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An inexact alternating direction method with SQP regularization for the structured variational inequalities

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Abstract

In this paper, we propose an inexact alternating direction method with square quadratic proximal (SQP) regularization for the structured variational inequalities. The predictor is obtained via solving SQP system approximately under significantly relaxed accuracy criterion and the new iterate is computed directly by an explicit formula derived from the original SQP method. Under appropriate conditions, the global convergence of the proposed method is proved. We show the $O(1/t)$ convergence rate for the inexact SQP alternating direction method. We also reported some numerical results to illustrate the efficiency of the proposed method.

Keywords: Variational inequalities; monotone operator; square quadratic proximal method; logarithmic-quadratic proximal method; alternating direction method.
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1. Introduction

Let $\mathbb{R}$ stand for the real axis; and $\mathbb{R}^n_+ = \{x \in \mathbb{R}; x \geq 0\}, \mathbb{R}^n_{++} = \{x \in \mathbb{R}; x > 0\}$, denote the positive half-axis and strict positive half-axis, respectively.
Further, given $n \in \mathbb{N}$, put

$$\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n)^\top; x_1, \ldots, x_n \in \mathbb{R}_+\}$$

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and

$$\mathbb{R}^n_{++} = \{x = (x_1, \dots, x_n)^\top; x_1, \dots, x_n \in \mathbb{R}_{++}\}$$

where $(.)^\top$ denotes the transpose.
Finally, define

$$\Omega := \left\{(u,v)^\top; u \in \mathbb{R}^n_+, v \in \mathbb{R}^m_+, A_1u + A_2v = b\right\}$$

where $A_1 \in \mathbb{R}^{l\times n}$, $A_2 \in \mathbb{R}^{l\times m}, b \in \mathbb{R}^l$ are given matrices and vectors, respectively.
Consider the variational inequality problem:
find $x \in \Omega$ such that

$$(x' - x)^\top F(x) \geq 0, \quad \forall\, x' \in \Omega, \tag{1.1}$$

where

$$F(x) = (f_1(u), f_2(v))^\top. \tag{1.2}$$

and $f_1 : \mathbb{R}^n_+ \to \mathbb{R}^n$, $f_2 : \mathbb{R}^m_+ \to \mathbb{R}^m$ are given monotone operators. Studies and applications of such problems can be found in [14, 18, 19, 20, 21]. By attaching a Lagrange multiplier vector $\lambda \in \mathbb{R}^l$ to the linear constraints $A_1u + A_2v = b$, the problem (1.1)+(1.2) may be expressed as find $z \in \mathcal{Z}' := \mathbb{R}^n_+ \times \mathbb{R}^m_+ \times \mathbb{R}^l$ such that

$$(z' - z)^\top Q(z) \geq 0, \quad \forall z' \in \mathcal{Z}', \tag{1.3}$$

where

$$z = (u, v, \lambda)^\top,\; Q(z) = \left(f_1(u) - A_1^\top\lambda, f_2(v) - A_2^\top\lambda, A_1u + A_2v - b\right)^\top. \tag{1.4}$$

Various methods have been suggested to find the solution of problem (1.3)+(1.4). A popular approach is the alternating direction method (ADM) which was proposed by Gabay and Mercier [18] and Gabay [19]. Typically, problems in applications for example, network economics [32] and nonlinear mechanics [17, 20, 21] are quite large and are often solved by ADM. The ADM can reduce the scale of variational inequalities by decomposing the original problem into a series of subproblems with a lower scale (see [11, 20, 21, 27, 29, 37], for example). The classical proximal alternating directions method (PADM) is one of the attractive ADMs. From a given $(u^k, v^k, \lambda^k) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ \times \mathbb{R}^l$, the new iterate $(u^{k+1}, v^{k+1}, \lambda^{k+1})$ is obtained via solving the following problem

$$(u' - u)^\top\left\{f_1(u) - A_1^\top[\lambda^k - \beta(A_1u + A_2v^k - b)] + r(u - u^k)\right\} \geq 0, \quad \forall\;\; u' \in \mathbb{R}^n_+, \tag{1.5a}$$

$$(v' - v)^\top\left\{f_2(v) - A_2^\top[\lambda^k - \beta(A_1u^{k+1} + A_2v - b)] + s(v - v^k)\right\} \geq 0, \quad \forall\;\; v' \in \mathbb{R}^m_+, \tag{1.5b}$$

$$\lambda^{k+1} = \lambda^k - \beta(A_1u + A_2v - b). \tag{1.5c}$$

Here $r > 0, s > 0$ are given proximal parameters and $\beta > 0$ is a given penalty parameter for the linearly constrained equation $A_1u + A_2v - b = 0$. Several works have been concentrated on the generalization of the proximal algorithm replacing in the alternating directions method (1.5a)-(1.5b) the proximal term $r(u - u^k)$ and $s(v - v^k)$ by some nonlinear functionals. Very recently, some alternating direction methods with logarithmic-quadratic proximal regularization [3, 4, 5, 6, 7, 8, 9, 30, 36, 39] have been developed by substituting in the alternating directions method (1.5a)-(1.5b) the term $r(u - u^k)$ and $s(v - v^k)$ by $R[(u - u^k) + \mu(u^k - U_k^2u^{-1})]$ and $S[(v - v^k) + \mu(v^k - V_k^2v^{-1})]$, respectively. The predictor $\tilde{z}^k = (\tilde{u}^k, \tilde{v}^k, \tilde{w}^k, \tilde{\lambda}^k)$ in [3, 4, 5, 6, 7, 10, 30, 36, 39] is obtained via the following procedure: From a given $z^k = (u^k, v^k, \lambda^k) \in \mathbb{R}^n_{++} \times \mathbb{R}^m_{++} \times \mathbb{R}^l$, and $\mu \in (0, 1)$, $(\tilde{u}^k, \tilde{v}^k, \tilde{\lambda}^k)$ is obtained via solving the following system:

$$f_1(u) - A_1^\top\left[\lambda^k - H(A_1u + A_2v^k - b)\right] + R\left[(u - u^k) + \mu(u^k - U_k^2u^{-1})\right] = 0, \tag{1.6a}$$

$$f_2(v) - A_2^\top \left[\lambda^k - H(A_1 u + A_2 v - b)\right] + S\left[(v - v^k) + \mu(v^k - V_k^2 v^{-1})\right] = 0, \tag{1.6b}$$

$$\lambda^{k+1} = \lambda^k - H(A_1 u^k + A_2 v^k - b), \tag{1.6c}$$

where $H \in \mathbb{R}^{l\times l}$, $R \in \mathbb{R}^{n\times n}$, and $S \in \mathbb{R}^{m\times m}$ are symmetric positive definite matrices.

Define

$$\Omega := \left\{(u, v, w)^\top : \ u \in \mathbb{R}_+^{n_1},\ v \in \mathbb{R}_+^{n_2},\ w \in \mathbb{R}_+^{n_3},\ A_1 u + A_2 v + A_3 w = b\right\},$$

where $A_1 \in \mathbb{R}^{m\times n_1}, A_2 \in \mathbb{R}^{m\times n_2}, A_3 \in \mathbb{R}^{m\times n_3}, b \in \mathbb{R}^m$ are given matrices and vectors, respectively. In this paper, we consider the following structured variational inequality with three separable operators: find $y \in \Omega$ such that

$$(y' - y)^\top F(y) \geq 0, \qquad \forall\, y' \in \Omega, \tag{1.7}$$

where

$$F(y) = (f_1(u), f_2(v), f_3(w))^\top, \tag{1.8}$$

and $f_1 : \mathbb{R}_+^{n_1} \to \mathbb{R}^{n_1}$, $f_2 : \mathbb{R}_+^{n_2} \to \mathbb{R}^{n_2}$, $f_3 : \mathbb{R}_+^{n_3} \to \mathbb{R}^{n_3}$ are given monotone operators. By attaching a Lagrange multiplier vector $\lambda \in \mathbb{R}^m$ to the linear constraints $A_1 u + A_2 v + A_3 w = b$, the problem (1.7)+(1.8) can be explained in terms of finding $z \in \mathcal{Z} := \mathbb{R}_+^{n_1} \times \mathbb{R}_+^{n_2} \times \mathbb{R}_+^{n_3} \times \mathbb{R}^m$ such that

$$(z' - z)^\top Q(z) \geq 0, \quad \forall z' \in \mathcal{Z}, \tag{1.9}$$

where

$$z = (u, v, w, \lambda)^\top,\ Q(z) = \left(f_1(u) - A_1^\top\lambda, f_2(v) - A_2^\top\lambda, f_3(w) - A_3^\top\lambda, A_1 u + A_2 v + A_3 w - b\right)^\top. \tag{1.10}$$

The problem (1.9)+(1.10) is referred to as SVI_3.

