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New existence results for a coupled system of nonlinear differential equations of arbitrary order

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Abstract

This paper studies the existence of solutions for a coupled system of nonlinear fractional differential equations. New existence and uniqueness results are established using Banach fixed point theorem. Other existence results are obtained using Schaefer and Krasnoselskii fixed point theorems. Some illustrative examples are also presented.

Keywords: Caputo derivative; Coupled system; Fractional differential equation; Fixed point.
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1. Introduction

The differential equations of fractional order arise in many scientific disciplines, such as physics, chemistry, control theory, signal processing and biophysics. For more details, we refer the reader to [7, 9, 12] and the references therein. Recently, there has been an important progress in the investigation of these equations, (see [3, 4, 13]). On the other hand, the study of coupled systems of fractional differential equations is also of a great importance. These systems occur in various problems of applied science and engineering. For some recent results, we refer the interested reader to ([1, 2, 5, 6, 11].

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In this paper, we discuss the existence and uniqueness of solutions for the following coupled system:

$$\begin{cases} D^{\alpha}u\left(t\right)=f_{1}\left(t,v\left(t\right),D^{\alpha-1}v\left(t\right)\right),t\in\left[0,1\right], \\ D^{\beta}v\left(t\right)=f_{2}\left(t,u\left(t\right),D^{\beta-1}u\left(t\right)\right),t\in\left[0,1\right], \\ u\left(0\right)=v\left(0\right)=0, \\ u'\left(0\right)=\gamma I^{p}u\left(\eta\right),\eta\in\left]0,1\right[, \\ v'\left(0\right)=\delta I^{q}v\left(\zeta\right),\zeta\in\left]0,1\right[, \end{cases} \tag{1.1}$$

where D^{α}, D^{β} denote the Caputo fractional derivatives, p, q are non negative reals numbers, $1<\alpha<2, 1<\beta<2$, f_1 and f_2 are two functions that will be specified later.
The paper is organized as follows: In section 2, we present some preliminaries and lemmas. In Section 3, we prove our main results for the existence of solutions of problem (1.1). In the last section, some examples are presented to illustrate our results.

2. Preliminaries

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha>0$, for a continuous function f on $[a,b]$ is defined as:

$$J^{\alpha}f\left(t\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{t}\left(t-\tau\right)^{\alpha-1}f\left(\tau\right)d\tau,\alpha>0,a\leq t\leq b \tag{2.1}$$

where $\Gamma\left(\alpha\right):=\int_{0}^{\infty}e^{-u}u^{\alpha-1}du.$

Definition 2.2. The fractional derivative of $f\in C^{n}([a,b])$ in the sense of Caputo is defined as:

$$D^{\alpha}f\left(t\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\int_{a}^{t}\left(t-\tau\right)^{n-\alpha-1}f^{(n)}\left(\tau\right)d\tau,n-1<\alpha<n,n\in N^{*},t\in[a,b]. \tag{2.2}$$

For more details about fractional calculus, we refer the interested reader to [10].

The following lemmas give some properties of fractional calculus theory [7, 9]:

Lemma 2.3. Let $r,s>0, f\in L_1([a,b])$. Then $I^{r}I^{s}f(t)=I^{r+s}f(t), D^{s}I^{s}f(t)=f(t), t\in\left[a,b\right].$

Lemma 2.4. Let $s>r>0, f\in L_1([a,b])$. Then $D^{r}I^{s}f(t)=I^{s-r}f(t), t\in\left[a,b\right].$

We need the following two lemmas [7]:

Lemma 2.5. Let $\alpha>0$. The general solution of the equation $D^{\alpha}x\left(t\right)=0$ is given by

$$x\left(t\right)=c_{0}+c_{1}t+c_{2}t^{2}+...+c_{n-1}t^{n-1}, \tag{2.3}$$

where $c_i\in\mathbb{R}, i=0,1,2,..,n-1, n=[\alpha]+1$.

