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Subordination and superordination properties for convolution operator

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Abstract

In the present paper, a certain convolution operator of analytic functions is defined. Subordination and superordination- preserving properties for a useful class of analytic operators on the space of normalized analytic functions in the open unit disk are obtained. Sandwich- type results are also obtained.

Keywords: Convolution operator; differential subordination; differential superordination; best dominant; best subordinant.
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1. Introduction and preliminaries

Let $H(\Delta)$ denote the class of analytic functions in the open unit disk $\Delta = \{z : |z| < 1\}$, and normalized by $f(0) = f'(0) - 1 = 0$. Also let *A(p)* be the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad p \in \mathbb{N},$$

and let $A(1) = A$. For a positive integer number n and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in H(\Delta) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\}$$

Let f and F be members of the analytic function class $H(\Delta)$. The function f is said to be subordinate to F or F is said to be superordinate of f, if there exist a function w analytic in Δ with

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$w(0)=0$, and $|w(z)|<1$ such that $f(z)=F(w(z))$ and we write $f(z)\prec F(z)$ or $f\prec F$. If function F is univalent, then we have $f\prec F$ if and only if $f(0)=F(0)$ and $f(\Delta)\subset F(\Delta)$.

Let $\varphi:\mathbb{C}^2\times\Delta\longrightarrow\mathbb{C}$ and h be analytic in Δ. If p is analytic in Δ and satisfies the (first-order) differential subordination

$$\varphi(p(z),zp'(z);z)\prec h(z), \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solution of the differential subordination, or dominant if $p\prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $q\prec\tilde{q}$ for all dominant of q of (1.1) is called the best dominant.

Let $\varphi:\mathbb{C}^2\times\Delta\longrightarrow\mathbb{C}$ and h be analytic in Δ. If p and $\varphi(p(z),zp'(z);z)$ are univalent and p satisfies the (first-order) differential superordination

$$h(z)\prec\varphi\left(p(z),zp'(z);z\right) \tag{1.2}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solution of the differential superordinate, or more simply a subordinant if $q\prec p$ for all q satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $\tilde{q}\prec q$ for all subordinant of q of (1.2) is called the best subordinant.

Ali et al [2] have obtained sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy $q_1(z)\prec\frac{zf'(z)}{f(z)}\prec q_2(z)$, where q_1 and q_2 are given univalent functions in Δ with $q_1(0)=q_2(0)=1$.

For two functions $f_j(z)$, $j=1,2$, given by

$$f_j(z)=z+\sum_{k=2}^{\infty}a_{k,j}z^k$$

we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1*f_2)(z):=z+\sum_{k=2}^{\infty}a_{k,1}a_{k,2}z^k=(f_2*f_1)(z),\qquad z\in\Delta.$$

In terms of the Pochhammer symbol (or the shifted factorial), define $(\kappa)_n$ by

$$(\kappa)_0=1,\quad \textit{and}\quad (\kappa)_n=\kappa(\kappa+1)(\kappa+2)\ldots(\kappa+n-1),\qquad n\in\mathbb{N}:=\{1,2,\ldots\}.$$

Also, Aghalary et al [1] have defined a function $\phi_a^\lambda(b,c;z)$ by

$$\phi_a^\lambda(b,c;z):=1+\sum_{n=1}^{\infty}\left(\frac{a}{a+n}\right)^\lambda\frac{(b)_n}{(a)_n}z^n,\qquad z\in\Delta, \tag{1.3}$$

where $b\in\mathbb{C}$, $c\in\mathbb{R}\backslash Z_0^-$, $a\in\mathbb{C}\backslash Z_0^-(Z_0^-=\{0,-1,-2,\ldots\})$ and $\lambda\geq 0$. Corresponding to the function $\phi_a^\lambda(b,c;z)$, given by (1.3), they have introduced the following convolution operator

$$L_a^\lambda(b,c;\beta)f(z):=\phi_a^\lambda(b,c;z)*\left(\frac{f(z)}{z}\right)^\beta,\qquad f\in A,\quad \beta\in\mathbb{C}\backslash\{0\}. \tag{1.4}$$