By combining the ADM and parallel splitting augmented Lagrangian method [33], Cao *et al.* [12] proposed a new partial splitting augmented Lagrangian method for solving SVI_3. The predictor $\tilde{z}^k = (\tilde{u}^k, \tilde{v}^k, \tilde{w}^k, \tilde{\lambda}^k)$ in [12] is obtained via the following procedure: From a given $z^k = (u^k, v^k, w^k, \lambda^k) \in \mathcal{Z}$, $(\tilde{u}^k, \tilde{v}^k, \tilde{w}^k, \tilde{\lambda}^k)$ is obtained via solving the following system:

$$(u - \tilde{u}^k)^\top (f_1(\tilde{u}^k) - A_1^\top[\lambda^k - \beta H(A_1\tilde{u}^k + A_2 v^k + A_3 w^k - b)]) \geq 0, \quad \forall u \in \mathbb{R}_+^{n_1}, \tag{1.11a}$$

$$(v - \tilde{v}^k)^\top (f_1(\tilde{v}^k) - A_2^\top[\lambda^k - \beta H(A_1\tilde{u}^k + A_2\tilde{v}^k + A_3 w^k - b)]) \geq 0, \quad \forall v \in \mathbb{R}_+^{n_2}, \tag{1.11b}$$

$$(w - \tilde{w}^k)^\top (f_3(\tilde{w}^k) - A_3^\top[\lambda^k - \beta H(A_1\tilde{u}^k + A_2 v^k + A_3\tilde{w}^k - b)]) \geq 0, \quad \forall w \in \mathbb{R}_+^{n_3}, \tag{1.11c}$$

$$\tilde{\lambda}^k = \lambda^k - \beta H(A_1\tilde{u}^k + A_2\tilde{v}^k + A_3\tilde{w}^k - b). \tag{1.11d}$$

In this paper, we suggest that the complementarity subproblems arising in ADM (1.11a)-(1.11d) could be regularized by the square quadratic proximal (SQP) regularization, the SQP regularization forces the solutions of ADM subproblems to be interior points of $\mathbb{R}_+^{n_1}$, $\mathbb{R}_+^{n_2}$ and $\mathbb{R}_+^{n_3}$, respectively; thus the complementarity subproblems (1.11a), (1.11b) and (1.11c) reduce to three easier systems of nonlinear equations. More specifically, the iterative scheme of ADM with SQP regularization is as follows: From a given $z^k = (u^k, v^k, w^k, \lambda^k) \in \mathbb{R}_{++}^{n_1} \times \mathbb{R}_{++}^{n_2} \times \mathbb{R}_{++}^{n_3} \times \mathbb{R}^m$, $(\tilde{u}^k, \tilde{v}^k, \tilde{w}^k, \tilde{\lambda}^k)$ is obtained via solving the following system:

$$f_1(u) - A_1^\top[\lambda^k - \beta H(A_1 u + A_2 v + A_3 w - b)] + R_1[\frac{1}{2}(u - u^k) + \mu(u^k - U_k(\sqrt{u})^{-1})] = 0, \tag{1.12a}$$

$$f_2(v) - A_2^\top[\lambda^k - \beta H(A_1 u + A_2 v + A_3 w - b)] + R_2[\frac{1}{2}(v - v^k) + \mu(v^k - V_k(\sqrt{v})^{-1})] = 0, \tag{1.12b}$$

$$f_3(w) - A_3^\top[\lambda^k - \beta H(A_1u + A_2v + A_3w - b)] + R_3[\frac{1}{2}(w - w^k) + \mu(w^k - W_k(\sqrt{w})^{-1}))] = 0, \quad (1.12c)$$
$$\tilde{\lambda}^k = \lambda^k - \beta H(A_1\tilde{u}^k + A_2\tilde{v}^k + A_3\tilde{w}^k - b),$$

where $\mu \in (0,1)$ and $\beta > 0$ are given constants; $H \in \mathbb{R}^{m\times m}$, $R_1 \in \mathbb{R}^{n_1\times n_1}$, $R_2 \in \mathbb{R}^{n_2\times n_2}$ and $R_3 \in \mathbb{R}^{n_3\times n_3}$ are positive definite diagonal matrices; U_k, V_k and W_k are positive definite diagonal matrices defined by

$$U_k = diag(u_1^k\sqrt{u_1^k}, \dots, u_n^k\sqrt{u_n^k}) := \begin{pmatrix} u_1^k\sqrt{u_1^k} & & \\ & \ddots & \\ & & u_n^k\sqrt{u_n^k} \end{pmatrix},$$
$$V_k = diag(v_1^k\sqrt{v_1^k}, \dots, v_n^k\sqrt{v_n^k})$$

and
$$W_k = diag(w_1^k\sqrt{w_1^k}, \dots, w_n^k\sqrt{w_n^k}),$$

$(\sqrt{u})^{-1} \in \mathcal{R}^{n_1}_{++}$ is a vector whose j-th element is $1/\sqrt{u_j}$, $(\sqrt{v})^{-1} \in \mathcal{R}^{n_2}_{++}$ is a vector whose j-th element is $1/\sqrt{v_j}$, $(\sqrt{w})^{-1} \in \mathcal{R}^{n_3}_{++}$ is a vector whose j-th element is $1/\sqrt{w_j}$.
Since (1.12a)–(1.12c) include both square and quadratic terms, the method is called the Square-Quadratic Proximal (SQP) method, and (1.12a), (1.12b) and (1.12c) are called the SQP system of nonlinear equations (SQP system).

By combining the ADM and SQP method, we propose an inexact alternating direction method for SVI_3. Each iteration of the proposed method contains a prediction and a correction, the predictor is obtained via solving the SQP system approximately under significantly relaxed accuracy criterion and the new iterate is computed directly by an explicit formula derived from the original SQP method. We also study the global convergence of the proposed method under certain conditions. Our results can be viewed as significant extensions of the previously known results.

2. Inexact SQP alternating direction method

In this section, we suggest and consider the inexact SQP alternating direction method for solving SVI_3. First, for any vector $u \in \mathbb{R}^n$, $\|u\|_\infty = \max\{|u_1| \dots, |u_n|\}$. Let $D \in \mathbb{R}^{n\times n}$ be a symmetry positive definite matrix, we denote the D-norm of u by $\|u\|_D^2 = u^T Du$.
we make the following standard assumptions.

Assumption 2.1. f_1 is monotone with respect to $\mathbb{R}^{n_1}_+$, that is, $(f_1(x) - f_1(y))^T(x - y) \geq 0, \forall x, y \in \mathbb{R}^{n_1}_+$, f_2 is monotone with respect to $\mathbb{R}^{n_2}_+$ and f_3 is monotone with respect to $\mathbb{R}^{n_3}_+$.

Assumption 2.2. The solution set of SVI_3, denoted by $\mathcal{Z}^*$, is nonempty.

Now, we are ready to present the inexact SQP alternating direction method for solving SVI_3.

Algorithm 2.1.
Prediction step: For a given $z^k = (u^k, v^k, w^k, \lambda^k) \in \mathbb{R}^{n_1}_{++} \times \mathbb{R}^{n_2}_{++} \times \mathbb{R}^{n_3}_{++} \times \mathbb{R}^m$, $\mu \in (0,1)$ and $\beta > 0$, the predictor $\tilde{z}^k = (\tilde{u}^k, \tilde{v}^k, \tilde{w}^k, \tilde{\lambda}^k) \in \mathbb{R}^{n_1}_{++} \times \mathbb{R}^{n_2}_{++} \times \mathbb{R}^{n_3}_{++} \times \mathbb{R}^m$ is obtained via solving the following system:

$$f_1(u) - A_1^\top[\lambda^k - \beta H(A_1u + A_2v + A_3w - b)] + R_1[\frac{1}{2}(u - u^k) + \mu(u^k - U_k(\sqrt{u})^{-1})] =: \xi_u^k \approx 0, \quad (2.1a)$$

$$f_2(v) - A_2^\top[\lambda^k - \beta H(A_1u + A_2v + A_3w - b)] + R_2[\frac{1}{2}(v - v^k) + \mu(v^k - V_k(\sqrt{v})^{-1})] =: \xi_v^k \approx 0, \quad (2.1b)$$

$$f_3(w) - A_3^\top[\lambda^k - \beta H(A_1u + A_2v + A_3w - b)] + R_3[\frac{1}{2}(w - w^k) + \mu(w^k - W_k(\sqrt{w})^{-1}))] =: \xi_w^k \approx 0, \quad (2.1c)$$

$$\tilde{\lambda}^k = \lambda^k - \beta H(A_1\tilde{u}^k + A_2\tilde{v}^k + A_3\tilde{w}^k - b), \quad (2.1d)$$

where $H \in \mathbb{R}^{m\times m}$, $R_1 \in \mathbb{R}^{n_1\times n_1}$, $R_2 \in \mathbb{R}^{n_2\times n_2}$ and $R_3 \in \mathbb{R}^{n_3\times n_3}$ and

$$\|G^{-1}\xi^k\|_G^2 \leq \frac{1-\mu}{1+\mu}\eta^2\|z^k - \tilde{z}^k\|_G^2, \quad \eta \in (0,1), \quad (2.2)$$

$$\xi^k = \begin{pmatrix} \xi_u^k \\ \xi_v^k \\ \xi_w^k \\ 0 \end{pmatrix} \quad (2.3)$$

and

$$G = \begin{pmatrix} \frac{(1+\mu)}{2}R_1 & & & \\ & \frac{(1+\mu)}{2}R_2 & & \\ & & \frac{(1+\mu)}{2}R_3 & \\ & & & \frac{1}{\beta}H^{-1} \end{pmatrix}. \quad (2.4)$$

Correction step: The new iterate $z^{k+1}(\alpha_k) = (u^{k+1}, v^{k+1}, w^{k+1}, \lambda^{k+1})$ is the solution of the following system:

$$\frac{1-\mu}{1+\mu}\alpha_k[f_1(\tilde{u}^k) - A_1^\top\tilde{\lambda}^k] + R_1[\frac{1}{2}(u - u^k) + \mu(u^k - U_k(\sqrt{u})^{-1})] = 0, \quad (2.5a)$$

$$\frac{1-\mu}{1+\mu}\alpha_k[f_2(\tilde{v}^k) - A_2^\top\tilde{\lambda}^k] + R_2[\frac{1}{2}(v - v^k) + \mu(v^k - V_k(\sqrt{v})^{-1})] = 0, \quad (2.5b)$$

$$\frac{1-\mu}{1+\mu}\alpha_k[f_3(\tilde{w}^k) - A_3^\top\tilde{\lambda}^k] + R_3[\frac{1}{2}(w - w^k) + \mu(w^k - W_k(\sqrt{w})^{-1})] = 0, \quad (2.5c)$$

$$\lambda^{k+1} = \lambda^k - \frac{1-\mu}{1+\mu}\alpha_k\beta H(A_1\tilde{u}^k + A_2\tilde{v}^k + A_3\tilde{w}^k - b), \quad (2.5d)$$

where

$$\alpha_k = \frac{\varphi(z^k, \tilde{z}^k)}{\|d(z^k, \tilde{z}^k)\|_G^2}, \quad (2.6)$$

$$\varphi(z^k, \tilde{z}^k) := \frac{1}{2}\|u^k - \tilde{u}^k\|_{R_1}^2 + \frac{1}{2}\|v^k - \tilde{v}^k\|_{R_2}^2 + \frac{1}{2}\|w^k - \tilde{w}^k\|_{R_3}^2 + \frac{1}{\beta}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2 + (z^k - \tilde{z}^k)^\top\xi^k \quad (2.7)$$

and

$$d(z^k, \tilde{z}^k) := z^k - \tilde{z}^k + G^{-1}\xi^k. \quad (2.8)$$

Remark 2.1. The proposed method can be viewed as a prediction-correction method which uses the SQP systems in both the prediction and correction steps.
The main task of the prediction is to find an approximate solution of the following equations

$$f_1(u) - A_1^\top[\lambda^k - \beta H(A_1u + A_2v + A_3w - b)] + R_1[\frac{1}{2}(u - u^k) + \mu(u^k - U_k(\sqrt{u})^{-1})] = 0, \quad (2.9a)$$

$$f_2(v) - A_2^\top[\lambda^k - \beta H(A_1u + A_2v + A_3w - b)] + R_2[\frac{1}{2}(v - v^k) + \mu(v^k - V_k(\sqrt{v})^{-1})] = 0, \quad (2.9b)$$

$$f_3(w) - A_3^\top[\lambda^k - \beta H(A_1u + A_2v + A_3w - b)] + R_3[\frac{1}{2}(w - w^k) + \mu(w^k - W_k(\sqrt{w})^{-1})] = 0. \quad (2.9c)$$