Lemma 2.6. Let $\alpha>0$. Then

$$J^{\alpha}D^{\alpha}x\left(t\right)=x\left(t\right)+c_{0}+c_{1}t+c_{2}t^{2}+...+c_{n-1}t^{n-1}, \tag{2.4}$$

with $c_i\in\mathbb{R}, i=0,1,2,...,n-1, n=[\alpha]+1.$

We prove the following result:

Lemma 2.7. *Let $g \in C([0,1], \mathbb{R})$. The solution of the problem*

$$D^{\alpha} x(t) = g(t),\ 1 < \alpha < 2, \tag{2.5}$$

associated with the conditions

$$x(0) = 0, x'(0) = \gamma I^{p} x(\eta), \eta \in]0,1[, p > 0,$$

is given by

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds \\ &+ \frac{\gamma \Gamma(p+2)\, t}{\Gamma(p+2) - \gamma \eta^{p+1}} \int_0^{\eta} \frac{(\eta - s)^{p+\alpha-1}}{\Gamma(p+\alpha)} g(s)\, ds \end{aligned} \tag{2.6}$$

such that $\gamma \neq \frac{\Gamma(p+2)}{\eta^{p+1}}$.

Proof . By Lemmas 2.5 and 2.6, we can write

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g(\tau)\, d\tau - c_0 - c_1 t. \tag{2.7}$$

Thus,

$$x'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-\tau)^{\alpha-2} g(\tau)\, d\tau - c_1.$$

It is clear that $c_0 = 0$.

On the other hand, by Lemma 2.3, we obtain

$$I^{p} x(t) = \frac{1}{\Gamma(p+\alpha)} \int_0^t (t-s)^{p+\alpha-1} g(s)\, ds - c_1 \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} s ds.$$

Also, we have

$$c_1 = -\frac{\gamma \Gamma(p+2)}{\Gamma(p+2) - \gamma \eta^{p+1}} I^{p+\alpha} g(\eta).$$

Substituting c_1 in (2.7), we obtain the desired quantity (2.6). Lemma 2.7 is thus proved. □

Let us now introduce the spaces

$$X := \left\{ u \in C([0,1], \mathbb{R}), D^{\alpha-1} u \in C([0,1], \mathbb{R}) \right\},$$

$$Y := \left\{ v \in C([0,1], \mathbb{R}), D^{\beta-1} v \in C([0,1], \mathbb{R}) \right\}.$$

For $1 < \alpha < 2$, we define on X the norm

$$\|u\|_1 := \max\left(\|u\|, \left\| D^{\alpha-1} u \right\| \right); ||u|| = \sup_{t \in [0,1]} |u(t)|, ||D^{\alpha-1}u|| = \sup_{t \in [0,1]} \left| D^{\alpha-1} u(t) \right|.$$

We also define on Y the norm

$$\|v\|_{1*} := \max\Big(\|v\| , \left\|D^{\beta-1}v\right\| \Big); ||v|| = \underset{t\in[0,1]}{sup} |v(t)| , ||D^{\beta-1}v|| = \underset{t\in[0,1]}{sup} \left|D^{\beta-1}v(t)\right| ,$$

where $1 < \beta < 2$.

For the space $X \times Y$, we define the norm

$$\|(u, v)\|_2 := \max\Big(\|u\|_1 , \|v\|_{1*} \Big).$$

It is clear that $(X \times Y, \|.\|_2)$ is a Banach space.

3. Main Results

We introduce the following quantities:

$$\begin{aligned}
M_1 &:= \frac{1}{\Gamma(\alpha+1)} + \frac{|\gamma| \Gamma(p+2) \eta^{p+\alpha}}{|\Gamma(p+2) - \gamma\eta^{p+1}| \Gamma(p+\alpha+1)},\\
M_2 &:= \frac{1}{\Gamma(\beta+1)} + \frac{|\delta| \Gamma(q+2) \zeta^{q+\beta}}{|\Gamma(q+2) - \delta\zeta^{q+1}| \Gamma(q+\beta+1)},\\
M_1' &:= \left(1 + \frac{|\gamma| \Gamma(p+2) \eta^{p+\alpha}}{|\Gamma(p+2) - \gamma\eta^{p+1}| \Gamma(3-\alpha) \Gamma(p+\alpha+1)}\right),\\
M_2' &:= \left(1 + \frac{|\delta| \Gamma(p+2) \zeta^{p+\beta}}{|\Gamma(p+2) - \delta\zeta^{p+1}| \Gamma(3-\beta) \Gamma(p+\beta+1)}\right).
\end{aligned}$$