It is easy to see that

$$z(\phi_a^\lambda(b,c;z))'=a\phi_a^\lambda(b,c;z)-a\phi_a^{\lambda+1}(b,c;z), \tag{1.5}$$

and

$$z(L_a^{\lambda+1}(b,c;\beta)f(z))' = aL_a^{\lambda}(b,c;\beta)f(z) - aL_a^{\lambda+1}(b,c;\beta)f(z). \tag{1.6}$$

The operator $L_a^{\lambda}(b,c;\beta)f(z)$ includes, as its special cases, Komatu integral operator (see [4], [5], [10]), some fractional calculus operators (see [4], [12], [13]) and Carlson-Shaffer operator (see [3]).

Making use of the principle of subordinant between analytic functions Miller et all [8] obtained some interesting subordination theorems involving certain operators. Also Miller and Mocanu [7] considered subordination-preserving properties of certain integral operator investigations as the dual concept of differential subordination. In the present investigation, we obtain the subordination and superordination-preserving properties of the convolution operator L_a^{λ} defined by (1.4) with the Sandwich-type theorems.

2. Definitions and Preliminaries

The following definitions and Lemmas will be required in our present investigation.

Definition 2.1. If $0 \le \alpha < 1$, $\lambda \ge 0$ and $a \in \mathbb{C}\backslash Z_0^-(Z_0^- = \{0,-1,-2,\ldots\})$, let $\mathcal{L}_a^{\lambda}(\alpha)$ denote the class of functions $f \in A$ wich satisfies the inequality

$$Re[L_a^{\lambda}(b,c;\beta)f(z)] > \alpha$$

For $a = 1$, we set $\mathcal{L}_1^{\lambda}(\alpha) = \mathcal{L}^{\lambda}(\alpha)$.

Definition 2.2. [6] Denote by Q the set of all functions q that are analytic and injective on $\overline{\Delta}\backslash E(q)$ where

$$E(q) = \{\xi \in \Delta : \lim_{z\to\xi} q(z) = \infty\}$$

and are such that $h'(\xi) \ne 0$ for $\xi \in \partial\Delta\backslash E(q)$.

Lemma 2.3. [6] Let $h(z)$ be analytic and convex univalent in Δ and $h(0) = a$. Also $p(z)$ be analytic in Δ with $p(0) = a$. If

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z), \quad \gamma \ne 0, \quad Re\gamma \ge 0,$$

then $p(z) \prec q(z) \prec h(z)$, where

$$q(z) = \frac{\gamma}{z^{\gamma}} \int_0^z h(t)t^{\gamma-1}dt.$$

Furthermore $q(z)$ is a convex function and is the best dominant.

Lemma 2.4. [7] Let $h(z)$ be a convex in Δ, $h(0) = a, \gamma \ne 0$ and $\Re\gamma \ge 0$. Also $p \in \mathcal{H}[a,n] \cap Q$. If $p(z) + \frac{zp'(z)}{\gamma}$ is univalent in Δ, $h(z) \prec p(z) + \frac{zp'(z)}{\gamma}$ and $q(z) = \frac{\gamma}{z^{\gamma}} \int_0^z h(t)t^{\gamma-1}dt$ then $q(z) \prec p(z)$, and $q(z)$ is a convex function and is the best subordinant.

Lemma 2.5. [11] Let $q(z)$ be a convex univalent function in Δ and $\psi, \gamma \in \mathbb{C}$ with $Re(1 + \frac{zq''(z)}{q'(z)}) > max\{0, -Re\frac{\psi}{\gamma}\}$, $q(0) = a, \gamma \ne 0$ and $Re\gamma \ge 0$. If $p(z)$ is analytic in Δ and $\psi p(z) + \gamma zp'(z) \prec \psi q(z) + \gamma zq'(z)$ then $p(z) \prec q(z)$, and $q(z)$ is the best dominant.