The exact solution of

$$f_1(u^k) - A_1^\top[\lambda^k - \beta H(A_1u^k + A_2v^k + A_3w^k - b)] + R_1[\frac{1}{2}(u - u^k) + \mu(u^k - U_k(\sqrt{u})^{-1})] = 0 \quad (2.10)$$

denoted by $\tilde{u}^k$, as the approximate solution of (2.9a). The exact solution of

$$f_2(v^k) - A_2^\top[\lambda^k - \beta H(A_1\tilde{u}^k + A_2v^k + A_3w^k - b)] + R_2[\frac{1}{2}(v - v^k) + \mu(v^k - V_k(\sqrt{v})^{-1})] = 0 \quad (2.11)$$

denoted by $\tilde{v}^k$, as the approximate solution of (2.9b). Then the exact solution of

$$f_3(w^k) - A_3^\top[\lambda^k - \beta H(A_1\tilde{u}^k + A_2\tilde{v}^k + A_3w^k - b)] + R_3[\frac{1}{2}(w - w^k) + \mu(w^k - W_k(\sqrt{w})^{-1})] = 0 \quad (2.12)$$

denoted by $\tilde{w}^k$, as the approximate solution of (2.9c).
It follows from (2.1) and (2.10)- (2.12) that

$$\xi^k = \begin{pmatrix} \xi_u^k \\ \xi_v^k \\ \xi_w^k \\ 0 \end{pmatrix} = \begin{pmatrix} f_1(\tilde{u}^k) - f_1(u^k) + \beta A_1^\top HA_1(\tilde{u}^k - u^k) + \beta A_1^\top HA_2(\tilde{v}^k - v^k) + \beta A_1^\top HA_3(\tilde{w}^k - w^k) \\ f_2(\tilde{v}^k) - f_2(v^k) + \beta A_2^\top HA_2(\tilde{v}^k - v^k) + \beta A_2^\top HA_3(\tilde{w}^k - w^k) \\ f_3(\tilde{w}^k) - f_3(w^k) + \beta A_3^\top HA_3(\tilde{w}^k - w^k) \\ 0 \end{pmatrix}.$$

We need the following result to study the convergence analysis of the proposed method.

Lemma 2.2. *Let $q(u) \in \mathbb{R}^n$ be a monotone mapping of u with respect to $\mathbb{R}^n_+$ and*

$$R := diag(r_1, \dots, r_n) \in \mathbb{R}^{n\times n}$$

be a positive definite diagonal matrix. For a given $u^k > 0$, $\mu > 0$, if

$$U_k := diag(u_1^k\sqrt{u_1^k}, \dots, u_n^k\sqrt{u_n^k}), \quad \sqrt{u} = (\sqrt{u_1}, \dots, \sqrt{u_n}),$$

and $(\sqrt{u})^{-1}$ be an n-vector whose j-th element is $1/\sqrt{u_j}$, then the equation

$$q(u) + R[\frac{1}{2}(u - u^k) + \mu(u^k - U_k(\sqrt{u})^{-1})] = 0 \quad (2.13)$$

has a unique positive solution u. Moreover, for any $v \geq 0$, we have

$$(v - u)^\top q(u) \geq \tfrac{1+\mu}{4}\left(\|u - v\|_R^2 - \|u^k - v\|_R^2\right) + \tfrac{1-\mu}{4}\|u^k - u\|_R^2. \quad (2.14)$$

Proof . The proof of the first assertion is similar as Proposition 2 in [1]; hence it is omitted. We now prove the second assertion. For each $t > 0$, we have $\frac{1}{2}\left(1 - \frac{1}{t}\right) \leq 1 - \frac{1}{\sqrt{t}} \leq \frac{1}{2}(t - 1)$, then we obtain after multiplication by $v_j u_j^k \geq 0$ for each $j = 1, \dots, n$,

$$v_j u_j^k(1 - \frac{\sqrt{u_j^k}}{\sqrt{u_j}}) \leq v_j u_j^k \frac{1}{2}\left(\frac{u_j}{u_j^k} - 1\right) = \frac{1}{2}v_j(u_j - u_j^k)$$

and after multiplication by $u_j u_j^k \geq 0$ for each $j = 1, \dots, n$,

$$-u_j u_j^k (1 - \frac{\sqrt{u_j^k}}{\sqrt{u_j}}) \leq u_j u_j^k \frac{1}{2} \left(\frac{u_j^k}{u_j} - 1 \right) = \frac{1}{2} u_j^k (u_j^k - u_j),$$

adding the two inequalities, then we obtained

$$(v_j - u_j) \left(\frac{1}{2}(u_j - u_j^k) + \mu \left(u_j^k - (\sqrt{u_j^k})^3 (\sqrt{u_j})^{-1} \right) \right) \leq \frac{1}{2}\mu (v_j - u_j^k)(u_j - u_j^k) + \frac{1}{2}(u_j - u_j^k)(v_j - u_j).$$

Using the identities

$$\frac{1}{2}(v_j - u_j^k)(u_j - u_j^k) = \frac{1}{4}\left((u_j - u_j^k)^2 - (u_j - v_j)^2 + (v_j - u_j^k)^2 \right)$$

$$\frac{1}{2}(u_j - u_j^k)(v_j - u_j) = \frac{1}{4}\left((v_j - u_j^k)^2 - (v_j - u_j)^2 - (u_j - u_j^k)^2 \right)$$

and recalling (2.13), thus we obtained

$$(u_j - v_j)(-q_j) \geq r_j \left(\frac{1+\mu}{4} \left((u_j - v_j)^2 - (u_j^k - v_j)^2 \right) + \frac{1-\mu}{4} (u_j^k - u_j)^2 \right).$$

Summing over $j = 1, \dots, n$, encountered (2.14). □

3. Basic results

In this section, we prove some basic properties, which will be used to establish the sufficient and necessary conditions for the convergence of the proposed method. The following theorem explains the reason of choosing α_k in the form (2.6).

Theorem 3.1. *For given* $z^k = (u^k, v^k, w^k, \lambda^k) \in \mathbb{R}^{n_1}_{++} \times \mathbb{R}^{n_2}_{++} \times \mathbb{R}^{n_3}_{++} \times \mathbb{R}^m$, *let* $z^{k+1}(\alpha_k) = (u^{k+1}, v^{k+1}, w^{k+1}, \lambda^{k+1})$ *be generated by* (2.5a)-(2.5d). *Then for any* $z^* = (u^*, v^*, w^*, \lambda^*) \in \mathcal{Z}^*$, *we have*

$$\|z^k - z^*\|_G^2 - \|z^{k+1}(\alpha_k) - z^*\|_G^2 \geq \frac{1-\mu}{1+\mu} \Phi(\alpha_k) \tag{3.1}$$

where

$$\Phi(\alpha_k) := 2\alpha_k \varphi(z^k, \tilde{z}^k) - \alpha_k^2 \|d(z^k, \tilde{z}^k)\|_G^2. \tag{3.2}$$

Proof . Applying Lemma 2.2 to (2.1a), we get

$$\begin{aligned}(u^{k+1} - \tilde{u}^k)^\top &\left\{ f_1(\tilde{u}^k) - A_1^\top [\lambda^k - \beta H(A_1\tilde{u}^k + A_2\tilde{v}^k + A_3\tilde{w}^k - b)] - \xi_u^k \right\} \\ &\geq \tfrac{1+\mu}{4}\left(\|\tilde{u}^k - u^{k+1}\|_{R_1}^2 - \|u^k - u^{k+1}\|_{R_1}^2 \right) + \tfrac{1-\mu}{4} \|u^k - \tilde{u}^k\|_{R_1}^2. \end{aligned} \tag{3.3}$$

Since

$$\|u^k - u^{k+1}\|_{R_1}^2 = \|u^k - \tilde{u}^k\|_{R_1}^2 + \|\tilde{u}^k - u^{k+1}\|_{R_1}^2 + 2(\tilde{u}^k - u^{k+1})^\top R_1 (u^k - \tilde{u}^k).$$

Then

$$\frac{1}{2}(u^{k+1}-\tilde{u}^k)^\top R_1(u^k-\tilde{u}^k)=\frac{1}{4}\Big(\|\tilde{u}^k-u^{k+1}\|_{R_1}^2-\|u^k-u^{k+1}\|_{R_1}^2\Big)+\frac{1}{4}\|u^k-\tilde{u}^k\|_{R_1}^2. \tag{3.4}$$

