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Orthogonal metric space and convex contractions

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Abstract

In this paper, generalized convex contractions on orthogonal metric spaces are stablished in whath might be called their definitive versions. Also, we show that there are examples which show that our main theorems are genuine generalizations of Theorem 3.1 and 3.2 of [M.A. Miandaragh, M. Postolache and S. Rezapour, *Approximate fixed points of generalized convex contractions*, Fixed Point Theory and Applications 2013, 2013:255].

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1. Introduction and preliminaries

M.A. Miandaragh et. al. ([3]) introduced the concepts of generalized convex contraction and generalized convex contraction of order 2 and then stablished some interesting fixed point theorems.

We summaize in the following the basic notions and results established in [3].

Let (X, d) be a metric space and $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be two mappings.

Definition 1.1. The mapping T is called α-admissible whenever $\alpha(x, y) \geq 1$ implies $\alpha(T(x), T(y)) \geq 1$.

Definition 1.2. We say that X has the property (H) whenever for each $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.

Definition 1.3. The mapping T is called generalized convex contraction (briefly GCC) if there exist $\alpha : X \times X \to [0, \infty)$ and $a, b \in [0, 1)$ with $a + b < 1$, such that

$$\alpha(x, y)d(T^2(x), T^2(y)) \leq ad(T(x), T(y)) + bd(x, y) \qquad for \quad each \quad x, y \in X.$$

We say that α is called based mapping.

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Definition 1.4. The mapping T is called generalized convex contraction of order 2 (briefly 2-GCC) if there exist $\alpha : X \times X \to [0,\infty)$ and $a_1, a_2, b_1, b_2 \in [0,1)$ with $a_1 + a_2 + b_1 + b_2 < 1$, such that for all $x, y \in X$,

$$\alpha(x,y)d(T^2(x),T^2(y)) \leq a_1 d(x,T(x)) + a_2 d(T(x),T^2(x)) + b_1 d(y,T(y)) + b_2 d(T(y),T^2(y)).$$

Definition 1.5. We say that the mapping T has the approximate fixed point, if for all $\epsilon > 0$ there exists $x \in X$ such that $d(x,T(x)) < \epsilon$.

Theorem 1.6. (Miandaragh et al.([3], Theorem 2.1)). Let (X,d) be a metric space and $T : X \to X$. Assume that the following conditions hold:
(i) T is a GCC with the based mapping α;
(ii) T is α-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, T(x_0)) \geq 1$.
Then T has an approximate fixed point. Also, if T is continuous and X is a complete metric space then T has a fixed point. Moreover, if X has the propert (H), then T has a unique fixed point.

Theorem 1.7. (Miandaragh et al.([3], Theorem 2.2)). Let (X,d) be a metric space and $T : X \to X$. Assume that the following conditions hold:
(i) T is a 2-GCC with the based mapping α;
(ii) T is α-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, T(x_0)) \geq 1$.
Then T has an approximate fixed point. Also, if T is continuous and X is a complete metric space then T has a fixed point. Moreover, if X has the propert (H), then T has a unique fixed point.

On the other hand, very recently Eshaghi et. al. ([2]) introduced the notion of orthogonal sets and extended Banach's fixed point theorem in orthogonal metric spaces.

Definition 1.8. (Eshaghi et. al. [2]). Let $X \neq \emptyset$ and $\perp \subseteq X \times X$ be an binary relation. If $\perp$ satisfies the following condition

$$\exists x_0; (\forall y; y \perp x_0) \ \ or \ \ (\forall y; x_0 \perp y),$$

it is called an orthogonal set (briefly O-set). We denote this O-set by $(X,\perp)$.

Definition 1.9. Let $(X,\perp)$ be O-set. A sequence $\{x_n\}$ is called orthogonal sequence (briefly O-sequence) if

$$(\forall n; x_n \perp x_{n+1}) \ \ or \ \ (\forall n; \forall x_{n+1} \perp x_n).$$

Definition 1.10. Let $(X,\perp,d)$ be an orthogonal set with metric d. Then X is orthogonally complete (briefly O-complete) if every Cauchy O-sequence is convergent.

It is easy to see that every complete metric space is O-complete and we can show the converse is not true.

Definition 1.11. Let $(X,\perp,d)$ be an orthogonal metric space. Then $f : X \to X$ is orthogonality continuous (briefly $\perp-$continuous) in $a \in X$ if for each O-sequence $\{a_n\}$ in X if $a_n \to a$, then $f(a_n) \to f(a)$. Also f is $\perp-$continuous on X if f is $\perp-$continuous in each $a \in X$.

It is easy to see that every continuous mapping is $\perp-$continuous, but [2] shows that the convers is not true.