Lemma 2.6. [9] Let $q(z)$ be a convex univalent function in Δ and $\eta \in \mathbb{C}$, assume that $Re\eta > 0$. If $p(z) \in \mathcal{H}[a,n] \cap Q$ and $p(z) + \eta zp'(z) \prec q(z) + \eta zq'(z)$ which implies that $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

3. Differential subordination defined by convolution operator

Theorem 3.1. *If $0 \leq \alpha < 1$, $\lambda \geq 0$ and $a \in \mathbb{C}\backslash Z_0^-$, then we have*

$$\mathcal{L}_a^{\lambda}(\alpha) \subset \mathcal{L}_a^{\lambda+1}(\delta),$$

where

$$\delta(\alpha, a) = a\beta(a) + a(2\alpha - 1)\beta(a+1),$$

and

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t} dt.$$

The result is sharp.

Proof . First note that $f \in \mathcal{L}_a^{\lambda}(\alpha)$ and

$$z(L_a^{\lambda+1}(b,c;\beta)f(z))' = aL_a^{\lambda}(b,c;\beta)f(z) - aL_a^{\lambda+1}(b,c;\beta)f(z). \tag{3.1}$$

We define $p(z) = L_a^{\lambda+1}(b,c;\beta)f(z)$. From the relation (1.1) we have

$$L_a^{\lambda}(b,c;\beta)f(z) = p(z) + \frac{zp'(z)}{a}.$$

Now from Lemma 2.3, for $\gamma = a$, it follows that

$$p(z) = L_a^{\lambda+1}(b,c;\beta)f(z) \prec q(z) = \frac{a}{z^a}\int_0^z \frac{1+(2\alpha-1)t}{1+t} t^{a-1} dt,$$

therefore we have

$$\mathcal{L}_a^{\lambda}(\alpha) \subset \mathcal{L}_a^{\lambda+1}(\delta),$$

where

$$\delta = \min_{|z|\leq 1} Req(z) = q(1) = a\beta(a) + a(2\alpha-1)\beta(a+1).$$

Furthermore $q(z)$ is a convex function and is the best dominant. □

For the class $\mathcal{L}^{\lambda}$ we obtain the next corollary.

Corollary 3.2. If $0 \leq \alpha < 1$ and $\lambda \geq 0$, then we have

$$\mathcal{L}^{\lambda}(\alpha) \subset \mathcal{L}^{\lambda+1}(\delta),$$

where

$$\delta = \delta(\alpha) = 2\alpha - 1 + 2(1-\alpha)\ln 2,$$

and the result is sharp.

Theorem 3.3. *Let $h \in H(\Delta)$, with $h(0) = 1$ and $h'(0) \neq 0$, which verifies the inequality $Re[1 + \frac{zh''(z)}{h'(z)}] > -\frac{1}{2}$. If $f \in A$ and satisfies the differential subordination*

$$L_a^{\lambda}(b,c;\beta)f(z) \prec h(z), \tag{3.2}$$

then

$$L_a^{\lambda+1}(b,c;\beta)f(z) \prec q(z), \tag{3.3}$$

where

$$q(z) = \frac{a}{z^a}\int_0^z h(t)t^{a-1} dt.$$

The function $q(z)$ is convex and is the best dominant.