Adding (3.3) and (3.4), we obtain

$$(u^{k+1}-\tilde{u}^k)^\top\Big\{\frac{(1+\mu)}{2}R_1(u^k-\tilde{u}^k)-f_1(\tilde{u}^k)+A_1^\top\tilde{\lambda}^k+\xi_u^k\Big\}\le\frac{\mu}{2}\|u^k-\tilde{u}^k\|_{R_1}^2,$$

which implies

$$\begin{aligned}\frac{2(1-\mu)}{1+\mu}\alpha_k(u^{k+1}-\tilde{u}^k)^\top\Big\{\frac{(1+\mu)}{2}R_1(u^k-\tilde{u}^k)-f_1(\tilde{u}^k)+A_1^\top\tilde{\lambda}^k+\xi_u^k\Big\}&\\ -\frac{1-\mu}{1+\mu}\alpha_k\mu\|u^k-\tilde{u}^k\|_{R_1}^2\le 0.&\end{aligned} \tag{3.5}$$

Similarly, applying Lemma 2.2 to (2.1b), we get

$$\begin{aligned}&(v^{k+1}-\tilde{v}^k)^\top\big\{f_2(\tilde{v}^k)-A_2^\top[\lambda^k-\beta H(A_1\tilde{u}^k+A_2\tilde{v}^k+A_3\tilde{w}^k-b)]-\xi_v^k\big\}\\ \ge\;&\tfrac{1+\mu}{4}\Big(\|\tilde{v}^k-v^{k+1}\|_{R_2}^2-\|v^k-v^{k+1}\|_{R_2}^2\Big)+\tfrac{1-\mu}{4}\|v^k-\tilde{v}^k\|_{R_2}^2.\end{aligned} \tag{3.6}$$

Similar as (3.4), we have

$$\frac{1}{2}(v^{k+1}-\tilde{v}^k)^\top R_2(v^k-\tilde{v}^k)=\frac{1}{4}\Big(\|\tilde{v}^k-v^{k+1}\|_{R_2}^2-\|v^k-v^{k+1}\|_{R_2}^2\Big)+\frac{1}{4}\|v^k-\tilde{v}^k\|_{R_2}^2. \tag{3.7}$$

Adding (3.6) and (3.7), we have

$$(v^{k+1}-\tilde{v}^k)^\top\Big\{\frac{(1+\mu)}{2}R_2(v^k-\tilde{v}^k)-f_2(\tilde{v}^k)+A_2^\top\tilde{\lambda}^k+\xi_v^k\Big\}\le\frac{\mu}{2}\|v^k-\tilde{v}^k\|_{R_2}^2,$$

which implies

$$\begin{aligned}\frac{2(1-\mu)}{1+\mu}\alpha_k(v^{k+1}-\tilde{v}^k)^\top\Big\{\frac{(1+\mu)}{2}R_2(v^k-\tilde{v}^k)-f_2(\tilde{v}^k)+A_2^\top\tilde{\lambda}^k+\xi_v^k\Big\}&\\ -\frac{1-\mu}{1+\mu}\alpha_k\mu\|v^k-\tilde{v}^k\|_{R_2}^2\le 0.&\end{aligned} \tag{3.8}$$

Similarly, we have

$$\begin{aligned}\frac{2(1-\mu)}{1+\mu}\alpha_k(w^{k+1}-\tilde{w}^k)^\top\Big\{\frac{(1+\mu)}{2}R_3(w^k-\tilde{w}^k)-f_3(\tilde{w}^k)+A_3^\top\tilde{\lambda}^k+\xi_w^k\Big\}&\\ -\frac{1-\mu}{1+\mu}\alpha_k\mu\|w^k-\tilde{w}^k\|_{R_3}^2\le 0.&\end{aligned} \tag{3.9}$$

We apply again Lemma 2.2 to (2.5a), we get

$$\begin{aligned}&(u^{k+1}-u^*)^\top\Big(-\frac{1-\mu}{1+\mu}\alpha_k[f_1(\tilde{u}^k)-A_1^\top\tilde{\lambda}^k]\Big)\\ \ge\;&\tfrac{1+\mu}{4}\Big(\|u^{k+1}-u^*\|_{R_1}^2-\|u^k-u^*\|_{R_1}^2\Big)+\tfrac{1-\mu}{4}\|u^k-u^{k+1}\|_{R_1}^2,\end{aligned} \tag{3.10}$$

which implies

$$\begin{aligned}&\frac{1+\mu}{2}\Big(\|u^k-u^*\|_{R_1}^2-\|u^{k+1}-u^*\|_{R_1}^2\Big)\\ \geq\ & \frac{2(1-\mu)}{1+\mu}\alpha_k(u^{k+1}-u^*)^\top(f_1(\tilde{u}^k)-A_1^\top\tilde{\lambda}^k)+\frac{1-\mu}{2}\|u^k-u^{k+1}\|_{R_1}^2.\end{aligned} \tag{3.11}$$

Similarly, applying Lemma 2.2 to (2.5b), we obtain

$$\begin{aligned}&(v^{k+1}-v^*)^\top\Big(-\frac{1-\mu}{1+\mu}\alpha_k[f_2(\tilde{v}^k)-A_2^\top\tilde{\lambda}^k]\Big)\\ \geq\ & \tfrac{1+\mu}{4}\Big(\|v^{k+1}-v^*\|_{R_2}^2-\|v^k-v^*\|_{R_2}^2\Big)+\tfrac{1-\mu}{4}\|v^k-v^{k+1}\|_{R_2}^2,\end{aligned} \tag{3.12}$$

which implies

$$\begin{aligned}&\frac{1+\mu}{2}\Big(\|v^k-v^*\|_{R_2}^2-\|v^{k+1}-v^*\|_{R_2}^2\Big)\\ \geq\ & \frac{2(1-\mu)}{1+\mu}\alpha_k(v^{k+1}-v^*)^\top(f_2(\tilde{v}^k)-A_2^\top\tilde{\lambda}^k)+\frac{1-\mu}{2}\|v^k-v^{k+1}\|_{R_2}^2.\end{aligned} \tag{3.13}$$

Similarly, we have

$$\begin{aligned}&\frac{1+\mu}{2}\Big(\|w^k-w^*\|_{R_3}^2-\|w^{k+1}-w^*\|_{R_3}^2\Big)\\ \geq\ & \frac{2(1-\mu)}{1+\mu}\alpha_k(w^{k+1}-w^*)^\top(f_3(\tilde{w}^k)-A_3^\top\tilde{\lambda}^k)+\frac{1-\mu}{2}\|w^k-w^{k+1}\|_{R_3}^2.\end{aligned} \tag{3.14}$$

On the other hand, from (2.5d), we have

$$\begin{aligned}&\|\lambda^k-\lambda^*\|_{H^{-1}}^2-\|\lambda^{k+1}-\lambda^*\|_{H^{-1}}^2\\ &\quad=\|\lambda^k-\lambda^{k+1}\|_{H^{-1}}^2+\frac{2(1-\mu)}{1+\mu}\alpha_k\beta(\lambda^{k+1}-\lambda^*)^\top(A_1\tilde{u}^k+A_2\tilde{v}^k+A_3\tilde{w}^k-b).\end{aligned} \tag{3.15}$$

Since (u^*,v^*,w^*,λ^*) is a solution of SVI_3, $\tilde{u}^k\in\mathbb{R}^{n_1}_{++}$, $\tilde{v}^k\in\mathbb{R}^{n_2}_{++}$ and $\tilde{w}^k\in\mathbb{R}^{n_3}_{++}$, we have

$$\begin{aligned}(\tilde{u}^k-u^*)^\top(f_1(u^*)-A_1^\top\lambda^*)&\geq 0,\\ (\tilde{v}^k-v^*)^\top(f_2(v^*)-A_2^\top\lambda^*)&\geq 0,\\ (\tilde{w}^k-w^*)^\top(f_3(w^*)-A_3^\top\lambda^*)&\geq 0,\end{aligned}$$

and

$$A_1u^*+A_2v^*+A_3w^*-b=0.$$

Using the monotonicity of f_1, f_2 and f_3, we obtain

$$\begin{aligned}&\begin{pmatrix}\tilde{u}^k-u^*\\ \tilde{v}^k-v^*\\ \tilde{w}^k-w^*\\ \tilde{\lambda}^k-\lambda^*\end{pmatrix}^\top\begin{pmatrix}f_1(\tilde{u}^k)-A_1^\top\tilde{\lambda}^k\\ f_2(\tilde{v}^k)-A_2^\top\tilde{\lambda}^k\\ f_3(\tilde{w}^k)-A_3^\top\tilde{\lambda}^k\\ A_1\tilde{u}^k+A_2\tilde{v}^k+A_3\tilde{w}^k-b\end{pmatrix}\\ &\qquad\geq\begin{pmatrix}\tilde{u}^k-u^*\\ \tilde{v}^k-v^*\\ \tilde{w}^k-w^*\\ \tilde{\lambda}^k-\lambda^*\end{pmatrix}^\top\begin{pmatrix}f_1(u^*)-A_1^\top\lambda^*\\ f_2(v^*)-A_2^\top\lambda^*\\ f_3(w^*)-A_3^\top\lambda^*\\ A_1u^*+A_2v^*+A_3w^*-b\end{pmatrix}\geq 0.\end{aligned} \tag{3.16}$$

It follows from (3.11), (3.14)-(3.16) that

$$\begin{aligned}
&\|z^k - z^*\|_G^2 - \|z^{k+1}(\alpha_k) - z^*\|_G^2 \\
&\geq \frac{1-\mu}{2}\|u^k - u^{k+1}\|_{R_1}^2 + \frac{1-\mu}{2}\|v^k - v^{k+1}\|_{R_2}^2 + \frac{1-\mu}{2}\|w^k - w^{k+1}\|_{R_3}^2 \\
&+ \frac{1}{\beta}\|\lambda^k - \lambda^{k+1}\|_{H^{-1}}^2 + \frac{2(1-\mu)}{1+\mu}\alpha_k(u^{k+1} - \tilde{u}^k)^\top(f_1(\tilde{u}^k) - A_1^\top\tilde{\lambda}^k) \\
&+ \frac{2(1-\mu)}{1+\mu}\alpha_k(v^{k+1} - \tilde{v}^k)^\top(f_2(\tilde{v}^k) - A_2^\top\tilde{\lambda}^k) \\
&+ \frac{2(1-\mu)}{1+\mu}\alpha_k(w^{k+1} - \tilde{w}^k)^\top(f_3(\tilde{w}^k) - A_3^\top\tilde{\lambda}^k) \\
&+ \frac{2(1-\mu)}{1+\mu}\alpha_k(\lambda^{k+1} - \tilde{\lambda}^k)^\top(A_1\tilde{u}^k + A_2\tilde{v}^k + A_3\tilde{w}^k - b).
\end{aligned} \tag{3.17}$$

