the sequence of sets $\{Co^r(A)\}_{r \in \mathbb{Z}^-}$, denoted by $Co^{-\infty}(A)$ where A is a finite subset of $\mathbb{R}^n_*$, is called $\mathbb{B}^{-1}$-polytope of A.

The definition of $\mathbb{B}^{-1}$-polytope can be expressed in the following form in $\mathbb{R}^n_{++}$.

Theorem 2.7. [4] For all nonempty finite subsets $A = \{x^{(1)}, x^{(2)}, \ldots, x^{(m)}\} \subset \mathbb{R}^n_{++}$ we have

$$Co^{-\infty}(A) = \lim_{r \to -\infty} Co^r(A) = \left\{ \overset{m}{\underset{i=1}{\wedge}} t_i x^{(i)} : t_i \geq 1, \min_{1 \leq i \leq m} t_i = 1 \right\}\ .$$

Next, we give the definition of $\mathbb{B}^{-1}$-convex sets.

Definition 2.8. [4] A subset U of $\mathbb{R}^n_*$ is called a $\mathbb{B}^{-1}$-convex if for all finite subsets $A \subset U$ the $\mathbb{B}^{-1}$-polytope $Co^{-\infty}(A)$ is contained in U.

By Theorem 2.7, we can reformulate the above definition for subsets of $\mathbb{R}^n_{++}$:

Theorem 2.9. [4] A subset U of $\mathbb{R}^n_{++}$ is $\mathbb{B}^{-1}$-convex if and only if for all $x^{(1)}, x^{(2)} \in U$ and all $\lambda \in [1, \infty)$ one has $\lambda x^{(1)} \wedge x^{(2)} \in U$.

Remark 2.10. As a result of Theorem 2.9, we can say that $\mathbb{B}^{-1}$-convex sets in $\mathbb{R}_{++}$ are positive intervals.

Definition 2.11. [13] For $U \subset \mathbb{R}_*^n$, a function $f : U \to \mathbb{R}_*$ is called a $\mathbb{B}^{-1}$-convex function if $epi^*(f) = \{(x,\mu) \,|x \in U, \mu \in \mathbb{R}_*, \mu \geq f(x)\}$ is a $\mathbb{B}^{-1}$-convex set.

In $\mathbb{R}_{++}^n$, we can give the following fundamental theorem which provides a sufficient and necessary condition for $\mathbb{B}^{-1}$-convex functions [13].

Theorem 2.12. Let $U \subset \mathbb{R}_{++}^n$ and $f : U \to \mathbb{R}_{++}$. The function f is $\mathbb{B}^{-1}$-convex if and only if the set U is $\mathbb{B}^{-1}$-convex and one has the inequality

$$f(\lambda x \wedge y) \leq \lambda f(x) \wedge f(y) \tag{2.2}$$

for all $x, y \in U$ and all $\lambda \in [1, +\infty)$.

2.3. Abstract Convexity Classes and Hermite-Hadamard inequalities.

Recall that for a function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$, which is convex on $[a,b]$, we have the following

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{1}{2}\left(f(a) + f(b)\right) \ .$$

This inequality is well known as the Hermite-Hadamard inequality. Moreover, for different classes of abstract convex functions, Hermite-Hadamard inequalities which are suitable for these function classes are obtained. For example:

1) A function $f : [a,b] \to (0, +\infty)$ is said to be log-convex or multiplicatively convex if for all $x, y \in [a,b]$ and $\lambda \in [0,1]$ we have ([17])

$$f(\lambda x + (1-\lambda)y) \leq [f(x)]^{\lambda}\,[f(y)]^{1-\lambda}$$

and for f, we have that

$$f\left(\frac{a+b}{2}\right) \leq \exp\left[\frac{1}{b-a}\int_a^b f(t)\,dt\right] \leq \sqrt{(f(a) + f(b))}$$

which is an Inequality of Hermite-Hadamard for log-convex functions.

2) A function $p : [a,b] \to (0, +\infty)$ is said to be p-function if for all $x, y \in [a,b]$ and $\lambda \in [0,1]$ one has the inequality ([19])

$$p(\lambda x + (1-\lambda)y) \leq p(x) + p(y)$$

and Hermite-Hadamard inequality for the function p is

$$p\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_a^b p(t)\,dt \leq 2\left(p(a) + p(b)\right) \ .$$

3) A positive function f is quasi-convex on a real interval $[a,b]$ if for all $x, y \in [a,b]$ and $\lambda \in [0,1]$ we have ([7])

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\} \ .$$

We know that the class of p-functions consists of the class of nonnegative quasi-convex functions. Hence, the Hermite-Hadamard inequality for p-functions is also valid for nonnegative quasi-convex functions. Additionally, different inequalities for Jensen-quasi-convex functions which is a special form of quasi-convex functions were studied:

A function $f : [a,b] \to (0,+\infty)$ is Jensen or J-quasi-convex if for all $x, y \in [a,b]$ one has the inequality ([7])

$$f\left(\frac{x+y}{2}\right) \leq \max\left\{f(x), f(y)\right\}$$

and Hermite-Hadamard inequality for J-quasi-convex functions is

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{2(b-a)}\int_a^b \left|f(t) - f(a+b-t)\right| dt\ .$$

4) A function $f : [a,b] \to (0,+\infty)$ is said to belong to the class Q(I) if it is nonnegative and for all $x, y \in [a,b]$ and $\lambda \in (0,1)$, satisfies the inequality ([9])

$$f(\lambda x + (1-\lambda)y) \leq \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}$$

and for the Q(I) class of functions, one has the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{4}{b-a}\int_a^b f(t)\,dt$$

that is Hermite-Hadamard inequality for the Q(I) class of functions.