Proof . Let
$$p(z) = L_a^{\lambda+1}(b,c;\beta)f(z). \tag{3.4}$$
Differentiating (3.4) with respect to z, we have $p'(z) = (L_a^{\lambda+1}(b,c;\beta)f(z))'$. From the relation (1.1) we have
$$\frac{zp'(z)}{a} + p(z) = L_a^{\lambda}(b,c;\beta)f(z).$$
Now, in view of (2.4), we obtain the following subordination
$$\frac{zp'(z)}{a} + p(z) \prec h(z).$$
Then from Lemma 2.3 for $\gamma = a$ we conclude that
$$p(z) = L_a^{\lambda+1}(b,c;\beta)f(z) \prec q(z),$$
where
$$q(z) = \frac{a}{z^a}\int_0^z h(t)t^{a-1}dt$$
and $q(z)$ is the best dominant. □

Taking $\lambda = 0$ in Theorem 3.3, we arrive the following corollary.

Corollary 3.4. Let $h \in H(\Delta)$, with $h(0) = 1, h'(0) \neq 0$, and $Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$. If $f \in A$ and satisfies $(\frac{f(z)}{z})^\beta \prec h(z)$, then $L_a(b,c;\beta) \prec q(z)$ where $q(z) = \frac{a}{z^a}\int_0^z h(t)t^{a-1}dt$. The function $q(z)$ is the best dominant.

By setting $a = \gamma + \beta, \lambda = 0$ and $b = c = 1$ in Theorem 3.3, we get the following corollary.

Corollary 3.5. Let $h \in H(\Delta)$, with $h(0) = 1$ and $h'(0) \neq 0$, which satisfies the inequality $Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$. If $f \in A$ and satisfies the differential subordination $(\frac{f(z)}{z})^\beta \prec h(z)$, then
$$\frac{\gamma+\beta}{z^{\gamma+\beta}}\int_0^z u^{\gamma-1}(f(u))^\beta du \prec \frac{1}{z}\int_0^z h(u)du$$
The function $\frac{1}{z}\int_0^z h(u)du$ is the best dominant.

Corollary 3.6. Let $0 < R \leq 1$ and let $h(z)$ be convex in Δ, defined by $h(z) = 1 + Rz + \frac{Rz}{2+Rz}$, with $h(0) = 1$. If $f \in A$ satisfies in the following differential subordination
$$L_a^{\lambda}(b,c;\beta)f(z) \prec h(z),$$
then
$$L_a^{\lambda+1}(b,c;\beta)f(z) \prec q(z),$$
where
$$\begin{aligned} q(z) &= \frac{a}{z^a}\int_0^z \left(1 + Rt + \frac{Rt}{2+Rt}t^{a-1}\right)dt \\ &= z^{a-1} + Ra\left(\frac{z^a}{a+1} + \frac{M(z)}{z}\right), \end{aligned}$$
with
$$M(z) = \int_0^z \frac{t^a}{2+Rt}dt.$$
The function $q(z)$ is convex and is the best dominant.

If $a = 1$, Corollary 3.6 becomes:

Corollary 3.7. Let $0 < R \leq 1$ and let $h(z)$ be convex in Δ, defined by $h(z) = 1 + Rz + \frac{Rz}{2+Rz}$, with $h(0) = 1$. If $f \in A$ and suppose that

$$L^{\lambda}(b, c; \beta)f(z) \prec h(z),$$

then

$$L^{\lambda+1}(b, c; \beta)f(z) \prec q(z)(z \in \Delta),$$

where

$$\begin{aligned} q(z) \quad &= \frac{1}{z}\int_0^z \left(1 + Rt + \frac{Rt}{2 + Rt}\right) dt \\ &= 2 + \frac{Rz}{2} - \frac{2}{Rz}\log(2 + Rz), \end{aligned}$$

The function $q(z)$ is convex and is the best dominant.

By taking $R = 1$ in Corollary 3.7 we have the following corollaries.

Corollary 3.8. Let $h(z)$ be convex in Δ, defined by $h(z) = 1 + z + \frac{z}{2+z}$, with $h(0) = 1$. If $f \in A$, satisfies in the differential subordination

$$L^{\lambda}(b, c; \beta)f(z) \prec h(z),$$

then

$$L^{\lambda+1}(b, c; \beta)f(z) \prec q(z),$$

where

$$q(z) = 2 + \frac{z}{2} - \frac{2}{z}\log(2 + z).$$

The function $q(z)$ is convex and is the best dominant.

