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Application of fractional-order Bernoulli functions for solving fractional Riccati differential equation

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Abstract

In this paper, a new numerical method for solving the fractional Riccati differential equation is presented. The fractional derivatives are described in the Caputo sense. The method is based upon fractional-order Bernoulli functions approximations. First, the fractional-order Bernoulli functions and their properties are presented. Then, an operational matrix of fractional order integration is derived and is utilized to reduce the under study problem to a system of algebraic equations. Error analysis included the residual error estimation and the upper bound of the absolute errors are introduced for this method. The technique and the error analysis are applied to some problems to demonstrate the validity and applicability of our method.

Keywords: Fractional Riccati differential equation; Fractional-order Bernoulli functions; Caputo derivative; Operational matrix; Collocation method.
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1. Introduction

Fractional differential equations (FDEs) are generalizations of ordinary differential equations to an arbitrary order. A history of the development of fractional differential operators can be found in [20, 23].

In real world, for modeling and analyzing many problems we need fractional calculus. FDEs find their applications in many fields of sciences and engineering, including fluid-dynamic traffic model [8],

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continuum and statistical mechanics [17], anomalous transport [19], dynamics of interfaces between nanoparticles and substrates [4] and solid mechanics [27].

In this paper, we consider the fractional Riccati differential equation

$$\frac{d^{\nu}y(t)}{dt^{\nu}} = a(t) + r(t)y(t) + k(t)y^2(t), \qquad m-1 < \nu \leq m,\ 0 \leq t \leq 1, \tag{1.1}$$

subject to the initial conditions

$$y^{(j)}(0) = \lambda_j, \qquad j = 0, 1, \ldots, m-1. \tag{1.2}$$

Here, $a(t), r(t), k(t)$ are given functions and $\lambda_j, j = 0, 1, \ldots, m-1$, are arbitrary constants.

In recent years, the fractional Riccati differential equations have been solved by Adomian's decomposition method [21], homotopy perturbation method [9], enhanced homotopy perturbation method [9], modified homotopy perturbation method [22], He's variational iteration method [1] and Bernstein polynomials [33].

During the last decades, several methods have been used for solving fractional differential equations, fractional integro-differential equations, fractional partial differential equations and dynamic systems containing fractional derivatives, such as Adomian's decompositions method [30], Taylor polynomials method [13], Jacobi operational matrix method [6], homotopy perturbation method [28], Sumudu transform method [5], second kind Chebyshev wavelet method [31], Legendre wavelet method [10], Bessel collocation method [34] and Bernoulli wavelet method [24, 26].

Recently, in [11] Kazem et al. defined new orthogonal functions based on the shifted Legendre polynomials to obtain the numerical solution of fractional-order differential equations. Yin et al. [32] extended this definition and presented the operational matrix of fractional derivative and integration for such functions to construct a new Tau technique for solving fractional partial differential equations (FPDEs). Bhrawy et al. [2] proposed the fractional-order generalized Laguerre functions based on the generalized Laguerre polynomials. They used these functions to find numerical solution of systems of fractional differential equations. In [33] Yüzbasi presented a collocation method based on the Bernstein polynomials for the fractional Riccati type differential equations. Chen et al. [3] expanded the fractional Legendre functions to interval $[0, h]$ in order to obtain the numerical solution of FPDEs. In [15], Krishnasamy and Razzaghi defined the fractional Taylor vector approximation for solving the Bagley-Torvik equation. Moreover, Rahimkhani et al. [25] constructed the fractional-order Bernoulli wavelets for solving FDEs and system of FDEs.

In this paper, a new numerical method for solving the fractional Riccati differential equation is presented. The method is based upon fractional-order Bernoulli functions approximation. First, the fractional-order Bernoulli functions are constructed. Then, we obtain the operational matrix of fractional order integration for fractional-order Bernoulli functions. Finally, this matrix is utilized to reduce the solution of the fractional Riccati differential equation to the solution of a system of algebraic equations.

The remainder of this article is organized as follows. In section 2, we give the basic definitions of fractional calculus and define Bernoulli polynomials and some of their properties. In section 3 the fractional-order Bernoulli functions and their operational matrix of fractional integration are obtained. In section 4, a technique is defined for approximating solution of fractional problem (1.1) with initial conditions (1.2). In section 5, we provide error analysis including the residual error estimation and an upper bound of the absolute errors of our method. In section 6, we apply the proposed technique to some examples and report our numerical results. We end the article with some concluding remarks in section 7.

2. Basic definitions

In this section, we present some notations, definitions and properties of the fractional calculus theory and Bernoulli polynomials which will be used further in this work.

2.1. Fractional integral and derivative

There are different definitions of fractional integration and derivatives. The widely used definition of a fractional integration is the Riemann-Liouville definition and of a fractional derivative is the Caputo definition.

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\nu \geq 0$ is defined as [12]

$$I^{\nu} f(t) = \begin{cases} \frac{1}{\Gamma(\nu)} \int_0^t \frac{f(s)}{(t-s)^{1-\nu}} ds, & \nu > 0, \ t > 0, \\ f(t), & \nu = 0. \end{cases} \tag{2.1}$$

The properties of the operator I^{ν} which are needed in this paper as follows [33]:

1. $I^{\nu_1} I^{\nu_2} f(t) = I^{\nu_1 + \nu_2} f(t)$,
2. $I^{\nu}(\lambda_1 f(t) + \lambda_2 g(t)) = \lambda_1 I^{\nu} f(t) + \lambda_2 I^{\nu} g(t)$,
3. $I^{\nu} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\nu+1)} t^{\nu+\beta}, \qquad \beta > -1$,

where λ_1 and λ_2 are real constants.