Adding (3.5), (3.8), (3.9) and (3.17), we get

$$\begin{aligned}
&\|z^k - z^*\|_G^2 - \|z^{k+1}(\alpha_k) - z^*\|_G^2 \\
&\geq \frac{1-\mu}{2}\|u^k - u^{k+1}\|_{R_1}^2 + \frac{1-\mu}{2}\|v^k - v^{k+1}\|_{R_2}^2 + \frac{1-\mu}{2}\|w^k - w^{k+1}\|_{R_3}^2 \\
&+ \frac{1}{\beta}\|\lambda^k - \lambda^{k+1}\|_{H^{-1}}^2 + \frac{2(1-\mu)}{1+\mu}\alpha_k(u^{k+1} - \tilde{u}^k)^\top\left(\frac{(1+\mu)}{2}R_1(u^k - \tilde{u}^k) + \xi_u^k\right) \\
&+ \frac{2(1-\mu)}{1+\mu}\alpha_k(v^{k+1} - \tilde{v}^k)^\top\left(\frac{(1+\mu)}{2}R_2(v^k - \tilde{v}^k) + \xi_v^k\right) \\
&+ \frac{2(1-\mu)}{1+\mu}\alpha_k(w^{k+1} - \tilde{w}^k)^\top\left(\frac{(1+\mu)}{2}R_3(w^k - \tilde{w}^k) + \xi_w^k\right) \\
&+ \frac{2(1-\mu)}{1+\mu}\alpha_k(\lambda^{k+1} - \tilde{\lambda}^k)^\top(A_1\tilde{u}^k + A_2\tilde{v}^k + A_3\tilde{w}^k - b) \\
&- \frac{1-\mu}{1+\mu}\alpha_k\mu\|u^k - \tilde{u}^k\|_{R_1}^2 - \frac{1-\mu}{1+\mu}\alpha_k\mu\|v^k - \tilde{v}^k\|_{R_2}^2 - \frac{1-\mu}{1+\mu}\alpha_k\mu\|w^k - \tilde{w}^k\|_{R_3}^2 \\
&= \frac{2(1-\mu)}{1+\mu}\alpha_k(z^{k+1}(\alpha_k) - \tilde{z}^k)^\top G(z^k - \tilde{z}^k + G^{-1}\xi^k) \\
&+ \frac{1-\mu}{1+\mu}\Big(\frac{1+\mu}{2}\|u^k - u^{k+1}\|_{R_1}^2 + \frac{1+\mu}{2}\|v^k - v^{k+1}\|_{R_2}^2 \\
&+ \frac{1+\mu}{2}\|w^k - w^{k+1}\|_{R_3}^2 + \frac{1}{\beta}\|\lambda^k - \lambda^{k+1}\|_{H^{-1}}^2\Big) \\
&+ \frac{2\mu}{\beta(1+\mu)}\|\lambda^k - \lambda^{k+1}\|_{H^{-1}}^2 - \frac{1-\mu}{1+\mu}\alpha_k\mu\|u^k - \tilde{u}^k\|_{R_1}^2 \\
&- \frac{1-\mu}{1+\mu}\alpha_k\mu\|v^k - \tilde{v}^k\|_{R_2}^2 - \frac{1-\mu}{1+\mu}\alpha_k\mu\|w^k - \tilde{w}^k\|_{R_3}^2 \\
&\geq \frac{1-\mu}{1+\mu}\|z^k - z^{k+1}(\alpha_k)\|_G^2 + \frac{2(1-\mu)}{1+\mu}\alpha_k(z^{k+1}(\alpha_k) - \tilde{z}^k)^\top Gd(z^k, \tilde{z}^k) \\
&- \frac{1-\mu}{1+\mu}\alpha_k\mu\|u^k - \tilde{u}^k\|_{R_1}^2 - \frac{1-\mu}{1+\mu}\alpha_k\mu\|v^k - \tilde{v}^k\|_{R_2}^2 \\
&- \frac{1-\mu}{1+\mu}\alpha_k\mu\|w^k - \tilde{w}^k\|_{R_3}^2.
\end{aligned} \tag{3.18}$$

It follows from (3.18) that

$$\begin{aligned}
&\|z^k - z^*\|_G^2 - \|z^{k+1}(\alpha_k) - z^*\|_G^2 \\
&\geq \frac{1-\mu}{1+\mu}\Big(\|z^k - z^{k+1}(\alpha_k)\|_G^2 + 2\alpha_k(z^{k+1}(\alpha_k) - z^k)^\top G d(z^k, \tilde{z}^k) \\
&+ 2\alpha_k(z^k - \tilde{z}^k)^\top G d(z^k, \tilde{z}^k) - \alpha_k\mu\|u^k - \tilde{u}^k\|_{R_1}^2 - \alpha_k\mu\|v^k - \tilde{v}^k\|_{R_2}^2 \\
&- \alpha_k\mu\|w^k - \tilde{w}^k\|_{R_3}^2\Big) \\
&= \frac{1-\mu}{1+\mu}\Big(\|z^k - z^{k+1}(\alpha_k) - \alpha_k d(z^k, \tilde{z}^k)\|_G^2 - \alpha_k^2\|d(z^k, \tilde{z}^k)\|_G^2 \\
&+ 2\alpha_k(z^k - \tilde{z}^k)^\top G d(z^k, \tilde{z}^k) - \alpha_k\mu\|u^k - \tilde{u}^k\|_{R_1}^2 - \alpha_k\mu\|v^k - \tilde{v}^k\|_{R_2}^2 \\
&- \alpha_k\mu\|w^k - \tilde{w}^k\|_{R_3}^2\Big) \\
&\geq \frac{1-\mu}{1+\mu}\Big(-\alpha_k^2\|d(z^k, \tilde{z}^k)\|_G^2 + 2\alpha_k(z^k - \tilde{z}^k)^\top G d(z^k, \tilde{z}^k) \\
&- \alpha_k\mu\|u^k - \tilde{u}^k\|_{R_1}^2 - \alpha_k\mu\|v^k - \tilde{v}^k\|_{R_2}^2 - \alpha_k\mu\|w^k - \tilde{w}^k\|_{R_3}^2\Big) \\
&\geq \frac{1-\mu}{1+\mu}\Big(\alpha_k(\|u^k - \tilde{u}^k\|_{R_1}^2 + \|v^k - \tilde{v}^k\|_{R_2}^2 + \|w^k - \tilde{w}^k\|_{R_3}^2 \\
&+ \frac{2}{\beta}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2 + 2(z^k - \tilde{z}^k)^\top \xi^k) - \alpha_k^2\|d(z^k, \tilde{z}^k)\|_G^2\Big).
\end{aligned} \tag{3.19}$$

Using the definitions of Φ and $\varphi(z^k, \tilde{z}^k)$ the assertion of this theorem is proved. □

Theorem 3.1 shows that $\Phi(\alpha_k)$ is a lower bound of $\|z^k - z^*\|_G^2 - \|z^{k+1}(\alpha_k) - z^*\|_G^2$, and this motivates us to maximize $\Phi(\alpha_k)$ to accelerate the convergence of the new method. Note that $\Phi(\alpha_k)$ is a quadratic function of α_k and it reaches its maximum at α_k defined by (2.6). Then

$$\Phi(\alpha_k) = \alpha_k \varphi(z^k, \tilde{z}^k). \tag{3.20}$$

Next theorem is one of the keys to prove the global convergence results.

Theorem 3.2. *For given $z^k \in \mathbb{R}^{n_1}_{++} \times \mathbb{R}^{n_2}_{++} \times \mathbb{R}^{n_3}_{++} \times \mathbb{R}^m$, let $\tilde{z}^k$ be generated by* (2.1*a*)-(2.1*d*), *then we have the following*

$$\alpha_k \geq \frac{1}{2} \tag{3.21}$$

and

$$\Phi(\alpha_k) \geq \frac{(1-\eta^2)(1-\mu)}{4(1+\mu)}\|z^k - \tilde{z}^k\|_G^2. \tag{3.22}$$

Proof . It follows from (2.7), (2.8) and under condition (2.2), we have

$$\begin{aligned}
2\varphi(z^k, \tilde{z}^k) - \|d(z^k, \tilde{z}^k)\|_G^2 &= \|u^k - \tilde{u}^k\|_{R_1}^2 + \|v^k - \tilde{v}^k\|_{R_2}^2 + \|w^k - \tilde{w}^k\|_{R_3}^2 + \frac{2}{\beta}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2 \\
&\quad - \|z^k - \tilde{z}^k\|_G^2 - \|G^{-1}\xi^k\|_G^2 \\
&= \frac{1-\mu}{2}\|u^k - \tilde{u}^k\|_{R_1}^2 + \frac{1-\mu}{2}\|v^k - \tilde{v}^k\|_{R_2}^2 + \frac{1-\mu}{2}\|w^k - \tilde{w}^k\|_{R_3}^2 \\
&\quad + \frac{1}{\beta}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2 - \|G^{-1}\xi^k\|_G^2
\end{aligned}$$

and so

$$\begin{aligned}2\varphi(z^k,\tilde{z}^k)-\|d(z^k,\tilde{z}^k)\|_G^2 &\geq \tfrac{1-\mu}{1+\mu}\Big(\frac{1+\mu}{2}\|u^k-\tilde{u}^k\|_{R_1}^2+\frac{1+\mu}{2}\|v^k-\tilde{v}^k\|_{R_2}^2\\ &\quad+\frac{1+\mu}{2}\|w^k-\tilde{w}^k\|_{R_3}^2+\frac{1}{\beta}\|\lambda^k-\tilde{\lambda}^k\|_{H^{-1}}^2\Big)-\|G^{-1}\xi^k\|_G^2\\ &=\tfrac{1-\mu}{1+\mu}\|z^k-\tilde{z}^k\|_G^2-\|G^{-1}\xi^k\|_G^2\\ &\geq \tfrac{1-\mu}{1+\mu}(1-\eta^2)\|z^k-\tilde{z}^k\|_G^2.\end{aligned} \tag{3.23}$$

Therefore, it follows from (2.6) and (3.23) that

$$\alpha_k \geq \frac{1}{2}. \tag{3.24}$$

Consequently, from (3.20), (3.23) and (3.24) we obtain

$$\Phi(\alpha_k) \geq \frac{(1-\eta^2)(1-\mu)}{4(1+\mu)}\|z^k-\tilde{z}^k\|_G^2. \qquad \square$$

From the numerical point of view, it is necessary to attach a relax factor $\gamma \in (0,2)$ to the optimal step size α_k to achieve faster convergence. The following theorem shows that the sequence $\{z^k\}$ is Fejer monotone with respect to $\mathcal{Z}^*$.