Corollary 3.9. Let $h(z)$ be convex in Δ, defined by $h(z) = 1 + z + \frac{z}{2+z}$, with $h(0) = 1$. Suppose that $\gamma \in \mathbb{C}$, $a = \gamma + \beta$, $\lambda = 0$ and $b = c = 1$. If $f \in A$ and satisfies the differential subordination $(\frac{f(z)}{z})^{\beta} \prec h(z)$, then

$$\frac{\gamma + \beta}{z^{\gamma+\beta}}\int_0^z u^{\gamma-1}(f(u))^{\beta}du \prec q(z) = 2 + \frac{z}{2} - \frac{2}{z}\log(2 + z).$$

The function $q(z)$ is convex and is the best dominant.

Corollary 3.10. Let $h(z) = \frac{1+(2\alpha-1)z}{1+z}$ be convex function in Δ, with $h(0) = 1$. If $f \in \mathcal{L}^{\lambda}(\alpha)$ and $L^{\lambda}(b, c; \beta)f(z) \prec h(z)$ then

$$L^{\lambda+1}(b, c; \beta)f(z) \prec q(z),$$

where

$$q(z) = 2\alpha - 1 + 2(1 - \alpha)\frac{\log(1 + z)}{z}.$$

The function $q(z)$ is convex and is the best dominant.

Theorem 3.11. *Let $q(z)$ be a convex function with $q(0) = 1$, and let h be a function such that $h(z) = q(z) + \frac{zq'(z)}{q(z)}$. If $f \in H(\Delta)$ and satisfies the differential subordination*

$$L_a^\lambda(b, c; \beta)f(z) \prec h(z), \tag{3.5}$$

then

$$L_a^{\lambda+1}(b, c; \beta)f(z) \prec q(z)$$

and this result is sharp.

Proof . We have

$$z(L_a^{\lambda+1}(b, c; \beta)f(z))' = aL_a^\lambda(b, c; \beta)f(z) - aL_a^{\lambda+1}(b, c; \beta)f(z). \tag{3.6}$$

Let $p(z) = L_a^{\lambda+1}(b, c; \beta)f(z)$, then from (3.5) and (3.6) , we have

$$p(z) + \frac{zp'(z)}{a} \prec q(z) + \frac{zq'(z)}{a}.$$

An application of Lemma 2.6, we conclude that $p(z) \prec q(z)$ or $L_a^{\lambda+1}(b, c; \beta)f(z) \prec q(z)$ and this result is sharp. □

Theorem 3.12. *Let $h \in H(\Delta)$, with $h(0) = 1$, and $h'(0) \neq 0$, which satisfies in the inequality $Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$. If $f \in A$ and satisfies the differential subordination*

$$(L_a^{\lambda+1}(b, c; \beta)f(z))' \prec h(z),$$

then

$$\frac{L_a^{\lambda+1}(b, c; \beta)f(z)}{z} \prec q(z),$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)t^{a-1}dt,$$

the function $q(z)$ is the best dominant.

Proof . Let us define the function f by

$$f(z) = \frac{L_a^{\lambda+1}(b, c; \beta)f(z)}{z}. \tag{3.7}$$

Differentiating with respect to z logarithmically, we have

$$\frac{zp'(z)}{p(z)} = \frac{z(L_a^{\lambda+1}(b, c; \beta)f(z))'}{L_a^{\lambda+1}(b, c; \beta)f(z)} - 1$$

and

$$p(z) + zp'(z) = (L_a^{\lambda+1}(b, c; \beta)f(z))'$$

Now, from (3.7) we obtain

$$p(z) + zp'(z) \prec h(z)$$

Then, by Lemma 2.3 , for $\gamma = 1$ we have $p(z) \prec q(z)$ or

$$\frac{L_a^{\lambda+1}(b, c; \beta)f(z)}{z} \prec \frac{1}{z}\int_0^z h(t)dt$$

and the function $q(z)$ is the best dominant. Therefore, we complete the proof of theorem 3.12. □

Suppose that $\lambda = 0$ and in Theorem 3.12 we have the following result.