Starting from this bachground, our main aim in this paper is to extend and generalize both of the results of [3].

2. Main Results

Let $(X, \perp)$ be an O-set and d be a metric on X, $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be two mappings. We start with following definitions.

Definition 2.1. We say that T is orthogonally α-admissible whenever $x \perp y$ and $\alpha(x, y) \geq 1$ imply that $\alpha(T(x), T(y)) \geq 1$.

It is clear that every α-admissible mapping is orthogonally α-admissible. The following example shows that the convers is not true.

Example 2.2. Let $X = [0, \infty)$ and d be a usual metric. Let $T : X \to X$ be defined by $T(x) = \frac{x}{2}$ if $x \neq 1$ else $T(x) = 1$. Define now $x \perp y$ if $xy \leq \min\{x, y\}$. Not that $0 \perp x$ for all $x \in X$. Hence $(X, \perp)$ is an O-set.

At first, we shall show that T is orthogonally α-admissible. Indeed, if $x \perp y$ and $\alpha(x, y) \geq 1$, then $xy \leq x$ and $xy \leq y$ and $xy \geq 1$. This shows that $x = 1$ and $y = 1$. Thus $\alpha(T(x), T(y)) = \alpha(1, 1) = 1$. On the other hand, T is not α-admissible. Because $\alpha(\frac{3}{2}, 1) = \frac{3}{2}$ and $\alpha(T(\frac{3}{2}), T(1)) = \alpha(\frac{3}{4}, 1) = \frac{3}{4}$.
Observe that all assumptions of Theorem 2.7 are satisfied. Thus T has a unique fixed point $x = 1$. We also showed that the mapping T does not satisfy assumptions of Theorem 1.6.

Definition 2.3. We say that X has the property (OH) whenever for each $x, y \in X$, there exists $z \in X$ such that $x \perp z$, $y \perp z$, $\alpha(x, z) \geq 1$, and $\alpha(y, z) \geq 1$.

Clearly, the property (OH) implies the property (H).

Definition 2.4. The mapping T is said to be orthogonally generalized convex contraction (briefly OGCC) with based mapping α if there exist $\alpha : X \times X \to [0, \infty)$ and $a, b \in [0, 1)$ with $a + b < 1$, such that

$$\forall x, y\ ,\ x \perp y \quad (\alpha(x, y) d(T^2(x), T^2(y)) \leq a d(T(x), T(y)) + b d(x, y))$$

It is easy to see that every GCC is OGCC. But the convers is not true. To see this, we have the following example.

Example 2.5. Suppose X, $\perp$, d and α are defined as in the Example 2.2. we define $T : X \to X$ as $T(x) = 2x$ if $x > 1$ and $T(x) = \frac{x}{2}$ if $x \leq 1$. we shall show that T is OGCC. Let $x \perp y$, then the following cases are satisfied:

case1) If $x = 0$ or $y = 0$, then $\alpha(x, y) d(T^2(x), T^2(y)) = 0$.

case2) If $x \neq 0$ and $y \neq 0$, then $x \leq 1$ and $y \leq 1$, and we have

$$\alpha(x, y) d(T^2(x), T^2(y)) \leq d(T^2(x), T^2(y)) = \frac{|x - y|}{4}$$

$$= \frac{1}{4} \frac{|x - y|}{2} + \frac{1}{8} |x - y| = \frac{1}{4} d(T(x), T(y)) + \frac{1}{8} d(x, y).$$

Therefore, $\alpha(x, y) d(T^2(x), T^2(y)) \leq \frac{1}{4} d(T(x), T(y)) + \frac{1}{8} d(x, y)$ for all $x, y \in X$ with $x \perp y$. But T is not GCC. To see this, for each $a, b \in [0, 1)$ with $a + b < 1$ we have

$$\alpha(2, 3) d(T^2(2), T^2(3)) = 24 > 2a + b = a d(T(2), T(3)) + b d(2, 3).$$

Definition 2.6. The mapping T is said to be orthogonally generalized convex contraction of order 2 (briefly 2-OGCC) if there exist $\alpha : X \times X \to [0,\infty)$ and $a_1, a_2, b_1, b_2 \in [0,1)$ with $a_1+a_2+b_1+b_2 < 1$, such that for all $x, y \in X$ with $x \perp y$,

$$\alpha(x,y)d(T^2(x),T^2(y)) \leq a_1 d(x,T(x)) + a_2 d(T(x),T^2(x)) + b_1 d(y,T(y)) + b_2 d(T(y),T^2(y)).$$

Clearly, every 2-GCC is 2-OGCC and the convers always is not true the example 2.5 shows this.