Definition 2.2. Caputo's fractional derivative of order ν is defined as [12]

$$D^{\nu} f(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\nu+1-n}} ds, \tag{2.2}$$

for $n - 1 < \nu \leq n$, $n \in \mathbb{N}$, $t > 0$. For the Caputo derivative we have [33, 12]

1. $D^{\nu} I^{\nu} f(t) = f(t)$,
2. $I^{\nu} D^{\nu} f(t) = f(t) - \sum_{i=0}^{n-1} f^{(i)}(0) \frac{t^i}{i!}$,
3. $D^{\nu} \lambda = 0$,

where λ is constant.

2.2. Bernoulli polynomials and their properties

The Bernoulli polynomials play an important role in different areas of mathematics, including number theory and the theory of finite differences. The classical Bernoulli polynomial produce the following exponential generating function [29]:

$$\frac{z e^{tz}}{e^z - 1} = \sum_{i=0}^{\infty} \beta_i(t) \frac{z^i}{i!}, \qquad (|z| < 2\pi), \ \ 0 \leq t \leq 1. \tag{2.3}$$

The following familiar expansion [29]

$$\sum_{i=0}^{m} \left(\begin{array}{c} m+1 \\ i \end{array} \right) \beta_i(t) = (m+1) t^m, \qquad 0 \leq t \leq 1, \tag{2.4}$$

is the most primary property of the Bernoulli polynomials. Also, the Bernoulli polynomials can be represented in the form [7]

$$\beta_m(t)=\sum_{i=0}^{m}\left(\begin{array}{c} m \\ i \end{array}\right)\beta_{m-i}t^i, \qquad 0\le t\le 1, \tag{2.5}$$

where $\beta_i := \beta_i(0),\ \ i=0,1,\dots,m$, are Bernoulli numbers. These numbers are a sequence of signed rational numbers which arise in the series expansion of trigonometric functions and can be defined by the identity [18]:

$$\frac{t}{e^t-1}=\sum_{i=0}^{\infty}\beta_i\frac{t^i}{i!}. \tag{2.6}$$

The first few Bernoulli numbers are

$$\beta_0=1, \qquad \beta_1=-\tfrac{1}{2}, \qquad \beta_2=\tfrac{1}{6}, \qquad \beta_4=-\tfrac{1}{30},\dots$$

with $\beta_{2i+1}=0, i=1,2,3,\dots$.

The first few Bernoulli polynomials are
$\beta_0(t)=1$,
$\beta_1(t)=t-\frac{1}{2}$,
$\beta_2(t)=t^2-t+\frac{1}{6}$,
$\beta_3(t)=t^3-\frac{3}{2}t^2+\frac{1}{2}t$.
These polynomials satisfy the following formula [7]:

$$\int_0^1\beta_n(t)\beta_m(t)dt=(-1)^{n-1}\frac{m!n!}{(m+n)!}\beta_{m+n}, \qquad m,n\ge 1. \tag{2.7}$$

According to [14], the Bernoulli polynomials form a complete basis over the interval $[0,1]$.

3. Main results

In this section, first we introduce the fractional-order Bernoulli functions and their properties. Then, we obtain their operational matrix of fractional integration.

3.1. Fractional-order Bernoulli functions

The fractional-order Bernoulli functions (FBFs) can be defined by introducing the change of variable $t\to(t-c)^\alpha$ (c is a real constant and $\alpha>0$) based on the Bernoulli polynomials. Let the FBFs be denoted by $F\beta_m^{\alpha,c}(t)$. By using (2.5) the analytic form of $F\beta_m^{\alpha,c}(t)$ of order $m\alpha$, is given by

$$F\beta_m^{\alpha,c}(t)=\sum_{i=0}^{m}\left(\begin{array}{c} m \\ i \end{array}\right)\beta_{m-i}(t-c)^{i\alpha}, \qquad 0\le t\le 1. \tag{3.1}$$

Thus, the first four such functions are
$F\beta_0^{\alpha,c}(t)=1$,
$F\beta_1^{\alpha,c}(t)=(t-c)^\alpha-\frac{1}{2}$,
$F\beta_2^{\alpha,c}(t)=(t-c)^{2\alpha}-(t-c)^\alpha+\frac{1}{6}$,
$F\beta_3^{\alpha,c}(t)=(t-c)^{3\alpha}-\frac{3}{2}(t-c)^{2\alpha}+\frac{1}{2}(t-c)^\alpha$.
By using Eq. (2.7) for the fractional-order Bernoulli functions, we have

$$\int_0^1 F\beta_n^{\alpha,c}(t)F\beta_m^{\alpha,c}(t)(t-c)^{\alpha-1}dt=\frac{1}{\alpha}(-1)^{n-1}\frac{m!n!}{(m+n)!}\beta_{m+n}, \qquad m,n\ge 1. \tag{3.2}$$

An arbitrary function $y \in L^2[0,1]$, can be expanded into the fractional-order Bernoulli functions as

$$y(t) \simeq \sum_{i=0}^{N-1} a_i F\beta_i^{\alpha,c}(t) = A^T \Phi(t), \tag{3.3}$$

where the fractional-order Bernoulli functions coefficient vector A and the fractional-order Bernoulli functions vector $\Phi(t)$ are given by

$$A = [a_0, a_1, \ldots, a_{N-1}]^T, \quad \Phi(t) = [F\beta_0^{\alpha,c}(t), F\beta_1^{\alpha,c}(t), \ldots, F\beta_{N-1}^{\alpha,c}(t)]^T. \tag{3.4}$$