Also, we consider the following hypotheses:

(H1): There exist non negative reals numbers $m_i, n_i, i = 1, 2,$ such that for all $t \in [0, 1] , (u_1, v_1) , (u_2, v_2) \in \mathbb{R}^2$, we have

$$\begin{aligned}
|f_1(t, u_2, v_2) - f_1(t, u_1, v_1)| &\leq m_1 |u_2 - u_1| + m_2 |v_2 - v_1| ,\\
|f_2(t, u_2, v_2) - f_2(t, u_1, v_1)| &\leq n_1 |u_2 - u_1| + n_2 |v_2 - v_1| ,
\end{aligned}$$

with $m := \max(m_1, m_2) , n := \max(n_1, n_2)$.

(H2): The functions $f_1, f_2 : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous.

(H3) : There exist positive constants L_1 and L_2, such that

$$|f_1(t, u, v)| \leq L_1, |f_2(t, u, v)| \leq L_2, \text{ for all } t \in [0, 1], u, v \in \mathbb{R}.$$

The first main result is given by:

Theorem 3.1. *Suppose that* $\gamma \neq \frac{\Gamma(p+2)}{\eta^{p+1}}, \delta \neq \frac{\Gamma(q+2)}{\zeta^{q+1}}$ *and assume that* (H1) *holds. If*

$$\max(m, n) \max(M_1', M_2') < \frac{1}{2}, \tag{3.1}$$

then the fractional system (1.1) *has a unique solution.*

Proof . Consider the operator $T : X \times Y \to X \times Y$ defined by

$$T(u,v)(t) = \left(\; T_1(v)(t), T_2(u)(t) \; \right), \tag{3.2}$$

where

$$\begin{aligned} T_1(v)(t) &= \int_0^t \frac{(t-s)}{\Gamma(\alpha)}^{\alpha-1} f_1\left(s, v(s), D^{\alpha-1}v(s)\right) ds \\ &+ \frac{\gamma\Gamma(p+2)t}{\Gamma(p+2)-\gamma\eta^{p+1}} \int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} f_1\left(s, v(s), D^{\alpha-1}v(s)\right) ds, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} T_2(u)(t) &= \int_0^t \frac{(t-s)}{\Gamma(\beta)}^{\beta-1} f_2\left(s, u(s), D^{\beta-1}u(s)\right) ds \\ &+ \frac{\delta\Gamma(q+2)t}{\Gamma(q+2)-\delta\zeta^{p+1}} \int_0^\zeta \frac{(\zeta-s)^{q+\beta-1}}{\Gamma(q+\beta)} f_2\left(s, u(s), D^{\beta-1}u(s)\right) ds. \end{aligned} \tag{3.4}$$

Thanks to Lemma 2.4, we get

$$\begin{aligned} D^{\alpha-1}T_1(v)(t) &= \int_0^t f_1\left(s, v(s), D^{\alpha-1}v(s)\right) ds \\ &+ \frac{\gamma\Gamma(p+2)}{\Gamma(p+2)-\gamma\eta^{p+1}} \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} \int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} f_1\left(s, v(s), D^{\alpha-1}v(s)\right) ds \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} D^{\beta-1}T_2(u)(t) &= \int_0^t f_2\left(s, u(s), D^{\beta-1}u(s)\right) ds \\ &+ \frac{\delta\Gamma(q+2)}{\Gamma(q+2)-\delta\zeta^{p+1}} \frac{t^{2-\beta}}{\Gamma(3-\beta)} \int_0^\zeta \frac{(\zeta-s)^{q+\beta-1}}{\Gamma(q+\beta)} f_2\left(s, u(s), D^{\beta-1}u(s)\right) ds. \end{aligned} \tag{3.6}$$

We shall show that T is a contraction:

Let $(u_1, v_1), (u_2, v_2) \in X \times Y$. Then, for each $t \in [0,1]$, we have

$$\begin{aligned} &\left|T_1(v_2)(t) - T_1(v_1)(t)\right| \\ &\leq \left(\int_0^t \frac{(t-s)}{\Gamma(\alpha)}^{\alpha-1} ds + \frac{|\gamma|\Gamma(p+2)t}{|\Gamma(p+2)-\gamma\eta^{p+1}|} \int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} ds \right) \\ &\times \sup_{0\leq s\leq 1} \left| f_1\left(s, v_2(s), D^{\alpha-1}v_2(s)\right) - f_1\left(s, v_1(s), D^{\alpha-1}v_1(s)\right) \right|. \end{aligned} \tag{3.7}$$

Using $(H1)$, we can write:

$$\begin{aligned} &\left|T_1(v_2) - T_1(v_1)\right| \\ &\leq \left(\frac{1}{\Gamma(\alpha+1)} + \frac{|\gamma|\Gamma(p+2)\eta^{p+\alpha}}{|\Gamma(p+2)-\gamma\eta^{p+1}|\Gamma(p+\alpha+1)} \right) \\ &\times \left(m_1 \|v_2 - v_1\| + m_2 \left\|D^{\alpha-1}(v_2 - v_1)\right\| \right). \end{aligned} \tag{3.8}$$

Consequently,

$$\|T_1(v_2) - T_1(v_1)\| \leq 2M_1 m \|v_2 - v_1\|_1. \tag{3.9}$$

Similarly,

$$\|T_2(u_2) - T_2(u_1)\| \leq 2M_2 n \|u_2 - u_1\|_{1*}. \tag{3.10}$$

On the other hand,

$$\left|D^{\alpha-1}T_1(v_2)(t)-D^{\alpha-1}T_1(v_1)(t)\right|$$

$$\le \int_0^t \left|f_1\left(s,v_2(s),D^{\alpha-1}v_2(s)\right)-f_1\left(s,v_1(s),D^{\alpha-1}v_1(s)\right)\right| ds+\frac{|\gamma|\Gamma(p+2)}{|\Gamma(p+2)-\gamma\eta^{p+1}|}\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} \tag{3.11}$$

$$\times\int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)}\left|f_1\left(s,v_2(t),D^{\alpha-1}v_2(t)\right)-f_1\left(s,v_1(s),D^{\alpha-1}v_1(s)\right)\right| ds.$$

This implies that

$$\left|D^{\alpha-1}T_1(v_2)(t)-D^{\alpha-1}T_1(v_1)(t)\right|$$

$$\le\left(1+\frac{|\gamma|\Gamma(p+2)\eta^{p+\alpha}}{|\Gamma(p+2)-\gamma\eta^{p+1}|\Gamma(3-\alpha)\Gamma(p+\alpha+1)}\right)\left(m_1\left\|v_2-v_1\right\|+m_2\left\|D^{\alpha-1}(v_2-v_1)\right\|\right) \tag{3.12}$$

Therefore,

$$\left\|D^{\alpha-1}T_1(v_2)-D^{\alpha-1}T_1(v_1)\right\|\le 2M_1'm\left\|v_2-v_1\right\|_1. \tag{3.13}$$

With the same arguments, we get

$$\left\|D^{\beta-1}T_2(u_2)-D^{\beta-1}T_2(u_1)\right\|\le 2M_2'm\left\|u_2-u_1\right\|_{1*}. \tag{3.14}$$

Since $M_i<M_i{}'; i=1,2$, then thanks to (3.9) and (3.13), we obtain

$$\left\|T_1(v_2)-T_1(v_1)\right\|_1\le 2M_1'm\left\|v_2-v_1\right\|_1 \tag{3.15}$$

and by (3.10) and (3.14), we get

$$\left\|T_2(u_2)-T_2(u_1)\right\|_1\le 2M_2'n\left\|u_2-u_1\right\|_{1*}. \tag{3.16}$$

Using (3.15) and (3.16), we deduce that

$$\left\|T(u_2,v_2)-T(u_1,v_1)\right\|_2\le$$

$$2\max(m,n)\max(M_1',M_2')\left\|(u_2-u_1),(v_2-v_1)\right\|_2.$$

Thanks to (3.1), we conclude that T is a contraction mapping. Hence by Banach fixed point theorem, there exists a unique fixed point which is a solution of (1.1). □

The second result is the following:

Theorem 3.2. *Suppose that* $\gamma\neq\frac{\Gamma(p+2)}{\eta^{p+1}}$, $\delta\neq\frac{\Gamma(q+2)}{\zeta^{q+1}}$ *and assume that* $(H2)$ *and* $(H3)$ *are satisfied. Then the boundary value problem* (1.1) *has at least one solution.*

Proof . First of all, we show that the operator T is completely continuous.
Step 1: Let us take $\sigma>0$ and $B_\sigma:=\{(u,v)\in X\times Y;\|(u,v)\|_2\le\sigma\}$. For $(u,v)\in B_\sigma$, using (H_3), we find that

$$\left\|T_1(v)\right\|\le\frac{L_1}{\Gamma(\alpha+1)}+\frac{L_1|\gamma|\Gamma(p+2)\eta^{p+\alpha}}{|\Gamma(p+2)-\gamma\eta^{p+1}|\Gamma(p+\alpha+1)}=L_1M_1 \tag{3.17}$$

and

$$\left\|T_2(u)\right\|\le L_2M_2. \tag{3.18}$$

Further, we have

$$\left\|D^{\alpha-1}T_1(v)\right\| \leq L_1\Big(1+\frac{|\gamma|\,\Gamma(p+2)\,\eta^{p+\alpha}}{|\Gamma(p+2)-\gamma\eta^{p+1}|\,\Gamma(3-\alpha)\,\Gamma(p+\alpha+1)}\Big)=L_1M_1' \tag{3.19}$$

and

$$\left\|D^{\beta-1}T_2(u)\right\| \leq L_2M_2'. \tag{3.20}$$

Since $M_i < {M_i}'; i=1,2$, then we can write

$$\|T_1(v)\|_1 \leq L_1M_1', \|T_2(u)\|_{1*} \leq L_2M_2'. \tag{3.21}$$

Consequently,

$$\|T(u,v)\|_2 \leq \max(L_1M_1', L_2M_2') < \infty. \tag{3.22}$$

Step 2: Let $t_1, t_2 \in [0,1], t_1 < t_2$ and $(u,v) \in B_\sigma$. We have

$$\begin{aligned} &| \, T_1(v)(t_2) - T_1(v)(t_1) \, | \\ &\leq \frac{L_1}{\Gamma(\alpha+1)}(t_2^\alpha - t_1^\alpha) + \frac{L_1\,|\gamma|\,\Gamma(p+2)\,\eta^{p+\alpha}(t_2-t_1)}{|\Gamma(p+2)-\gamma\eta^{p+1}|\,\Gamma(p+\alpha+1)}. \end{aligned} \tag{3.23}$$

Analogously, we can obtain

$$\begin{aligned} &| \, T_2(u)(t_2) - T_2(u)(t_1) \, | \\ &\leq \tfrac{L_2}{\Gamma(\beta+1)}\left(t_2^\beta - t_1^\beta\right) + \tfrac{L_2|\gamma|\Gamma(q+2)\zeta^{q+\beta}(t_2-t_1)}{|\Gamma(q+2)-\delta\zeta^{q+1}|\Gamma(q+\beta+1)}. \end{aligned} \tag{3.24}$$

On the other hand,

$$\begin{aligned} \left|D^{\alpha-1}T_1(v)(t_2) - D^{\alpha-1}T_1(v)(t_1)\right| &\leq M_1'(t_2-t_1), \\ \left|D^{\beta-1}T_2(u)(t_2) - D^{\beta-1}T_2(u)(t_1)\right| &\leq M_2'(t_2-t_1). \end{aligned} \tag{3.25}$$

The inequalities (3.23), (3.24) and (3.25) imply that T is equi-continuous. Then, by Arzela-Ascoli theorem, we conclude that T is completely continuous.

Next, we consider

$$\Omega := \{(u,v) \in X \times Y, (u,v) = \lambda T(u,v), 0 < \lambda < 1\}. \tag{3.26}$$

We show that Ω is bounded.