Corollary 3.13. Let $h \in H(\Delta)$, with $h(0) = 1$ and $h'(0) \neq 0$, which satisfies in the inequality

$$Re(1 + \frac{zh''(z)}{h'(z)}) > -\frac{1}{2}$$

. If $f \in A$ and $(L_a(b, c; \beta)f(z))' \prec h(z)$ then $\frac{L_a(b,c;\beta)f(z)}{z} \prec \frac{1}{z}\int_0^z h(t)dt$, and the function $\frac{1}{z}\int_0^z h(t)dt$ is the best dominant.

By taking $\gamma \in \mathbb{C}$, $a = \gamma + \beta$, $\lambda = 0$, and $b = c = 1$ in the Theorem 3.12 we get the following result.

Corollary 3.14. Let $f \in A$, $h \in H(\Delta)$ and $h(0) = 1, h'(0) \neq 0$. If $Re(1 + \frac{zh''(z)}{h'(z)}) > -\frac{1}{2}$

$$\frac{-(\gamma+\beta)}{z^{\gamma+\beta+1}}\int_0^z u^{\gamma-1}(f(u))^{\beta}du + \frac{\gamma+\beta}{z^{\beta+1}} \prec h(z),$$

then

$$\frac{\gamma+\beta}{z^{\gamma+\beta-1}}\int_0^z u^{\gamma-1}(f(u))^{\beta}du \prec \frac{1}{z}\int_0^z h(u)du.$$

The function $\frac{1}{z}\int_0^z h(u)du$ is the best dominant.

Corollary 3.15. Let $0 < R \leq 1$ and let $h(z)$ be convex in Δ, defined by $h(z) = 1 + Rz + \frac{Rz}{2+Rz}$, with $h(0) = 1$. If $f \in A$ satisfies in the following differential subordination

$$(L^{\lambda+1}(b, c; \beta)f(z))' \prec h(z),$$

then

$$\frac{L^{\lambda+1}(b, c; \beta)f(z)}{z} \prec q(z),$$

where

$$\begin{aligned} q(z) &= \frac{1}{z}\int_0^z \left(1 + Rt + \frac{Rt}{2+Rt}\right)dt \\ &= 1 + \frac{Rz}{2} + \frac{RM(z)}{z}, \end{aligned}$$

with

$$M(z) = \frac{z}{R} - \frac{2}{R^2}(\ln(2 + Rz)) - \frac{2}{R}\ln 2.$$

The function $q(z)$ is convex and is the best dominant.

Suppose that $\gamma \in \mathbb{C}$, $a = \gamma + \beta$, $\lambda = 0$ and $b = c = 1$ in Corollary 3.15 we have the following corollary.

Corollary 3.16. Let $h(z)$ be convex in Δ, defined by $h(z) = 1 + z + \frac{z}{2+z}$, with $h(0) = 1$. If $f \in A$, satisfies in the differential subordination

$$\frac{-(\gamma+\beta)}{z^{\gamma+\beta+1}}\int_0^z u^{\gamma-1}(f(u))^{\beta}du + \frac{\gamma+\beta}{z^{\beta+1}} \prec h(z),$$

then

$$\frac{\gamma+\beta}{z^{\gamma+\beta-1}}\int_0^z u^{\gamma-1}(f(u))^{\beta}du \prec \frac{1}{z}\int_0^z h(u)du,$$

where

$$q(z) = 2 + \frac{z}{2} - \frac{2}{z}\log(2 + z).$$

The function $q(z)$ is convex and is the best dominant.