Theorem 2.7. *Let $(X,\perp,d)$ be an orthogonal metric space and $T : X \to X$ be a mapping. Assume that the following conditions hold:*
(i) T is $\perp$-preserving, that is, $x\perp y$ implies $T(x)\perp T(y)$;
(ii) T is OGCC with the based mapping α;
(iii) T is orthogonally α-admissible;
(iv) there exists $x_0 \in X$ such that $x_0 \perp T(x_0)$ and $\alpha(x_0, T(x_0)) \geq 1$.
Then T has an approximate fixed point. Also, if T is $\perp$-continuous and X is an O- complete metric space then T has a fixed point. Moreover, if X has the property (OH), then T has a unique fixed point.

Proof . By condition (iv), there exists $x_0 \in X$ such that $x_0 \perp T(x_0)$ and $\alpha(x_0, T(x_0)) \geq 1$. Let $x_n = T^n(x_0)$ for all $n \geq 0$. Since T is $\perp$- preserving, then $\{x_n\}$ is an O-sequence in X. Condition (ii) follows that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$.

We shall assume that $x_n \neq x_{n+1}$ for all $n \geq 0$, since if $x_n = x_{n+1}$ for some n, then $x_n = T(x_n)$, that is x_n is fixed point of T and the assertion of Theorem is proved. Now, for $x = x_0$ and $y = T(x_0)$, we get

$$d(x_3,x_2) \leq \alpha(x_0,T(x_0))d(T^2(x_0),T^3(x_0)) \leq ad(T(x_0),T^2(x_0)) + bd(x_0,T(x_0)) \leq c\beta$$

where $\beta = d(T(x_0),T^2(x_0)) + d(x_0,T(x_0))$ and $c = a+b$. By using a similar technique to that in the proof of Theorem 3 in [1], we can see that

$$d(x_{n+1},x_n) \leq 2c^{l-1}\beta \tag{2.1}$$

where $n = 2l$ or $2l-1$ for all $l \geq 2$. Therefore, $d(x_{n+1},x_n) \to 0$ as $n \to \infty$. This implies that T has an approximate fixed point. It followes from 2.1, $\{x_n\}$ is a Cauchy O-sequence in X (see Theorem 3 in [1]). Since X is an O-complete metric space, then there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Using the $\perp$-continuity of T, we get $\lim_{n\to\infty} x_{n+1} = \lim_{n+1} T(x_n) = x^*$. Thus $T(x^*) = x^*$ and x^* is a fixed point of T.

In Theorem 2.1 of [3] it is proved that if X has the property (H), then T has a unique fixed point. By following this technique, we can prove, if X has the property (OH), then T has a unique fixed point. □

Remark 2.8. Theorem 2.7 is a generalization of Theorem 1.6 of Miandaragh et. al.. In fact, we suppose that

$$x\perp y \iff \alpha(x,y)d(T^2(x),T^2(y)) \leq ad(T(x),T(y)) + bd(x,y).$$

Since T is GCC, then for each x and y in X, we see that $x\perp y$. Hence $(X,\perp)$ is an O-set. We have the following statements:

1) Since T is GCC, then T is $\perp$-preserving and OGCC.

2) Since T is α-admissible, then T is orthogonally α-admissible.

3) By using Condition (iii) in Theorem 1.6 and definition of $\perp$, there exists $x_0 \in X$ such that $x_0 \perp T(x_0)$ and $\alpha(x_0, T(x_0)) \geq 1$.
4) If X is complete, then X is also O-complete.
5) If T is continuous, then T is also $\perp$-continuous.
6) If X has the property (H), using definition of $\perp$, we see that X has the property (OH).

Therefore, All conditions of Theorem 2.7 are hold. Applying Theorem 2.7 we can see the results.

Now, we shall show that there is an example which shows that Theorem 2.7 is a genuine generalization of Theorem 1.6 of Miandaragh et al..

Example 2.9. Let $X = (0, \infty)$ and d be a usual metric. Suppose $x \perp y$ if $xy = x$. It is easy to see that $(X, \perp)$ is an O-set. Let $T : X \to X$ defined by $T(x) = \frac{x+1}{2}$ if $x \leq 1$ and $T(x) = \frac{1}{2}$ if $x > 1$ and define $\alpha : X \times X \to [0, \infty)$ by $\alpha(x, y) = y$

Now, we shall show that T satisfies all assumptions of our Theorem 2.7. We have the following steps:

step1) X is O-complete (not complete). In fact, if $\{x_n\}$ is an arbitrary Cauchy O-sequence in X, then there exists $n_0 \in \mathbb{N}$ such that $x_n = 1$ for all $n \geq n_0$. It follows that $\{x_n\}$ is the constant sequence 1 and hence x_n converges to $x = 1$.