Theorem 3.3. *Let $z^* \in \mathcal{Z}^*$ be a solution of SVI_3 and let $z^{k+1}(\gamma\alpha_k)$ be generated by (2.5a)-(2.5d). Then z^k and $\tilde{z}^k$ are bounded, and*

$$\|z^{k+1}(\gamma\alpha_k)-z^*\|_G^2 \leq \|z^k-z^*\|_G^2 - c\|z^k-\tilde{z}^k\|_G^2, \qquad \forall z^* \in \mathcal{Z}^* \tag{3.25}$$

where

$$c := \frac{\gamma(2-\gamma)(1-\eta^2)(1-\mu)^2}{4(1+\mu)^2} > 0.$$

Proof . It follows from Theorem 3.1 and Theorem 3.2 that

$$\|z^{k+1}(\gamma\alpha_k)-z^*\|_G^2 \leq \|z^k-z^*\|_G^2 - c\|z^k-\tilde{z}^k\|_G^2, \qquad \forall z^* \in \mathcal{Z}^*. \tag{3.26}$$

Since $\gamma \in (0,2)$ we have

$$\|z^{k+1}(\gamma\alpha_k)-z^*\|_G \leq \|z^k-z^*\|_G \leq \ldots \leq \|z^0-z^*\|_G$$

and thus $\{z^k\}$ is a bounded sequence.
It follows from (3.25) that

$$\sum_{k=0}^{\infty} c\|z^k-\tilde{z}^k\|_G^2 < +\infty$$

which means that

$$\lim_{k\to\infty}\|z^k-\tilde{z}^k\|_G = 0. \tag{3.27}$$

Since $\{z^k\}$ is a bounded sequence, we conclude that $\{\tilde{z}^k\}$ is also bounded. $\square$

4. Convergence of the proposed method

We start this section with the following lemma which plays an important role in consequent convergence analysis.

Lemma 4.1. *For a given* $z^k = (u^k, v^k, w^k, \lambda^k) \in \mathbb{R}^{n_1}_{++} \times \mathbb{R}^{n_2}_{++} \times \mathbb{R}^{n_3}_{++} \times \mathbb{R}^m$, *let* $\tilde{z}^k = (\tilde{u}^k, \tilde{v}^k, \tilde{w}^k, \tilde{\lambda}^k)$ *be generated by* (2.1a)-(2.1d). *Then for any* $z = (u, v, w, \lambda) \in \mathcal{Z}$, *we have*

$$(u-\tilde{u}^k)^\top\Big(f_1(\tilde{u}^k) - A_1^\top\tilde{\lambda}^k - \xi_u^k\Big) \geq \frac{1}{2}(u^k-\tilde{u}^k)^\top R_1\big\{(1+\mu)u - (\mu u^k + \tilde{u}^k)\big\}, \tag{4.1}$$

$$(v-\tilde{v}^k)^\top\Big(f_2(\tilde{v}^k) - A_2^\top\tilde{\lambda}^k - \xi_v^k\Big) \geq \frac{1}{2}(v^k-\tilde{v}^k)^\top R_2\big\{(1+\mu)v - (\mu v^k + \tilde{v}^k)\big\} \tag{4.2}$$

and

$$(w-\tilde{w}^k)^\top\Big(f_3(\tilde{w}^k) - A_3^\top\tilde{\lambda}^k - \xi_w^k\Big) \geq \frac{1}{2}(w^k-\tilde{w}^k)^\top R_3\big\{(1+\mu)w - (\mu w^k + \tilde{w}^k)\big\}. \tag{4.3}$$

Proof . Applying Lemma 2.2 to to prediction step, it follows that

$$(u-\tilde{u}^k)^\top\Big(f_1(\tilde{u}^k) - A_1^\top\tilde{\lambda}^k - \xi_u^k\Big) \geq \tfrac{1+\mu}{4}\Big(\|\tilde{u}^k-u\|_{R_1}^2 - \|u^k-u\|_{R_1}^2\Big) + \tfrac{1-\mu}{4}\|u^k-\tilde{u}^k\|_{R_1}^2.$$

By a simple manipulation, we have

$$\begin{aligned}
&\tfrac{1+\mu}{4}\Big(\|\tilde{u}^k-u\|_{R_1}^2 - \|u^k-u\|_{R_1}^2\Big) + \tfrac{1-\mu}{4}\|u^k-\tilde{u}^k\|_{R_1}^2\\
&= \frac{(1+\mu)}{2}u^\top R_1 u^k - \frac{(1+\mu)}{2}u^\top R_1\tilde{u}^k - \frac{(1-\mu)}{2}(\tilde{u}^k)^\top R_1 u^k - \frac{\mu}{2}\|u^k\|_{R_1}^2 + \frac{1}{2}\|\tilde{u}^k\|_{R_1}^2\\
&= \frac{(1+\mu)}{2}u^\top R_1(u^k-\tilde{u}^k) - (u^k-\tilde{u}^k)^\top R_1(\frac{\mu}{2}u^k + \frac{1}{2}\tilde{u}^k)\\
&= \frac{1}{2}(u^k-\tilde{u}^k)^\top R_1\big\{(1+\mu)u - \big(\mu u^k + \tilde{u}^k\big)\big\},
\end{aligned}$$

and the assertion (4.1) is proved. Similarly, we can prove the assertions (4.2) and (4.3). □

Now, we are ready to prove the convergence of the proposed method.

Theorem 4.2. *The sequence* $\{z^k\}$ *generated by the proposed method converges to some* z^∞ *which is a solution of* SVI_3.

Proof . It follows from (3.27) that

$$\lim_{k\to\infty}\|u^k-\tilde{u}^k\|_{R_1} = 0, \qquad \lim_{k\to\infty}\|v^k-\tilde{v}^k\|_{R_2} = 0, \qquad \lim_{k\to\infty}\|w^k-\tilde{w}^k\|_{R_3} = 0 \tag{4.4}$$

and

$$\lim_{k\to\infty}\|\lambda^k-\tilde{\lambda}^k\|_{H^{-1}} = \lim_{k\to\infty}\|A_1\tilde{u}^k + A_2\tilde{v}^k + A_3\tilde{w}^k - b\|_H = 0. \tag{4.5}$$

Moreover, (4.1), (4.2) and (4.3) imply that

$$(u-\tilde{u}^k)^\top(f_1(\tilde{u}^k) - A_1^\top\tilde{\lambda}^k) \geq \frac{1}{2}(u^k-\tilde{u}^k)^\top R_1\big\{(1+\mu)u - (\mu u^k + \tilde{u}^k)\big\} + (u-\tilde{u}^k)^\top\xi_u^k,$$

$$(v-\tilde{v}^k)^\top(f_2(\tilde{v}^k)-A_2^\top\tilde{\lambda}^k)\geq\frac{1}{2}(v^k-\tilde{v}^k)^\top R_2\big\{(1+\mu)v-(\mu v^k+\tilde{v}^k)\big\}+(v-\tilde{v}^k)^\top\xi_v^k$$

and

$$(w-\tilde{w}^k)^\top(f_2(\tilde{w}^k)-A_3^\top\tilde{\lambda}^k)\geq\frac{1}{2}(w^k-\tilde{w}^k)^\top R_3\big\{(1+\mu)w-(\mu w^k+\tilde{w}^k)\big\}+(w-\tilde{w}^k)^\top\xi_w^k.$$

We deduce from (4.4) that

$$\begin{cases}\lim_{k\to\infty}(u-\tilde{u}^k)^\top\{f_1(\tilde{u}^k)-A_1^\top\tilde{\lambda}^k\}\geq 0, & \forall u\in\mathbb{R}^{n_1}_{++},\\ \lim_{k\to\infty}(v-\tilde{v}^k)^\top\{f_2(\tilde{v}^k)-A_2^\top\tilde{\lambda}^k\}\geq 0, & \forall v\in\mathbb{R}^{n_2}_{++},\\ \lim_{k\to\infty}(w-\tilde{w}^k)^\top\{f_3(\tilde{w}^k)-A_3^\top\tilde{\lambda}^k\}\geq 0, & \forall w\in\mathbb{R}^{n_3}_{++}.\end{cases}\tag{4.6}$$

Since $\{z^k\}$ is bounded, so it has at least one cluster point. Let z^∞ be a cluster point of $\{z^k\}$ and the subsequence $\{z^{k_j}\}$ converges to z^∞. It follows from (4.5) and (4.6) that

$$\begin{cases}\lim_{j\to\infty}(u-u^{k_j})^\top\{f_1(u^{k_j})-A_1^\top\lambda^{k_j}\}\geq 0, & \forall u\in\mathbb{R}^{n_1}_{++},\\ \lim_{j\to\infty}(v-v^{k_j})^\top\{f_2(v^{k_j})-A_2^\top\lambda^{k_j}\}\geq 0, & \forall v\in\mathbb{R}^{n_2}_{++},\\ \lim_{j\to\infty}(w-w^{k_j})^\top\{f_3(w^{k_j})-A_3^\top\lambda^{k_j}\}\geq 0, & \forall w\in\mathbb{R}^{n_3}_{++},\\ \lim_{j\to\infty}(A_1u^{k_j}+A_2v^{k_j}+A_3w^{k_j}-b)=0.\end{cases}$$

and consequently

$$\begin{cases}(u-u^\infty)^\top\{f_1(u^\infty)-A_1^\top\lambda^\infty\}\geq 0, & \forall u\in\mathbb{R}^{n_1}_{++},\\ (v-v^\infty)^\top\{f_2(v^\infty)-A_2^\top\lambda^\infty\}\geq 0, & \forall v\in\mathbb{R}^{n_2}_{++},\\ (w-w^\infty)^\top\{f_3(w^\infty)-A_3^\top\lambda^\infty\}\geq 0, & \forall w\in\mathbb{R}^{n_3}_{++},\\ A_1u^\infty+A_2v^\infty+A_3w^\infty-b=0,\end{cases}$$

which means that z^∞ is a solution of SVI_3.