Accordingly, as we take importance of Hermite-Hadamard inequality and its applications into consideration, we prove the Hermite-Hadamard inequalities for $\mathbb{B}$-convex and $\mathbb{B}^{-1}$-convex functions which are new abstract convex function classes in this paper.

3. Hermite-Hadamard inequality for $\mathbb{B}$-convex Functions.

Theorem 3.1. *Let $f : [a,b] \subset \mathbb{R}_+ \to \mathbb{R}_+$ be a $\mathbb{B}$-convex function. Then one has the inequalities*

$$\frac{1}{b-a}\int_a^b f(t)\,dt \leq \begin{cases} f(a), & f(a) \geq f(b) \\ \frac{b\left([f(a)]^2 + [f(b)]^2\right) - 2af(a)f(b)}{2(b-a)f(b)}, & f(a) < f(b) \end{cases}. \tag{3.1}$$

Proof . Since $a \leq b$, for all $\lambda \in [0,1]$ we have $\max\{\lambda a, b\} = b$. Then, from the inequality (2.1) in Theorem 2.5

$$f(b) = f(\max\{\lambda a, b\}) \leq \max\{\lambda f(a), f(b)\}$$

is obtained. Since it is valid for all functions, there is no point in examining this case.

Thus, let us examine the case of $\max\{a, \lambda b\}$. If we make the substitution $t = \lambda b$, we obtain that

$$\begin{aligned}\int_0^1 f(\max\{a,\lambda b\})\,d\lambda &= \int_0^{a/b} f(\max\{a,\lambda b\})\,d\lambda + \int_{a/b}^1 f(\max\{a,\lambda b\})\,d\lambda \\ &= \int_0^{a/b} f(a)\,d\lambda + \int_{a/b}^1 f(\lambda b)\,d\lambda \\ &= f(a)\frac{a}{b} + \frac{1}{b}\int_a^b f(t)\,dt\ .\end{aligned}$$

From $\mathbb{B}$-convexity of the function f, following inequality holds

$$\int_0^1 f(\max\{a, \lambda b\})\, d\lambda \le \int_0^1 \max\{f(a), \lambda f(b)\}\, d\lambda \ .$$

For the right-hand side of the inequality, there are the following two possible cases:
1) It can be $f(a) \ge f(b)$. In this case, we have

$$\int_0^1 \max\{f(a), \lambda f(b)\}\, d\lambda = \int_0^1 f(a)\, d\lambda = f(a) \ .$$

Hence, we obtain that

$$f(a)\frac{a}{b} + \frac{1}{b}\int_a^b f(t)\, dt \le f(a) \ \Rightarrow \ \frac{1}{b-a}\int_a^b f(t)\, dt \le f(a) \ .$$

2) If $f(a) < f(b)$, we deduce that

$$\begin{aligned} \int_0^1 \max\{f(a), \lambda f(b)\}\, d\lambda &= \int_0^{f(a)/f(b)} \max\{f(a), \lambda f(b)\}\, d\lambda \\ &\quad + \int_{f(a)/f(b)}^1 \max\{f(a), \lambda f(b)\}\, d\lambda \\ &= \int_0^{f(a)/f(b)} f(a)\, d\lambda + \int_{f(a)/f(b)}^1 \lambda f(b)\, d\lambda \\ &= \frac{(f(a))^2 + (f(b))^2}{2f(b)} \ . \end{aligned}$$

Thus,

$$\frac{1}{b-a}\int_a^b f(t)\, dt \le \frac{b\left([f(a)]^2 + [f(b)]^2\right) - 2af(a)f(b)}{2(b-a)f(b)}$$

is obtained. Thence, Hermite-Hadamard inequality for $\mathbb{B}$-convex functions is

$$\frac{1}{b-a}\int_a^b f(t)\, dt \le \begin{cases} f(a), & f(a) \ge f(b) \\ \frac{b\left([f(a)]^2+[f(b)]^2\right)-2af(a)f(b)}{2(b-a)f(b)}, & f(a) < f(b) \ . \end{cases}$$