To evaluate A we get

$$A^T = F^T D^{-1},$$

where

$$D =< \Phi, \Phi >= \int_0^1 \Phi(t)\Phi^T(t)(t-c)^\alpha dt,$$

and

$$F = [f_0, f_1, \ldots, f_{N-1}]^T,$$

where

$$f_i = \int_0^1 y(t) F\beta_i^{\alpha,c}(t)(t-c)^\alpha dt, \qquad i = 0, 1, \ldots, N-1.$$

3.2. Operational matrix of fractional integration

The Riemann-Liouville fractional integration of the vector $\Phi(t)$ given in Eq. (3.4) can be expressed by

$$I^\nu \Phi(t) = P^{(\nu,\alpha,c)} \Phi(t), \tag{3.5}$$

where $P^{(\nu,\alpha,c)}$ is the $N \times N$ operational matrix of fractional integration. Using Eq. (3.1) and the properties of the operator I^ν in Definition 1, for $i = 0, 1, \ldots, N-1$, we have

$$\begin{aligned} I^\nu F\beta_i^{\alpha,c}(t) &= I^\nu \left(\sum_{r=0}^{i} \binom{i}{r} \beta_{i-r} (t-c)^{r\alpha} \right) = \sum_{r=0}^{i} \binom{i}{r} \beta_{i-r} I^\nu (t-c)^{r\alpha} \\ &= \sum_{r=0}^{i} \binom{i}{r} \beta_{i-r} \frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha+1+\nu)} (t-c)^{r\alpha+\nu} = \sum_{r=0}^{i} b_{i,r}^{(\nu,\alpha)} (t-c)^{r\alpha+\nu}, \end{aligned} \tag{3.6}$$

where

$$b_{i,r}^{(\nu,\alpha)} = \binom{i}{r} \frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha+1+\nu)} \beta_{i-r}.$$

Assume $(t-c)^{r\alpha+\nu}$ can be expanded in N terms of the fractional-order Bernoulli functions as

$$(t-c)^{r\alpha+\nu} \simeq \sum_{j=0}^{N-1} \eta_{r,j}^{(\nu,\alpha,c)} F\beta_j^{\alpha,c}(t). \tag{3.7}$$

By using Eqs. (3.6) and (3.7) for $i = 0, 1, \ldots, N-1$, we get

$$I^\nu F\beta_i^{\alpha,c}(t) \simeq \sum_{r=0}^{i} b_{i,r}^{(\nu,\alpha)} \sum_{j=0}^{N-1} \eta_{r,j}^{(\nu,\alpha,c)} F\beta_j^{\alpha,c}(t) = \sum_{j=0}^{N-1} \left(\sum_{r=0}^{i} \Theta_{i,j,r}^{(\nu,\alpha,c)} \right) F\beta_j^{\alpha,c}(t), \tag{3.8}$$

where

$$\Theta_{i,j,r}^{(\nu,\alpha,c)} = b_{i,r}^{(\nu,\alpha)} \eta_{r,j}^{(\nu,\alpha,c)}.$$

Eq. (3.8) can be rewritten as

$$I^\nu F\beta_i^{\alpha,c}(t) \simeq [\sum_{r=0}^{i} \Theta_{i,0,r}^{(\nu,\alpha,c)}, \sum_{r=0}^{i} \Theta_{i,1,r}^{(\nu,\alpha,c)}, \ldots, \sum_{r=0}^{i} \Theta_{i,N-1,r}^{(\nu,\alpha,c)}]\Phi(t), \qquad i = 0, 1, \ldots, N-1. \tag{3.9}$$

Therefore, we have

$$P^{(\nu,\alpha,c)} = \begin{bmatrix} \Theta_{0,0,0}^{(\nu,\alpha,c)} & \Theta_{0,1,0}^{(\nu,\alpha,c)} & \cdots & \Theta_{0,N-1,0}^{(\nu,\alpha,c)} \\ \sum_{r=0}^{1} \Theta_{1,0,r}^{(\nu,\alpha,c)} & \sum_{r=0}^{1} \Theta_{1,1,r}^{(\nu,\alpha,c)} & \cdots & \sum_{r=0}^{1} \Theta_{1,N-1,r}^{(\nu,\alpha,c)} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{r=0}^{N-1} \Theta_{N-1,0,r}^{(\nu,\alpha,c)} & \sum_{r=0}^{N-1} \Theta_{N-1,1,r}^{(\nu,\alpha,c)} & \cdots & \sum_{r=0}^{N-1} \Theta_{N-1,N-1,r}^{(\nu,\alpha,c)} \end{bmatrix}.$$

For example, for $N = 3$ the operational matrix of the fractional integration can be expressed as

$$P^{(1,1,0)} = \begin{bmatrix} 0.5 & 1 & 0 \\ -0.0833333 & 0 & 0.5 \\ 0 & -0.0333333 & 0 \end{bmatrix},$$

$$P^{(\frac{1}{2},\frac{1}{2},0)} = \begin{bmatrix} 0.56419 & 1.12838 & 5.63739 \times 10^{-15} \\ 0.0133142 & 0.322037 & 0.886227 \\ -0.0133142 & -0.0211362 & 0.242152 \end{bmatrix},$$

$$P^{(2,2,0)} = \begin{bmatrix} 0.25 & 0.5 & 0 \\ -0.0972222 & -0.166667 & 0.0833333 \\ 0.0222222 & 0.03 & -0.0333333 \end{bmatrix}.$$