Corollary 3.17. Let $h(z) = \frac{1+(2\alpha-1)z}{1+z}$ be convex function in Δ, with $h(0) = 1$. If $f \in \mathcal{L}^{\lambda}(\alpha)$ and

$$(L^{\lambda+1}(b, c; \beta)f(z))' \prec h(z),$$

then

$$\frac{L^{\lambda+1}(b, c; \beta)f(z)}{z} \prec q(z),$$

where

$$q(z) = 2\alpha - 1 + 2(1-\alpha)\frac{\log(1+z)}{z}.$$

The function $q(z)$ is convex and is the best dominant.

Theorem 3.18. *Let $q(z)$ be a convex function in Δ, $q(0) = 1$ and $h(z) = q(z) + \frac{zq'(z)}{q(z)}$. If $f \in H(\Delta)$ and satisfies the differential subordination*

$$(L_a^{\lambda+1}(b, c; \beta)f(z))' \prec h(z), \tag{3.8}$$

then

$$\frac{L_a^{\lambda+1}(b, c; \beta)f(z)}{z} \prec q(z)$$

and this result is sharp.

Proof . Let

$$p(z) = \frac{L_a^{\lambda+1}(b, c; \beta)f(z)}{z}. \tag{3.9}$$

Logarithmic differentiation of (3.9) and through a little simplification we obtain

$$p(z) + zp'(z) = (L_a^{\lambda+1}(b, c; \beta)f(z))'.$$

Now by using Lemma 2.6, we conclude that the differential equation

$$\frac{L_a^{\lambda+1}(b, c; \beta)f(z)}{z} \prec q(z)$$

and this result is sharp. □

4. Differential superordination defined by convolution operator

The results this section are obtained with differential superordination method.

Theorem 4.1. *Let h be convex function in Δ, with $h(0) = 1$, and $f \in A$. Assume that $L_a^{\lambda}(b, c; \beta)f(z)$ is univalent with $L_a^{\lambda+1}(b, c; \beta)f(z) \in \mathcal{H}[1, n] \cap Q$. If $h(z) \prec L_a^{\lambda}(b, c; \beta)f(z)$ then*

$$q(z) \prec L_a^{\lambda+1}(b, c; \beta)f(z), \tag{4.1}$$

where

$$q(z) = \frac{a}{z^a}\int_0^z h(t)t^{a-1}dt.$$

The function $q(z)$ is the best subordinant.

Proof . If we let

$$p(z) = L_a^{\lambda+1}(b, c; \beta)f(z),$$

then from the relation (1.6) we have $p(z) + \frac{zp'(z)}{a} = L_a^{\lambda}(b, c; \beta)f(z)$. Now according to Lemma 2.4 we get the desired result (4.1). □

Corollary 4.2. Suppose that $\gamma \in \mathbb{C}$, $a = \gamma + \beta$ $\lambda = 0$ and $b = c = 1$. Let $h \in H(\Delta)$ be convex function in Δ, with $h(0) = 1$, and $f \in A$. Assume that $(\frac{f(z)}{z})^{\beta}$ is univalent with $\frac{\gamma+\beta}{z^{\gamma+\beta}} \int_0^z u^{\gamma-1}(f(u))^{\beta} du \in \mathcal{H}[1, n] \cap Q$. If $h(z) \prec (\frac{f(z)}{z})^{\beta}$ then

$$\frac{1}{z} \int_0^z h(u)du \prec \frac{\gamma + \beta}{z^{\gamma+\beta}} \int_0^z u^{\gamma-1}(f(u))^{\beta} du$$

and $\frac{1}{z} \int_0^z h(u)du$ is the best subordinant.