□

4. Hermite-Hadamard inequality for $\mathbb{B}^{-1}$-convex Functions

Theorem 4.1. *Suppose $f : [a,b] \subset \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ is a $\mathbb{B}^{-1}$-convex function. Then the following inequalities hold*

$$\frac{1}{b-a}\int_a^b f(t)\, dt \le \begin{cases} \frac{2bf(a)f(b) - a\left[(f(a))^2 + (f(b))^2\right]}{2(b-a)f(a)}, & 1 \le \frac{f(b)}{f(a)} < \frac{b}{a} \\ \frac{f(a)(a+b)}{2a}, & \frac{b}{a} \le \frac{f(b)}{f(a)} \ . \end{cases} \tag{4.1}$$

Proof . Because of $a \le b$, for all $\lambda \in [1,+\infty)$ we have $\min\{a,\lambda b\} = a$. Hence, using the inequality (2.2) in Theorem 2.12, for all $\mathbb{B}^{-1}$-convex functions f it can be seen that

$$f(a) = f(\min\{a,\lambda b\}) \le \min\{f(a), \lambda f(b)\} \ .$$

Namely, for all $\lambda \in [1,+\infty)$ we get $f(a) \le \lambda f(b)$. Since it is valid for all λ, also it holds for $\lambda = 1$. Because this investigation is provided for every $x, y \in [a,b]$, $x \le y$; we obtain that a $\mathbb{B}^{-1}$-convex function is monotone nondecreasing function.

Now, let us examine the case of $\min\{\lambda a, b\}$. Then, we have

$$\begin{aligned} \int_1^{+\infty} f(\min\{\lambda a,b\})\, d\lambda &= \int_1^{b/a} f(\min\{\lambda a,b\})\, d\lambda + \int_{b/a}^{+\infty} f(\min\{\lambda a,b\})\, d\lambda \\ &= \int_1^{b/a} f(\lambda a)\, d\lambda + \int_{b/a}^{+\infty} f(b)\, d\lambda \ . \end{aligned}$$

Here, $\int_{b/a}^{+\infty} f(b)\, d\lambda = +\infty$ and a similar case occurs in the right side of the inequality. Therefore, since $\int_1^{+\infty} f(\min\{\lambda a,b\})\, d\lambda = \int_1^{+\infty} \min\{\lambda f(a), f(b)\}\, d\lambda = +\infty$, the inequality holds when we take the region of integration as $[1,+\infty)$. Let's get the region of integration as $\left[1, \frac{b}{a}\right]$. Thus, from the $\mathbb{B}^{-1}$-convexity of f, we deduce that

$$\int_1^{b/a} f(\min\{\lambda a,b\})\, d\lambda \le \int_1^{b/a} \min\{\lambda f(a), f(b)\}\, d\lambda \ .$$

If we make the substitution $t = \lambda a$, we obtain that

$$\begin{aligned} \int_1^{b/a} f(\min\{\lambda a,b\})\, d\lambda &= \int_1^{b/a} f(\lambda a)\, d\lambda \\ &= \frac{1}{a}\int_a^b f(t)\, dt \le \int_1^{b/a} \min\{\lambda f(a), f(b)\}\, d\lambda \ . \end{aligned}$$

To this inequality, there are two possibilities:

1) It can be $1 \le \frac{f(b)}{f(a)} < \frac{b}{a}$. Then, we have that

$$\begin{aligned} \int_1^{b/a} \min\{\lambda f(a), f(b)\}\, d\lambda &= \int_1^{\frac{f(b)}{f(a)}} \min\{\lambda f(a), f(b)\}\, d\lambda \\ &\quad + \int_{\frac{f(b)}{f(a)}}^{b/a} \min\{\lambda f(a), f(b)\}\, d\lambda \\ &= \int_1^{\frac{f(b)}{f(a)}} \lambda f(a)\, d\lambda + \int_{\frac{f(b)}{f(a)}}^{b/a} f(b)\, d\lambda \\ &= \frac{f(a)}{2}\frac{(f(b))^2 - (f(a))^2}{(f(a))^2} + f(b)\frac{bf(a) - af(b)}{af(a)} \\ &= \frac{2bf(a)f(b) - a(f(a))^2 - a(f(b))^2}{2af(a)} \ . \end{aligned}$$

Thereby, we get

$$\frac{1}{b-a}\int_a^b f(t)\, dt \le \frac{2bf(a)f(b) - a\left[(f(a))^2 + (f(b))^2\right]}{2(b-a)f(a)} \ .$$

2) If $\frac{f(b)}{f(a)} \geq \frac{b}{a}$, then we deduce that

$$\int_1^{b/a} \min\{\lambda f(a), f(b)\}\, d\lambda = \int_1^{b/a} \lambda f(a)\, d\lambda = f(a)\frac{b^2-a^2}{2a^2}.$$

Thus, we have the following inequalities:

$$\frac{1}{b-a}\int_a^b f(t)\, dt \leq \frac{f(a)(a+b)}{2a}.$$

Consequently, as we take all of the foregoing inequalities into consideration, Hermite-Hadamard inequality for $\mathbb{B}^{-1}$-convex functions is obtained as in the inequality (4.1). □

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