case 2) The function T is orthogonally α-admissible but is not α-admissible. To see this, let $x \perp y$ and $\alpha(x, y) \geq 1$. Then $y = 1$, $T(y) = 1$ and $\alpha(T(x), T(y)) = T(y) = 1 \geq 1$. On the other hand, we have $\alpha(2, 3) = 3 \geq 1$ and $\alpha(T(2), T(3)) = \alpha(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2} < 1$.

case 3) T is $\perp$-continuous but is not continuous. In fact, let $\{x_n\}$ be an O-sequence converging to a point $x \in X$. By using the step 1, there exists $n_0 \in \mathbb{N}$ such that $x_n = 1$ for all $n \geq n_0$ and $x = 1$. This implies that $T(x_n) \to 1 = T(x)$.

step 4) T is $\perp$-preserving. In fact, if $x \perp y$, then $y = 1$. By definition of T, we see that $T(y) = 1$ and $T(x)\ T(y) = T(x)$, this implies that $T(x) \perp T(y)$.

step 5) T is OGCC. To see this, let $x \perp y$, it followes that $y = 1$. If $x > 1$ then

$$\alpha(x, y) d(T^2(x), T^2(y)) = \frac{1}{4} = \frac{1}{2} d(T(x), T(y)).$$

If $x < 1$, then

$$\alpha(x, y) d(T^2(x), T^2(y)) = \frac{|x-1|}{4} = \frac{1}{2} d(T(x), T(y)).$$

If $x = 1$, then $T^2(x) = 0$, hence $\alpha(x, y) d(T^2(x), T^2(y)) = 0$.
Putting $a = \frac{1}{2}$ and $b = 0$, we see that T is OGCC.

case 6) X has the property (OH). In fact, for all $x, y \in X$ there exists an element $z = 1$ such that $x \perp z$ and $y \perp z$ and $\alpha(x, z) = 1 \geq 1$ and $\alpha(y, z) = 1 \geq 1$.

Observe that all assumptions of Theorem 2.7 are satisfied. Thus T has a unique fixed point $x = 1$. We also showed that the mapping T does not satisfy assumptions of Theorem 1.6.

Theorem 2.10. *Let $(X, \perp, d)$ be an orthogonal metric space and $T : X \to X$ be a mapping. Assume that the following conditions hold:*

(i) T is $\perp$-preserving;

(ii) T is a 2-OGCC with the based mapping α;

(iii) T is orthogonally α-admissible;

(iv) there exists $x_0 \in X$ such that $x_0 \perp T(x_0)$ and $\alpha(x_0, T(x_0)) \geq 1$.

Then T has an approximate fixed point. Also, if T is $\perp$-continuous and X is an O- complete metric space then T has a fixed point. Moreover, if X has the propert (OH), then T has a unique fixed point.

Proof . Replacing 2-OGCC with OGCC and following the lines in the proof of Theorem 2.7, one can constract O-sequence $\{x_n\}$ in X such that $x_{n+1} \neq x_n$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$. Let $\beta = d(T(x_0), T^2(x_0)) + d(x_0, T(x_0))$ and $c = a_1 + a_2 + b_1$ and $d = 1 - b_2$, then

$$d(x_3, x_2) \leq \alpha(x_0, T(x_0)) d(T^3(x_0), T^2(x_0)) \leq a_1 d(x_0, T(x_0)) + a_2 d(T(x_0), T^2(x_0))$$

$$+ b_1 d(T(x_0), T^2(x_0)) + b_2 d(T^2(x_0), T^3(x_0)).$$

Hence, $d(x_3, x_2) \leq (\frac{c}{d})\beta$. Follwing the proof of Theorem 2.1 of [3], one can prove that $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$. Hence, T has an approximate fixed point. Proceeding as in proof of Theorem 2.7, one can prove the results. □

Remark 2.11. Theorem 2.10 is a real extention of Theorem 1.7 of Miandaragh et al.. It is enough to define $x \perp y$ if

$$\alpha(x, y) d(T^2(x), T^2(y)) \leq a_1 d(x, T(x)) + a_2 d(T(x), T^2(x)) + b_1 d(y, T(y)) + b_2 d(T(y), T^2(y)).$$

Proceeding as in the Corollarly 2.8 , we can see the result. Also, in the Example 2.9, put $a_1 = \frac{1}{2}$ and $a_2 = b_1 = b_2 = 0$, then we can see that the mapping T is 2-OGCC. This implies that Theorem 2.10 is a real extention of Theorem 1.7 of Miandaragh et al..

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