Let $(u,v) \in \Omega$, then $(u,v) = \lambda T(u,v)$, for some $0 < \lambda < 1$. Hence, for $t \in [0,1]$, we have:

$$u(t) = \lambda T_1(v)(t), v(t) = \lambda T_2(u)(t).$$

Thanks to $(H3)$ and using (3.17) and (3.18), we conclude that

$$\|u\| \leq \lambda L_1M_1, \|v\| \leq \lambda L_2M_2. \tag{3.27}$$

By (3.19) and (3.20), we can state that

$$\left\|D^{\alpha-1}u\right\| \leq \lambda L_1M_1', \ \left\|D^{\beta-1}v\right\| \leq \lambda L_2M_2'. \tag{3.28}$$

Consequently,

$$\|u\|_1 \leq \lambda L_1 M_1', \|v\|_{1*} \leq \lambda L_2 M_2'. \tag{3.29}$$

Therefore,

$$\|(u,v)\|_2 \leq \lambda \max\left(L_1 M_1', L_2 M_2'\right). \tag{3.30}$$

This shows that Ω is bounded.
As a conclusion of Schaefer fixed point theorem, we deduce that T has at least one fixed point, which is a solution of (1.1). □

Our third main result is based on Krasnoselskii theorem [8]. We have:

Theorem 3.3. *Let $\gamma \neq \frac{\Gamma(p+2)}{\eta^{p+1}}, \delta \neq \frac{\Gamma(q+2)}{\zeta^{q+1}}$. Suppose that (H1), (H2) and (H3) are satisfied, and*

$$\max(m,n) < \frac{1}{2}. \tag{3.31}$$

Then, the fractional system (1.1) *has at least one solution.*

Proof . Let us fix $\theta \geq \max(L_1 M_1', L_2 M_2')$ and consider $B_\theta = \{(u,v) \in X \times Y, ||(u,v)||_2 \leq \theta\}$. On B_θ, we define the operators R and S as follows:

$$\begin{aligned} R(u,v)(t) &= \left(R_1(v)(t), R_2(u)(t)\right), \\ S(u,v)(t) &= \left(S_1(v)(t), S_2(u)(t)\right), \end{aligned} \tag{3.32}$$

where,

$$\begin{aligned} R_1 v(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\left(s, v(s), D^{\alpha-1}v(s)\right) ds, \\ R_2 u(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\left(s, u(s), D^{\beta-1}u(s)\right) ds, \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} S_1 v(t) &= \frac{\gamma \Gamma(p+2) t}{\Gamma(p+2) - \gamma \eta^{p+1}} \int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} f_1\left(s, v(s), D^{\alpha-1}v(s)\right) ds, \\ S_2 u(t) &= \frac{\delta \Gamma(q+2) t}{\Gamma(q+2) - \delta \zeta^{p+1}} \int_0^\zeta \frac{(\zeta-s)^{q+\beta-1}}{\Gamma(q+\beta)} f_2\left(s, u(s), D^{\beta-1}u(s)\right) ds. \end{aligned} \tag{3.34}$$

For $(u_1,v_1),(u_2,v_2) \in B_\theta, t \in [0,1]$, we find that

$$\begin{aligned} &|R_1(v_1)(t) + S_1(v_2)(t)| \\ &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left|f_1\left(s, v_1(s), D^{\alpha-1}v_1(s)\right)\right| ds \\ &+ \frac{|\gamma|\Gamma(p+2)t}{|\Gamma(p+2) - \gamma\eta^{p+1}|} \int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} \left|f_1\left(s, v_2(s), D^{\alpha-1}v_2(s)\right)\right| ds. \end{aligned} \tag{3.35}$$

Thanks to $(H3)$, we obtain

$$\|R_1(v_1) + S_1(v_2)\| \leq L_1 M_1, \tag{3.36}$$

and

$$\|R_2(u_1)+S_2(u_2)\| \leq L_2M_2. \tag{3.37}$$

Again, by (H3), yield

$$\left\|D^{\alpha-1}R_1(v_1)+D^{\alpha-1}S_1(v_2)\right\| \leq L_1M_1' \tag{3.38}$$

and

$$\left\|D^{\beta-1}R_2(u_1)+D^{\beta-1}S_2(u_2)\right\| \leq L_2M_2'. \tag{3.39}$$

Therefore,

$$\|R(u_1,v_1)+S(u_2,v_2)\|_2 \leq \max(L_1M_1', L_2M_2') \leq \theta. \tag{3.40}$$