4. Numerical method

In this paper, we consider the fractional Riccati differential equation

$$\frac{d^\nu y(t)}{dt^\nu} = a(t) + r(t)y(t) + k(t)y^2(t), \qquad m-1 < \nu \leq m, \ \ 0 \leq t \leq 1, \tag{4.1}$$

subject to the initial conditions

$$y^{(i)}(0) = \lambda_i, \qquad i = 0, 1, \ldots, m-1. \tag{4.2}$$

Here $y(t)$ is an unknown function; $a(t), r(t), k(t)$ are given functions, and $\lambda_i, i = 0, 1, \ldots, m-1$, are arbitrary constants. For this problem, we first expand $D^\nu y(t)$ by the fractional-order Bernoulli functions as

$$D^\nu y(t) \simeq A^T \Phi(t) = D^\nu y_N(t). \tag{4.3}$$

From Eqs. (3.5), (4.2) and (4.3), we obtain

$$y(t) \simeq I^\nu(A^T\Phi(t)) + \sum_{i=0}^{m-1} y^{(i)}(0)\frac{t^i}{i!} \simeq A^T P^{(\nu,\alpha,c)}\Phi(t) + \sum_{i=0}^{m-1} \lambda_i \frac{t^i}{i!} = y_N(t). \tag{4.4}$$

Similarly, the known functions $a(t), r(t)$, $k(t)$ and $\frac{t^i}{i!}, i = 0, 1, \ldots, m-1$, can be expanded by the fractional-order Bernoulli functions as

$$a(t) \simeq E_1^T \Phi(t) = a_N(t), \quad r(t) \simeq E_2^T \Phi(t) = r_N(t), \quad k(t) \simeq E_3^T \Phi(t) = k_N(t), \tag{4.5}$$

$$\lambda_i \frac{t^i}{i!} \simeq W_i^T \Phi(t), \ \ i = 0, 1, \ldots, m-1. \tag{4.6}$$

Substituting Eqs. (4.3) - (4.6) in Eq. (4.1), we get

$$A^T \Phi(t) = E_1^T \Phi(t) + (E_2^T \Phi(t))(A^T P^{(\nu,\alpha,c)} \Phi(t) + \sum_{i=0}^{m-1} W_i^T \Phi(t))^T + (E_3^T \Phi(t))(A^T P^{(\nu,\alpha,c)} \Phi(t)$$
$$+ \sum_{i=0}^{m-1} W_i^T \Phi(t))(A^T P^{(\nu,\alpha,c)} \Phi(t) + \sum_{i=0}^{m-1} W_i^T \Phi(t))^T.$$

Next, we collocate this equation at the N zeros of shifted Legendre polynomial $L_N(t)$. These equations, constitute a system of N nonlinear algebraic equations with N unknown coefficients, which can be solved by using any standard iterative method, such as Newton's iterative method.

5. Error analysis

In this section, error analysis of the method will be presented for the fractional Riccati differential equation. Firstly, an upper bound of the absolute errors will be given. Secondly, we introduce an error estimation by means of the norm of residual error.

(i) The upper bound of the absolute errors for the fractional-order Bernoulli series solution (4.4).

In this section, for simplicity prove of theorems, we can write Eqs. (1.1) and (1.2) in the following form

$$\frac{d^\nu y(t)}{dt^\nu} = a(t) + H(t, y(t)), \qquad m-1 < \nu \leq m, \ 0 \leq t \leq 1,$$
$$y^{(j)}(0) = \lambda_j, \qquad j = 0, 1, \ldots, m-1,$$

where $H(t, y(t))$ is a continuous function of unknown real function $y(t)$.

Theorem 5.1. *Consider $H(t, y(t))$, satisfying the Lipschitz condition ($\|H(t,y) - H(t,z)\| \leq \eta \|y - z\|, \eta > 0$) and $\frac{\eta}{\Gamma(\nu+1)} \neq 1$. Let y and y_N be the exact and approximate solution of (1.1). Then we have*

$$\|y - y_N\| \leq \frac{E(a)}{|\Gamma(\nu+1)(1 - \frac{\eta}{\Gamma(\nu+1)})|}, \tag{5.1}$$

where

$$E(a) = \|a - a_N\|.$$

Proof . According to the assumptions, we have

$$\begin{aligned} &\tfrac{d^\nu y(t)}{dt^\nu} = a(t) + H(t, y(t)), \qquad m-1 < \nu \leq m, \ 0 \leq t \leq 1, \\ &y^{(j)}(0) = \lambda_j, \qquad j = 0, 1, \ldots, m-1. \end{aligned} \tag{5.2}$$

Applying operator I^ν on both sides of (5.2), we yield

$$y(t)=\sum_{j=0}^{m-1}\lambda_j\frac{t^j}{j!}+\frac{1}{\Gamma(\nu)}\int_0^t(t-s)^{\nu-1}a(s)ds+\frac{1}{\Gamma(\nu)}\int_0^t(t-s)^{\nu-1}H(s,y(s))ds.$$