Corollary 4.3. Let $h(z)$ be a convex mapping in Δ, defined by $h(z) = 1 + z + \frac{z}{2+z}$, with $h(0) = 1$. Suppose that $\gamma \in \mathbb{C}, a = \gamma + \beta, \lambda = 0, b = c = 1$, and $f \in A$ and $(\frac{f(z)}{z})^{\beta}$ is univalent with $\frac{\gamma+\beta}{z^{\gamma+\beta}} \int_0^z u^{\gamma-1}(f(u))^{\beta} du \in \mathcal{H}[1, n] \cap Q$. If $h(z) \prec (\frac{f(z)}{z})^{\beta}$ then $q(z) \prec \frac{\gamma+\beta}{z^{\gamma+\beta}} \int_0^z u^{\gamma-1}(f(u))^{\beta} du$, where $q(z) = 2 + \frac{z}{2} - \frac{2}{z} \log(2 + z)$. The function $q(z)$ is the best subordinant.

Corollary 4.4. Let $h(z) = \frac{1+(2\alpha-1)z}{1+z}$ be a convex function in Δ with $h(0) = 1$. Assume that $f \in \mathcal{L}^{\lambda+1}(\alpha)$ and $L^{\lambda}(b, c; \beta)f(z)$ is univalent with $L^{\lambda+1}(b, c; \beta)f(z) \in \mathcal{H}[1, n] \cap Q$. If $h(z) \prec L^{\lambda}(b, c; \beta)f(z)$ then

$$q(z) \prec L_a^{\lambda+1}(b, c; \beta)f(z),$$

where

$$q(z) = 2\alpha - 1 + 2(1 - \alpha)\frac{\log(1 + z)}{z}.$$

The function $q(z)$ is the best subordinant.

Theorem 4.5. Let h be a convex function in Δ, with $h(0) = 1$, and $f \in A$. Assume that $(L_a^{\lambda+1}(b, c; \beta)f(z))'$ is univalent with $\frac{L_a^{\lambda+1}(b,c;\beta)f(z)}{z} \in \mathcal{H}[1, n] \cap Q$. If $h(z) \prec (L_a^{\lambda+1}(b, c; \beta)f(z))'$ then

$$q(z) \prec \frac{L_a^{\lambda+1}(b, c; \beta)f(z)}{z},$$

where

$$q(z) = \frac{1}{z} \int_0^z h(t)dt.$$

The function $q(z)$ is the best subordinant.

5. Sandwich results

Combining results of differential subordinations and superordinations, we arrive at the following "Sandwich results".

Theorem 5.1. Let $q_1(z)$ be convex univalent in the open unit disk Δ, and $q_2(z)$ be univalent in the open unite disk Δ and $f \in A$. Also let $L_a^{\lambda}(b, c; \beta)f(z)$ be univalent with $L_a^{\lambda+1}(b, c; \beta)f(z) \in \mathcal{H}[1, n] \cap Q$. The following subordinate relationship $q_1(z) \prec L_a^{\lambda}(b, c; \beta)f(z) \prec q_1(z)$ implies $q_1(z) \prec L_a^{\lambda+1}(b, c; \beta)f(z) \prec q_2(z)$. Moreover the functions $q_1(z)$ and $q_2(z)$ are the best subordinant and the best dominant respectively.

Theorem 5.2. Suppose that $q_1(z)$ is convex univalent, and let $q_2(z)$ be univalent in Δ and $f \in A$. Let $(L_a^{\lambda+1}(b,c;\beta)f(z))'$ be univalent with $\frac{L_a^{\lambda+1}(b,c;\beta)f(z)}{z} \in \mathcal{H}[1,n] \cap Q$. If $q_1(z) \prec (L_a^{\lambda+1}(b,c;\beta)f(z))' \prec q_2(z)$ then $q_1(z) \prec \frac{L_a^{\lambda+1}(b,c;\beta)f(z)}{z} \prec q_2(z)$. Moreover the functions $q_1(z)$ and $q_2(z)$ are the best subordinant and the best dominant respectively.

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