Now we prove that the sequence $\{z^k\}$ converges to z^∞. Since

$$\lim_{k\to\infty}\|z^k-\tilde{z}^k\|_G=0,\quad\text{and}\quad\{\tilde{z}^{k_j}\}\to z^\infty,$$

for any $\epsilon>0$, there exists an $l>0$ such that

$$\|\tilde{z}^{k_l}-z^\infty\|_G<\frac{\epsilon}{2}\quad\text{and}\quad\|z^{k_l}-\tilde{z}^{k_l}\|_G<\frac{\epsilon}{2}.\tag{4.7}$$

From (3.25), we have $\|z^{k+1}-z^*\|_G\leq\|z^k-z^*\|_G$. Therefore, for any $k\geq k_l$, it follows from (4.7) that

$$\|z^k-z^\infty\|_G\leq\|z^{k_l}-z^\infty\|_G\leq\|z^{k_l}-\tilde{z}^{k_l}\|_G+\|\tilde{z}^{k_l}-z^\infty\|_G<\epsilon.$$

This implies that the sequence $\{z^k\}$ converges to z^∞ which is a solution of SVI_3. □

5. Convergence Rate

Recall that $\mathcal{Z}^*$ can be characterized as (see (2.3.2) in pp. 159 of [16])

$$\mathcal{Z}^*=\bigcap_{z\in\mathcal{Z}}\left\{\tilde{z}\in\mathcal{Z}\;:\;(z-\tilde{z})^\top Q(z)\geq 0\right\}.$$

This implies that $\tilde{z}$ is an approximate solution of SVI$_3$ with the accuracy $\epsilon > 0$ if it satisfies

$$\tilde{z} \in \mathcal{Z} \quad \text{and} \quad \sup_{z\in\mathcal{Z}}\{(\tilde{z}-z)^\top Q(z)\} \le \epsilon. \tag{5.1}$$

Now, we show that after t iterations of the proposed method, we can find a $\tilde{z} \in \mathcal{Z}$ such that (5.1) is satisfied with $\epsilon = O(1/t)$.

Lemma 5.1. *Let $\tilde{z}^k$ be generated by (2.1b)-(2.1d) and $z^{k+1}(\gamma\alpha_k)$ be generated by (2.5a)-(2.5d). If we take $\tau_k = \frac{1-\mu}{1+\mu}\gamma\alpha_k$. Then for any $z = (u, v, w, \lambda) \in \mathcal{Z}$, we have*

$$\big(z - z^{k+1}(\gamma\alpha_k)\big)^\top Q(\tilde{z}^k) \ge \frac{1}{2\tau_k}\big(\|z^{k+1}-z\|_G^2 - \|z^k - z\|_G^2\big) + \frac{1-\mu}{2\tau_k(1+\mu)}\|z^k - z^{k+1}(\gamma\alpha_k)\|_G^2. \tag{5.2}$$

Proof . It follows from (3.10) and (3.12) that

$$\begin{aligned}&(u-u^{k+1})^\top\Big(f_1(\tilde{u}^k) - A_1^\top\tilde{\lambda}^k\Big)\\ \ge\;& \tfrac{1+\mu}{4\tau_k}\Big(\|u^{k+1}-u\|_{R_1}^2 - \|u^k-u\|_{R_1}^2\Big) + \tfrac{1-\mu}{4\tau_k}\|u^k - u^{k+1}\|_{R_1}^2\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}&(v-v^{k+1})^\top\Big(f_2(\tilde{v}^k) - A_2^\top\tilde{\lambda}^k\Big)\\ \ge\;& \tfrac{1+\mu}{4\tau_k}\Big(\|v^{k+1}-v\|_{R_2}^2 - \|v^k-v\|_{R_2}^2\Big) + \tfrac{1-\mu}{4\tau_k}\|v^k - v^{k+1}\|_{R_2}^2.\end{aligned} \tag{5.4}$$

Similarly, we have

$$\begin{aligned}&(w-w^{k+1})^\top\Big(f_3(\tilde{w}^k) - A_3^\top\tilde{\lambda}^k\Big)\\ \ge\;& \tfrac{1+\mu}{4\tau_k}\Big(\|w^{k+1}-w\|_{R_3}^2 - \|w^k-w\|_{R_3}^2\Big) + \tfrac{1-\mu}{4\tau_k}\|w^k - w^{k+1}\|_{R_3}^2.\end{aligned} \tag{5.5}$$

On the other hand, from (2.5d), we have

$$\begin{aligned}(\lambda-\lambda^{k+1})^\top(A_1\tilde{u}^k + A_2\tilde{v}^k + A_3\tilde{w}^k - b) &= \frac{1}{2\tau_k\beta}\Big(\|\lambda^{k+1}-\lambda\|_{H^{-1}}^2 - \|\lambda^k-\lambda\|_{H^{-1}}^2\Big)\\ &\quad + \frac{1}{2\tau_k\beta}\|\lambda^k - \lambda^{k+1}\|_{H^{-1}}^2\\ &\ge \frac{1}{2\tau_k\beta}\Big(\|\lambda^{k+1}-\lambda\|_{H^{-1}}^2 - \|\lambda^k-\lambda\|_{H^{-1}}^2\Big)\\ &\quad + \frac{1-\mu}{2(1+\mu))\tau_k\beta}\|\lambda^k - \lambda^{k+1}\|_{H^{-1}}^2.\end{aligned} \tag{5.6}$$

Recall the definition of Q in (1.10), we obtain the assertion (5.2). □

Lemma 5.2. *For given $z^k \in \mathbb{R}^{n_1}_{++} \times \mathbb{R}^{n_2}_{++} \times \mathbb{R}^{n_3}_{++} \times \mathbb{R}^m$, let $\tilde{z}^k$ be generated by (2.1b)-(2.1d) and $z^{k+1}(\gamma\alpha_k)$ be generated by (2.5a)-(2.5d). Then we have the following*

$$\big(z - \tilde{z}^k\big)^\top Q(z) \ge \frac{(2-\gamma)\alpha_k}{2}\|d(z^k, \tilde{z}^k)\|_G^2 + \frac{1}{2\tau_k}\big(\|z^{k+1}(\gamma\alpha_k) - z\|_G^2 - \|z^k - z\|_G^2\big). \tag{5.7}$$

Proof . It follows from (3.5), (3.8), (3.9) and

$$\big(\lambda^{k+1}-\tilde{\lambda}^k\big)^\top\Big(A_1\tilde{u}^k+A_2\tilde{v}^k+A_3\tilde{w}^k-b\Big)=\frac{1}{\beta}\big(\lambda^{k+1}-\tilde{\lambda}^k\big)^\top H^{-1}\Big(\lambda^k-\tilde{\lambda}^k\Big)$$

that

$$\begin{aligned}\big(z^{k+1}(\gamma\alpha_k)-\tilde{z}^k\big)^\top Q(\tilde{z}^k) &\geq \big(z^{k+1}(\gamma\alpha_k)-\tilde{z}^k\big)^\top Gd(z^k,\tilde{z}^k)-\frac{\mu}{2}\|u^k-\tilde{u}^k\|^2_{R_1}-\frac{\mu}{2}\|v^k-\tilde{v}^k\|^2_{R_2}\\ &\quad -\frac{\mu}{2}\|w^k-\tilde{w}^k\|^2_{R_3}\\ &= \|z^k-\tilde{z}^k\|^2_G+(z^k-\tilde{z}^k)^\top\xi^k-\frac{\mu}{2}\|u^k-\tilde{u}^k\|^2_{R_1}-\frac{\mu}{2}\|v^k-\tilde{v}^k\|^2_{R_2}\\ &\quad -\frac{\mu}{2}\|w^k-\tilde{w}^k\|^2_{R_3}+\big(z^{k+1}(\gamma\alpha_k)-z^k\big)^\top Gd(z^k,\tilde{z}^k)\\ &= \varphi(z^k,\tilde{z}^k)+\big(z^{k+1}(\gamma\alpha_k)-z^k\big)^\top Gd(z^k,\tilde{z}^k)\\ &= \alpha_k\|d(z^k,\tilde{z}^k)\|^2_G+\big(z^{k+1}(\gamma\alpha_k)-z^k\big)^\top Gd(z^k,\tilde{z}^k). \end{aligned} \tag{5.8}$$

Adding (5.2) and (5.8), we get

$$\begin{aligned}\big(z-\tilde{z}^k\big)^\top Q(\tilde{z}^k) &\geq \alpha_k\|d(z^k,\tilde{z}^k)\|^2_G+\big(z^{k+1}(\gamma\alpha_k)-z^k\big)^\top Gd(z^k,\tilde{z}^k)\\ &\quad +\frac{1}{2\tau_k}\big(\|z^{k+1}-z\|^2_G-\|z^k-z\|^2_G\big)+\frac{1-\mu}{2\tau_k(1+\mu)}\|z^k-z^{k+1}(\gamma\alpha_k)\|^2_G.\end{aligned}$$

Using the following inequality

$$\big(z^{k+1}(\gamma\alpha_k)-z^k\big)^\top Gd(z^k,\tilde{z}^k)\geq-\frac{1-\mu}{2\tau_k(1+\mu)}\|z^k-z^{k+1}(\gamma\alpha_k)\|^2_G-\frac{\tau_k(1+\mu)}{2(1-\mu)}\|d(z^k,\tilde{z}^k)\|^2_G,$$

we obtain

$$\big(z-\tilde{z}^k\big)^\top Q(\tilde{z}^k) \geq \frac{(2-\gamma)\alpha_k}{2}\|d(z^k,\tilde{z}^k)\|^2_G+\frac{1}{2\tau_k}\big(\|z^{k+1}-z\|^2_G-\|z^k-z\|^2_G\big),$$

and by using the monotonicity of Q, we obtain (5.7). □

Now, we are ready to present the $O(1/t)$ convergence rate of the proposed method.