Now, suppose that function a is expanded in terms of fractional-order Bernoulli functions, then the obtained solution is an approximated function; y_N. Our aim is to find an upper bound for the associated error between the exact solution y and the approximated solution y_N for Eq. (1.1). We get

$$\begin{aligned}\|y-y_N\| &\leq \|\frac{1}{\Gamma(\nu)}\int_0^t(t-s)^{\nu-1}(a(s)-a_N(s))ds\|\\ &\quad+\|\frac{1}{\Gamma(\nu)}\int_0^t(t-s)^{\nu-1}(H(s,y(s))-H(s,y_N(s)))ds\|\\ &\leq \frac{1}{\Gamma(\nu)}\int_0^t(t-s)^{\nu-1}\|a(s)-a_N(s)\|ds\\ &\quad+\frac{1}{\Gamma(\nu)}\int_0^t(t-s)^{\nu-1}\|H(s,y(s))-H(s,y_N(s))\|ds\\ &\leq \frac{E(a)}{\Gamma(\nu+1)}+\frac{1}{\Gamma(\nu+1)}\|H(t,y)-H(t,y_N)\|\\ &\leq \frac{E(a)}{\Gamma(\nu+1)}+\frac{\eta}{\Gamma(\nu+1)}\|y-y_N\|.\end{aligned}$$

In other words,

$$\|y(t)-y_N(t)\|\leq\frac{E(a)}{|\Gamma(\nu+1)(1-\frac{\eta}{\Gamma(\nu+1)})|},$$

and this completes the proof. □

(ii) Error estimation: Since the truncated fractional-order Bernoulli series is approximate solution of equation (1.1), so one has an error function for $y(t)$ as follows

$$E(y_N(t))=|y(t)-y_N(t)|,$$

where setting $t=t_j\in[0,1]$, the absolute error value of t_j can be obtained.

Mostly, the exact solutions for the non-integer values of ν are not known. Therefore, to show efficient of the present method for the fractional Riccati differential equation, we define the norm of residual error as follows:

$$\begin{aligned}E(y_N(t)) &= A^T\Phi(t)-E_1^T\Phi(t)-(E_2^T\Phi(t))(A^TP^{(\nu,\alpha,c)}\Phi(t)+\sum_{i=0}^{m-1}W_i^T\Phi(t))^T\\ &\quad-(E_3^T\Phi(t))(A^TP^{(\nu,\alpha,c)}\Phi(t)+\sum_{i=0}^{m-1}W_i^T\Phi(t))(A^TP^{(\nu,\alpha,c)}\Phi(t)+\sum_{i=0}^{m-1}W_i^T\Phi(t))^T,\end{aligned}$$

then, we let

$$\|E(y_N)\|^2=\int_0^1E^2(y_N(t))dt. \tag{5.3}$$

Therefore, if the exact solution of the problem is not known, the error estimation (5.3) can be used to test the reliability of the results.

6. Illustrative examples

In this section, we apply the method of section 4 for two different examples of the fractional Riccati differential equations to demonstrate advantages and accuracy of the present technique. We are calculated all numerical computations by using the Mathematica software.

Example 6.1. Firstly, we consider the following Riccati fractional differential equation [22]

$$\begin{aligned} &\tfrac{d^\nu y(t)}{dt^\nu} = 1 - y^2(t), \qquad 0 < \nu \le 1, \ \ 0 \le t \le 1, \\ &y(0) = 0. \end{aligned} \tag{6.1}$$

The exact solution of the problem (6.1), when $\nu = 1$ is

$$y(t) = \frac{e^{2t} - 1}{e^{2t} + 1}. \tag{6.2}$$

In this problem, $a(t) = 1, r(t) = 0, k(t) = -1$. By applying the technique described of section 4, the problem reduces to

$$A^T \Phi(t) - E_1^T \Phi(t) + (A^T P^{(\nu,\alpha,c)} \Phi(t))(A^T P^{(\nu,\alpha,c)} \Phi(t))^T = 0. \tag{6.3}$$

Then, we collocate Eq. (6.3) at the zeros of shifted Legendre polynomials, which can be solved for the unknown vector C, using Newton's iterative method. For this example, by using $y(0) = 0$, and $y(t) = A^T P^{(\nu,\alpha,c)} \Phi(t)$, we choose the initial guesses such that $A^T P^{(\nu,\alpha,c)} \Phi(0) = 0$.

Fig. 1 shows the numerical results of problem (6.1) for $N = 5, c = 0$ with

$$\alpha = \nu = 0.5, 0.75, 0.85, 0.95, 1$$

and the exact solution. We see that the approximate solutions are in high agreement with the exact solution, when $\nu = 1$. Therefore, we state the solution for $\nu = 0.5$ and $\nu = 0.75$ is also credible. In Tables 1−3, the numerical solutions of present method are compared with the modified homotopy perturbation method [22] by using fourth-order term. Table 1 demonstrates the values of the solutions for $\alpha = \nu = 0.5$, Table 2 shows them for $\alpha = \nu = 0.75$ and Table 3 gives the values of the solutions for $\alpha = \nu = 1$. Also, the approximate solutions for $\alpha = \nu = 1$ are compared with the exact solution in Table 3. Figs. 2(a) and 2(b) plot the absolute error at $\alpha = \nu = 1, c = 0$ for $N = 7, 9$ respectively. We know the exact solution for the values of $\nu \neq 1$ are not known. Therefore to show efficient of the present method for this problem, we use estimated error $\|E(y_N)\|^2$ in section 5. Table 4, displays $\|E(y_N)\|^2$ for some N and different values of ν. These tables and figures demonstrate the advantages and the accuracy of the fractional-order Bernoulli functions for solving the fractional Riccati differential equation. Also, Tables 5 and 6 demonstrate the effect of parameters α and c for this problem, respectively. From above tables and figures, we can say that the best cases of α and c for this problem are $\alpha = \nu$ and $c = 0$, respectively.