Thus, $R(u_1,v_1)+S(u_2,v_2) \in B_\theta$.
Now we prove the contraction of R. Using (H1), we can write

$$\|R_1(v_2)-R_1(v_1)\| \leq \frac{2m}{\Gamma(\alpha+1)}\|v_2-v_1\|_1, \tag{3.41}$$

$$\|R_2(u_2)-R_2(u_1)\| \leq \frac{2n}{\Gamma(\beta+1)}\|u_2-u_1\|_{1*}, \tag{3.42}$$

$$\left\|D^{\alpha-1}R_1(v_2)-D^{\alpha-1}R_1(v_1)\right\| \leq 2m\|v_2-v_1\|_1 \tag{3.43}$$

and

$$\left\|D^{\beta-1}R_2(u_2)-D^{\beta-1}R_2(u_1)\right\| \leq 2n\|u_2-u_1\|_{1*}. \tag{3.44}$$

Consequently,

$$\|R(u_2,v_2)-R(u_1,v_1)\|_2 \leq 2\max(m,n)\|(u_2-u_1,v_2-v_1)\|_2. \tag{3.45}$$

Thanks to (3.31), we conclude that R is a contraction mapping.

The Continuity of f_1 and f_2 given in (H2) implies that the operator S is continuous. Now, we prove the compactness of the operator S.

Let $t_1,t_2 \in [0,1], t_1<t_2$ and $(u,v) \in B_\theta$. We have

$$\begin{aligned} | S_1(v)(t_2)-S_1(v)(t_1) | &\leq \frac{L_1|\gamma|\Gamma(p+2)\eta^{p+\alpha}(t_2-t_1)}{|\Gamma(p+2)-\gamma\eta^{p+1}|\Gamma(p+\alpha+1)}, \\ | S_2(u)(t_2)-S_2(u)(t_1) | &\leq \frac{L_2|\gamma|\Gamma(q+2)\zeta^{q+\beta}(t_2-t_1)}{|\Gamma(q+2)-\delta\zeta^{q+1}|\Gamma(q+\beta+1)}. \end{aligned} \tag{3.46}$$

We have also

$$\begin{aligned} \left|D^{\alpha-1}S_1(v)(t_2)-D^{\alpha-1}S_1(v)(t_1)\right| &\leq \frac{L_1|\gamma|\Gamma(p+2)\eta^{p+\alpha}(t_2-t_1)}{|\Gamma(p+2)-\gamma\eta^{p+1}|\Gamma(p+\alpha+1)\Gamma(3-\alpha)}, \\ \left|D^{\beta-1}S_2(u)(t_2)-D^{\beta-1}S_2(u)(t_1)\right| &\leq \frac{L_2|\gamma|\Gamma(q+2)\zeta^{q+\beta}(t_2-t_1)}{|\Gamma(q+2)-\delta\zeta^{q+1}|\Gamma(q+\beta+1)\Gamma(3-\beta)}. \end{aligned} \tag{3.47}$$

Thus,

$$\|S_1(v)(t_2)-S_1(v)(t_1)\|_1 \leq \frac{L_1|\gamma|\Gamma(p+2)\eta^{p+\alpha}(t_2-t_1)}{|\Gamma(p+2)-\gamma\eta^{p+1}|\Gamma(p+\alpha+1)\Gamma(3-\alpha)}, \tag{3.48}$$

and

$$\|S_2(u)(t_2)-S_2(u)(t_1)\|_{1*} \leq \frac{L_2|\gamma|\Gamma(q+2)\zeta^{q+\beta}(t_2-t_1)}{|\Gamma(q+2)-\delta\zeta^{q+1}|\Gamma(q+\beta+1)\Gamma(3-\beta)}. \tag{3.49}$$

Therefore,

$$\|S(u,v)(t_2) - S(u,v)(t_1)\|_2 \\ \leq (t_2 - t_1) \max\left(\frac{L_1|\gamma|\Gamma(p+2)\eta^{p+\alpha}}{|\Gamma(p+2)-\gamma\eta^{p+1}|\Gamma(p+\alpha+1)\Gamma(3-\alpha)}, \frac{L_2|\gamma|\Gamma(q+2)\zeta^{q+\beta}}{|\Gamma(q+2)-\delta\zeta^{q+1}|\Gamma(q+\beta+1)\Gamma(3-\beta)} \right). \tag{3.50}$$