Theorem 5.3. *For any integer $t>0$, we have a $\tilde{z}_t\in\mathcal{Z}$ which satisfies*

$$(\tilde{z}_t-z)^\top Q(z)\leq\frac{1}{2\Upsilon_t}\|z-z^0\|^2_G,\quad\forall z\in\mathcal{Z},$$

where

$$\tilde{z}_t=\frac{1}{\Upsilon_t}\sum_{k=0}^t\tau_k\tilde{z}^k\quad\textit{and}\quad\Upsilon_t=\sum_{k=0}^t\tau_k.$$

Proof . Summing the inequality (5.7) over $k=0,\dots,t$, we obtain

$$\left(\left(\sum_{k=0}^t\tau_k\right)z-\sum_{k=0}^t\tau_k\tilde{z}^k\right)^\top Q(z)+\frac{1}{2}\|z-z^0\|^2_G\geq 0.$$

Using the notations of Υ_t and $\tilde{z}_t$ in the above inequality, we derive

$$(\tilde{z}_t - z)^\top Q(z) \leq \frac{1}{2\Upsilon_t}\|z - z^0\|_G^2, \quad \forall z \in \mathcal{Z}.$$

Indeed, $\tilde{z}_t \in \mathcal{Z}$ because it is a convex combination of $\tilde{z}^0, \tilde{z}^1, \ldots, \tilde{z}^t$.
The proof is completed. □

It follows from (3.21) that

$$\Upsilon_t \geq \frac{(1-\mu)\gamma}{2(1+\mu)}\,(t+1)\,.$$

Suppose that for any compact set $\mathcal{D} \subset \mathcal{Z}$, let $d = \sup\{\|z - z^0\|_G | z \in \mathcal{D}\}$. For any given $\epsilon > 0$, after most

$$t = \left[\frac{(1+\mu)d^2}{(1-\mu)\gamma\epsilon} - 1\right]$$

iterations, we have

$$(\tilde{z}_t - z)^T Q(z) \leq \epsilon, \forall z \in \mathcal{D}.$$

That is, the $O(1/t)$ convergence rate of the inexact SQP alternating direction method is established in an ergodic sense.

6. Preliminary Computational Results

In this section, we present some numerical experiments to illustrate our algorithm and convergence result.

We denote by $0_{n\times n} \in \mathbb{R}^{n\times n}$ the null matrix, and by $I_{n\times n} \in \mathbb{R}^{n\times n}$ the identity matrix. Let $S^n = \left\{X \in \mathbb{R}^{n\times n} : X^\top = X\right\}$, $S^n_+ = \{X \in S^n : X \succeq 0_{n\times n}\}$, $\mathbb{B} = \{X \in S^n : H_v \leq X \leq H_u\}$, and $H_v, H_u \in S^n$ are given proper matrices. The matrix inequality $S \preceq T$ means that T-S is a positive semi-definite matrix, while $S \leq T$ means that $S_{ij} \leq T_{ij}$ ($\forall i, j \in I = \{1, 2, \ldots, n\}$). For $C \in \mathbb{R}^{n\times n}$, we denote by $\|C\|_F$ the matrix Frobenius norm of C, i.e., $\|C\|_F = (\sum_{i=1}^n \sum_{j=1}^n |C_{ij}|^2)^{1/2}$. Note that the matrix Fröbenis norm is induced by the inner product

$$\langle A, B\rangle = \text{trace}(A^\top B).$$

We consider the following optimization problem with matrix variables, which is studied in [12] and [33]:

$$\min\left\{\frac{1}{2}\|U - Q\|_F^2 \,:\, 0_{n\times n} \preceq U \preceq M, U \in \mathbb{B}\right\}, \tag{6.1}$$

where $Q, M \in S^n$ are given proper matrices. Note that the problem (6.1) can be reformulated into the following separable form:

$$\min\left\{\tfrac{1}{2}\|U - Q\|_F^2 + \tfrac{1}{2}\|V + Q - M\|_F^2 + \tfrac{1}{2}\|W - Q\|_F^2\right\} \tag{6.2}$$

$$\text{such that } U + V = M, \tag{6.3}$$

$$V - W = 0_{n\times n}, \tag{6.4}$$

$$V + W = M, \tag{6.5}$$

where $U, V \in S^n_+$, $W \in \mathbb{B}$. Then, the problem (5.2)-(5.5) is equivalent to the following structured variational inequality problem: Find $z^* = (U^*, V^*, W^*, \lambda^*) \in \Omega := S^n_+ \times S^n_+ \times \mathbb{B} \times \mathbb{R}^{3n\times n}$ such that

$$\begin{cases} \langle U - U^*, f_1(U^*) - A^\top \lambda^*\rangle \geq 0, \\ \langle V - V^*, f_2(V^*) - A_2^\top \lambda^*\rangle \geq 0, \\ \langle W - W^*, f_3(W^*) - A_3^\top \lambda^*\rangle \geq 0, \qquad \forall z = (U, V, W, \lambda) \in \Omega, \\ A_1U^* + A_2V^* + A_3W^* - b = 0. \end{cases} \tag{6.6}$$

where

$$A_1 = \begin{pmatrix} I_{n\times n} \\ I_{n\times n} \\ 0_{n\times n} \end{pmatrix}, \quad A_2 = \begin{pmatrix} I_{n\times n} \\ 0_{n\times n} \\ I_{n\times n} \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0_{n\times n} \\ -I_{n\times n} \\ I_{n\times n} \end{pmatrix}, \quad b = \begin{pmatrix} M \\ 0_{n\times n} \\ M \end{pmatrix}$$

and

$$f_1(U) = U - Q, \quad f_2(V) = V + Q - M, \quad f_3(W) = W - Q.$$

The entries of Q are randomly with the restriction that $Q_{ii} \in (0, 2)$ and $Q_{ij} \in (-1, 1)$. The matrices H_v and H_u are given by

$$(H_u)_{jj} = (H_v)_{jj} = 1 \text{ and } (H_u)_{ij} = -(H_v)_{ij} = 0.1, \quad \forall i \neq j, i, j = 1, 2, \ldots, n.$$

The matrix M has the following form:

$$M = X\Sigma X, \quad X = I_{n\times n} - 2xx^\top, \quad \Sigma = \text{diag}(e_1, e_2, \ldots, e_n),$$

where x is a random unit vector, e_i $(i = 1, 2, \ldots, n)$ is a given eigenvalue of the matrix M. For simplification, we take $R_1 = r_1 I_{n\times n}, R_2 = r_2 I_{n\times n}$, $R_3 = r_3 I_{n\times n}$ and $H = I_{n\times n}$ where $r_1 > 0$, $r_2 > 0$ and $r_3 > 0$ are scalars. In all tests, we take $\mu = 0.01$, $\beta = 1, \gamma = 1.9$, $\sigma = 0.1, r_1 = r_2 = r_3 = 10$ and $(U^0, V^0, W^0, \lambda^0) = (I_{n\times n}, I_{n\times n}, I_{n\times n}, 0_{3n\times n})$ as the initial point in the test. The iteration is stopped as soon as

$$\frac{\max(abs(z^k - \tilde{z}^k))}{\max(abs(z^0 - \tilde{z}^0))} \leq 10^{-5}.$$

$abs(D)$ is the absolute value of matrix D, that is, if $D = [d_{ij}]$ where $d_{ij} \in \mathbb{R}, i = 1, \ldots, n, j = 1, \ldots, n$. Then, $abs(D) = [|d_{ij}|]$. All codes were written in Matlab, we compare the proposed method with those in [12] and [33]. The iteration numbers denoted by k, and the computational time for the problem (6.1) with different dimensions are given in Tables 1–4.

Table 1: Numerical results for problem (6.1) with $e_i \in (1.25, 2)$.

Dimension of the problem	The proposed method		The method in [12]		The method in [33]	
	k	CPU(Sec.)	k	CPU(Sec.)	k	CPU(Sec.)
100	185	10.43	623	17.87	736	19.61
200	212	63.88	529	108.41	620	117.26
300	256	220.24	618	383.39	722	398.69
400	271	469.20	641	900.70	747	1024.95

Table 2: Numerical results for problem (6.1) with $e_i \in (1.8, 2)$.

Dimension of the problem	The proposed method		The method in [12]		The method in [33]	
	k	CPU(Sec.)	k	CPU(Sec.)	k	CPU(Sec.)
100	81	4.64	296	9.95	358	10.92
200	77	21.51	310	66.41	362	79.51
300	87	61.40	283	162.11	330	164.78
400	100	183.41	286	304.73	333	363.32

Table 3: Numerical results for problem (6.1) with $e_i \in (2, 3)$.

Dimension of the problem	The proposed method		The method in [12]		The method in [33]	
	k	CPU(Sec.)	k	CPU(Sec.)	k	CPU(Sec.)
100	87	4.97	303	12.23	343	13.91
200	91	23.18	339	51.53	393	55.08
300	94	60.12	321	139.94	380	154.44
400	89	136.16	335	351.83	393	391.64

Table 4: Numerical results for problem (6.1) with $e_i \in (10, 12)$.

Dimension of the problem	The proposed method		The method in [12]		The method in [33]	
	k	CPU(Sec.)	k	CPU(Sec.)	k	CPU(Sec.)
100	81	4.61	196	7.06	213	7.13
200	78	19.46	304	57.93	325	59.88
300	88	57.81	356	141.58	377	149.47
400	99	168.63	389	422.36	417	480.56

Tables 1–4 show that the proposed method is more flexible and efficient for the problem tested. Moreover, it demonstrates computationally that the new method is more effective than those in [12] and [33] in the sense that the new method needs fewer iterations and less computational time.

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