Example 6.2. Now, let us consider the fractional Riccati differential equation [22, 16]:

$$\begin{aligned} &\tfrac{d^\nu y(t)}{dt^\nu} = 1 + 2y(t) - y^2(t), \qquad 0 < \nu \le 1, \quad 0 \le t \le 1, \\ &y(0) = 0. \end{aligned} \tag{6.4}$$

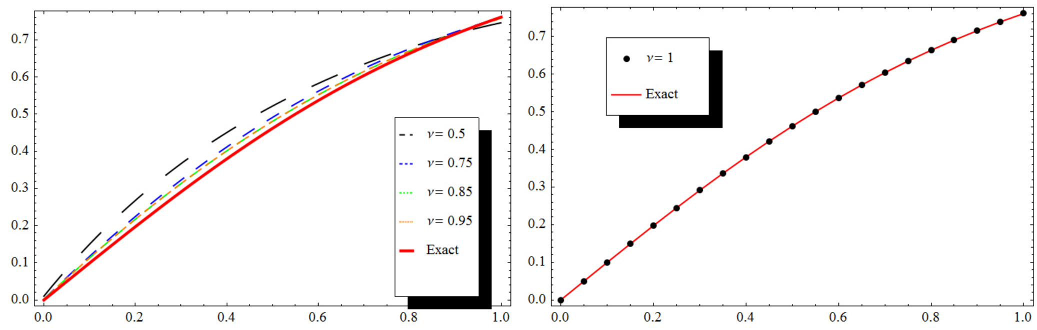

Figure 1: The comparison of y(t) for $N = 5, c = 0$ with $\alpha = \nu = 0.5, 0.75, 0.85, 0.95, 1$, and the exact solution, for Example 6.1.

Table 1: Comparison of the numerical solutions with the Ref. [22] for $\alpha = \nu = 0.5$ and $c = 0$ for Example 6.1.

t	*Present method*		Ref. [22]
	$N = 8$	$N = 10$	
0.1	0.330101	0.330112	0.273875
0.2	0.436844	0.436841	0.454125
0.3	0.504894	0.504891	0.573932
0.4	0.553776	0.553783	0.644422
0.5	0.591188	0.591195	0.674137
0.6	0.621017	0.621014	0.671987
0.7	0.645494	0.645486	0.648003
0.8	0.666018	0.666020	0.613306
0.9	0.683542	0.683552	0.579641
1	0.698768	0.698741	0.558557

Table 2: Comparison of the numerical solutions with the Ref. [22] for $\alpha = \nu = 0.75$ and $c = 0$ for Example 6.1.

t	*Present method*		Ref. [22]
	$N = 8$	$N = 10$	
0.1	0.190102	0.190101	0.184795
0.2	0.309975	0.309975	0.313795
0.3	0.404615	0.404615	0.414562
0.4	0.481633	0.481632	0.492889
0.5	0.545090	0.545089	0.462117
0.6	0.597781	0.597783	0.597393
0.7	0.641821	0.641820	0.631772
0.8	0.678851	0.678849	0.660412
0.9	0.710173	0.710175	0.687960
1	0.736843	0.736836	0.718260

The exact solution of the problem for $\nu = 1$ is given by

$$y(t) = 1 + \sqrt{2}\tanh(\sqrt{2}t + \frac{1}{2}\ln(\frac{\sqrt{2}-1}{\sqrt{2}+1})). \tag{6.5}$$

Table 3: Comparison of the numerical solutions with the Ref. [22] for $\alpha = \nu = 1$ and $c = 0$ for Example 6.1.

t	*Exact solution*	*Present method*			Ref. [22]
		$N = 8$	$N = 10$	$N = 13$	
0.1	0.0996679946	0.0996679151	0.0996679941	0.0996679946	0.099668
0.2	0.1973753202	0.1973752555	0.1973753200	0.1973753202	0.197375
0.3	0.2913126125	0.2913128313	0.2913126145	0.2913126125	0.291312
0.4	0.3799489623	0.3799488684	0.3799489595	0.3799489622	0.379944
0.5	0.4621171573	0.4621169753	0.4621171582	0.4621171573	0.462078
0.6	0.5370495670	0.5370497148	0.5370495656	0.5370495670	0.536857
0.7	0.6043677771	0.6043679153	0.6043677747	0.6043677771	0.603631
0.8	0.6640367703	0.6640365452	0.6640367721	0.6640367703	0.661706
0.9	0.7162978702	0.7162979772	0.7162978687	0.7162978702	0.709919
1	0.7615941560	0.7615934647	0.7615941524	0.7615941560	0.746032

Table 4: The $\|E(y_N)\|^2$ with some N and various values of ν for Example 6.1.

ν	$N = 5$		$N = 8$	
	$\alpha = 1$	$\alpha = \nu$	$\alpha = 1$	$\alpha = \nu$
0.5	2.35×10^{-5}	7.13×10^{-7}	2.63×10^{-6}	1.73×10^{-11}
0.6	1.75×10^{-5}	9.05×10^{-7}	1.14×10^{-6}	1.62×10^{-9}
0.7	8.04×10^{-6}	7.00×10^{-7}	3.12×10^{-7}	7.60×10^{-10}
0.8	1.92×10^{-6}	4.09×10^{-7}	6.50×10^{-8}	1.40×10^{-10}
0.9	7.31×10^{-8}	2.17×10^{-7}	7.53×10^{-9}	1.79×10^{-11}

Table 5: The absolute errors for $\nu = 1$ with $N = 9$ and various values of α for Example 6.1.