The right hand side of (3.50) is independent of (u, v) and tends to zero as $t_1 \to t_2$, so S is relatively compact on B_θ. Then by Ascolli-Arzella theorem, the operator S is compact. Finally, by Krasnoselskii theorem, we conclude that there exists a solution to (1.1). Theorem 3.3 is thus proved. □

4. Example

Example 4.1. Consider the following fractional differential system:

$$\begin{cases} D^{\frac{3}{2}} u(t) = \frac{e^{-t^2}|v(t)|}{16+e^t} + \frac{\sin\left(D^{\frac{1}{2}} v(t)\right)}{32(\pi t^2+1)}, t \in [0,1], \\ D^{\frac{3}{2}} v(t) = \frac{|u(t)|+\left|D^{\frac{1}{2}} u(t)\right|}{e(\pi t+20)\left(e^t+|u(t)|+\left|D^{\frac{1}{2}} u(t)\right|\right)}, t \in [0,1], \\ u(0) = 0, \ u'(0) = 4I^{\frac{1}{2}} u(\eta), \\ v(0) = 0, \ v'(0) = -8^3 I^{\frac{3}{2}} v(\xi), \end{cases}$$

where, $\alpha = \beta = \frac{3}{2}$, $p = \frac{1}{2}$, $q = \frac{3}{2}$, $\gamma = 4$, $\delta = -8^3$, $\eta = \frac{2}{5}$, $\xi = \frac{4}{5}$.
For $u_1, u_2, v_1, v_2 \in \mathbb{R}, t \in [0,1]$, we have

$$\begin{aligned} |f_1(t, u_2, v_2) - f_1(t, u_1, v_1)| &\leq \frac{1}{16}(|u_2 - u_1| + |v_2 - v_1|), \\ |f_2(t, u_2, v_2) - f_2(t, u_1, v_1)| &\leq \frac{1}{20}(|u_2 - u_1| + |v_2 - v_1|). \end{aligned}$$

We have also

$$\begin{aligned} M_1 &= \frac{4}{3\sqrt{\pi}} + \frac{3\sqrt{\pi}}{24\sqrt{\pi} - 16}, \\ M_1' &= 1 + \frac{3}{(12\sqrt{\pi} - 8)}, \\ M_2 &= \frac{4}{3\sqrt{\pi}} + \frac{3\sqrt{\pi}}{30\sqrt{\pi} + 32\sqrt{2}}, \\ M_2' &= 1 + \frac{3}{15\sqrt{\pi} + 16\sqrt{2}}. \end{aligned}$$

The conditions of the Theorem 3.1 hold. Therefore, the problem (3.41) has a unique solution on $[0, 1]$.

Example 4.2. Consider the following problem:

$$\begin{cases} D^{\frac{5}{4}} u(t) = \frac{e^{-t}}{16+|\sin(v(t))|+\left|\cos\left(D^{\frac{1}{4}} v(t)\right)\right|}, t \in [0,1], \\ D^{\frac{9}{7}} v(t) = \frac{e^{-2t}\sin(u(t))}{16+\left|\cos\left(D^{\frac{2}{7}} u(t)\right)\right|}, t \in [0,1], \\ u(0) = 0, \ u'(0) = I^3 u(\eta), \\ v(0) = 0, \ v'(0) = I^2 v(\xi). \end{cases}$$

For this example, we have $\alpha = \frac{5}{4}, \beta = \frac{9}{7}$, $p = 3$, $q = 2$, $\gamma = \delta = 1$, $\eta = \frac{4}{5}$, $\xi = \frac{1}{5}$, and

$$\begin{aligned} f_1(t,u,v) &= \frac{e^{-t}}{16+|\sin u|+|\cos v|}, \\ f_2(t,u,v) &= \frac{e^{-2t}\sin u}{16+|\cos v|}. \end{aligned}$$

It's clear that f_1 and f_2 are continuous and bounded functions. Thus the conditions of Theorem 3.2 hold, then the problem (3.42) has at least one solution on $[0, 1]$.

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