t	$\alpha = \frac{1}{3}$	$\alpha = \frac{1}{2}$	$\alpha = \frac{2}{3}$	$\alpha = \nu$	$\alpha = 2$
0	3.91×10^{-5}	1.90×10^{-5}	5.27×10^{-4}	2.25×10^{-7}	5.46×10^{-2}
0.2	2.35×10^{-4}	1.75×10^{-5}	3.19×10^{-5}	8.77×10^{-9}	3.49×10^{-3}
0.4	1.84×10^{-4}	2.02×10^{-5}	2.53×10^{-5}	5.51×10^{-8}	3.69×10^{-4}
0.6	1.77×10^{-4}	1.26×10^{-5}	3.30×10^{-5}	5.37×10^{-8}	5.55×10^{-4}
0.8	1.18×10^{-4}	1.51×10^{-5}	2.71×10^{-5}	5.64×10^{-9}	2.85×10^{-3}
1	1.13×10^{-4}	2.46×10^{-5}	9.04×10^{-5}	2.25×10^{-7}	2.42×10^{-3}

Table 6: The absolute errors for $\nu = 1$ with $N = 8$ and various values of α for Example 6.1.

t	$c = 0$	$c = 0.001$	$c = 0.01$	$c = 0.1$	$c = 0.5$
0	6.91×10^{-7}	1.00×10^{-3}	1.00×10^{-2}	9.97×10^{-2}	4.62×10^{-1}
0.2	6.47×10^{-8}	9.61×10^{-4}	9.63×10^{-3}	9.77×10^{-2}	4.89×10^{-1}
0.4	9.39×10^{-8}	8.56×10^{-4}	8.59×10^{-3}	8.86×10^{-2}	4.80×10^{-1}
0.6	1.48×10^{-7}	7.12×10^{-4}	7.15×10^{-3}	7.49×10^{-2}	4.37×10^{-1}
0.8	2.25×10^{-7}	5.60×10^{-4}	5.63×10^{-3}	5.97×10^{-2}	3.73×10^{-1}
1	6.91×10^{-7}	4.21×10^{-4}	4.23×10^{-3}	4.53×10^{-2}	2.99×10^{-1}

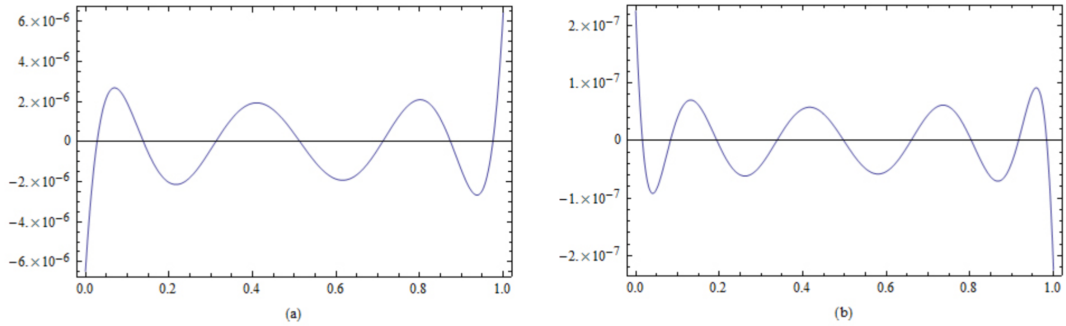

Figure 2: The absolute errors between the exact and approximate solutions for $c = 0, \alpha = \nu = 1$: (a) $N = 7$, (b) $N = 9$ for Example 6.1.

In this problem $a(t) = 1, r(t) = 2, k(t) = -1$. Using technique presented of section 4, the problem (6.4) reduces to

$$A^T\Phi(t) - E_1^T\Phi(t) - (E_2^T\Phi(t))(A^TP^{(\nu,\alpha,c)}\Phi(t))^T + (A^TP^{(\nu,\alpha,c)}\Phi(t))(A^TP^{(\nu,\alpha,c)}\Phi(t))^T = 0. \quad (6.6)$$

Then, we collocate Eq. (6.6) at the zeros of shifted Legendre polynomials, which can be solved for the unknown vector C, using Newton's iterative method. For this example, by using $y(0) = 0$, and $y(t) = A^TP^{(\nu,\alpha,c)}\Phi(t)$, we choose the initial guesses such that $A^TP^{(\nu,\alpha,c)}\Phi(0) = 0$.

Fig. 3 shows the numerical results of problem (6.4) for $N = 5, c = 0$ with $\alpha = \nu = 0.5, 0.75, 0.85, 0.95, 1$ and the exact solution. We see that the approximate solutions are in good agreement with the exact solution, when $\nu = 1$. Therefore, we state the solution for $\nu = 0.5$ and $\nu = 0.75$ is also credible. In Tables 7−9, the numerical solutions of present method are compared with the Chebyshev wavelet method [16] for $N = 192$ and the modified homotopy perturbation method [22] by using fourth-order term. Table 7 demonstrates the values of the solutions for $\alpha = \nu = 0.5$, Table 8 shows them for $\alpha = \nu = 0.75$ and Table 9 gives the values of the solutions for $\alpha = \nu = 1$. Also, the approximate solutions for $\nu = 1$ are compared with the exact solution in Table 9. The absolute errors for $N = 10$ and $N = 12$ are shown in Figs. 4(a) and 4(b), respectively. In Table 10, we list estimated error in section 5 for various choices of the ν and N. Also, Table 11 demonstrates the effect of parameter α for this problem. From above tables, we can say that the best case of α for this problem is $\alpha = \nu$.

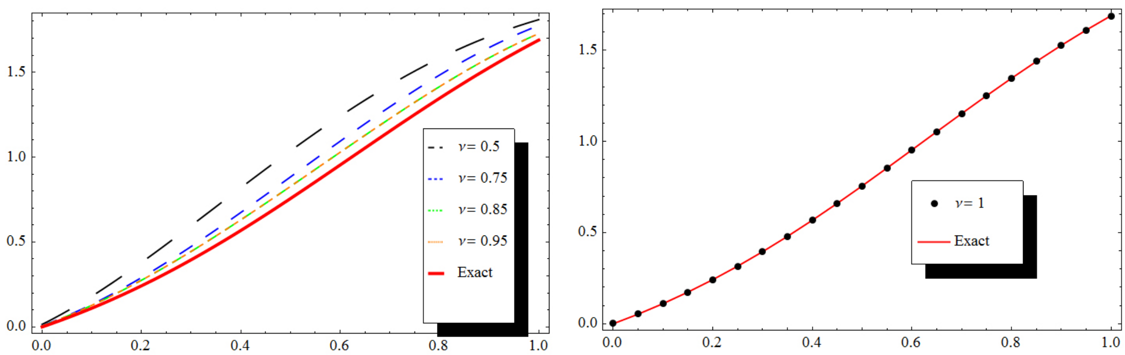

Figure 3: The comparison of y(t) for $N = 5, c = 0$ with $\alpha = \nu = 0.5, 0.75, 0.85, 0.95, 1$, and the exact solution, for Example 6.2.

Table 7: Comparison of the numerical solutions with the other methods for $\alpha = \nu = 0.5$, and $c = 0$ for Example 6.2.

t	*Present method*			Ref. [16]	Ref. [22]
	$N = 8$	$N = 10$	$N = 12$		
0.1	0.600020	0.134831	0.561615	0.592756	0.321730
0.2	0.939956	0.579573	0.938990	0.933179	0.629666
0.3	1.178645	0.916894	1.199375	1.173983	0.940941
0.4	1.349558	1.145601	1.375309	1.346654	1.250737
0.5	1.476024	1.329534	1.497081	1.473887	1.549439
0.6	1.572592	1.469785	1.583735	1.570571	1.825456
0.7	1.648047	1.559457	1.648506	1.646199	2.066523
0.8	1.708044	1.628104	1.700288	1.706880	2.260633
0.9	1.757149	1.708765	1.742512	1.756644	2.396839
1	1.800400	1.716043	1.780551	1.798220	2.466004

Table 8: Comparison of the numerical solutions with the other methods for $\alpha = \nu = 0.75$, and $c = 0$ for Example 6.2.

t	*Present method*			Ref. [16]	Ref. [22]
	$N = 8$	$N = 10$	$N = 12$		
0.1	0.245337	0.245440	0.250694	0.310732	0.216866
0.2	0.475010	0.475121	0.481754	0.584307	0.428892
0.3	0.709907	0.710043	0.715221	0.822173	0.654614
0.4	0.938359	0.938544	0.940851	1.024974	0.891404
0.5	1.148960	1.149082	1.148612	1.198621	1.132763
0.6	1.334330	1.334353	1.332241	1.349150	1.370240
0.7	1.491844	1.491923	1.489621	1.481449	1.594278
0.8	1.622824	1.623002	1.621639	1.599235	1.794879
0.9	1.730659	1.730621	1.730833	1.705303	1.962239
1	1.818092	1.818566	1.820368	1.801763	2.087384

Table 9: Comparison of the numerical solutions with the other methods for $\alpha = \nu = 1$, and $c = 0$ for Example 6.2.

t	*Exact solution*	*Present method*		Ref. [16]	Ref. [22]
		$N = 10$	$N = 16$		
0.1	0.1102951969	0.1102946907	0.1102951969	0.110311	0.110294
0.2	0.2419767996	0.2419772146	0.2419767996	0.241995	0.241965
0.3	0.3951048487	0.3951046565	0.3951048487	0.395123	0.395106
0.4	0.5678121663	0.5678119966	0.5678121663	0.567829	0.568115
0.5	0.7560143934	0.7560148034	0.7560143935	0.756029	0.757564
0.6	0.9535662165	0.9535659224	0.9535662164	0.953576	0.958259
0.7	1.1529489670	1.1529489050	1.1529489670	1.152955	1.163459
0.8	1.3463636554	1.3463640106	1.3463636554	1.346365	1.365240
0.9	1.5269113133	1.5269108395	1.5269113132	1.526909	1.554960
1	1.6894983916	1.6894966943	1.6894983918	1.689494	1.723810

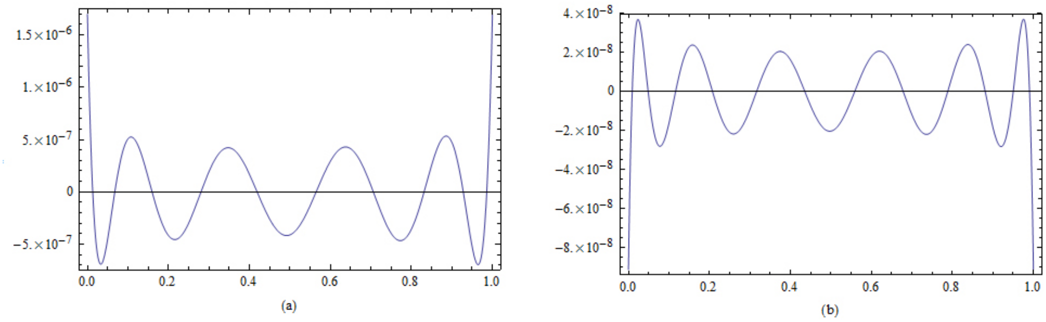

Figure 4: The absolute errors between the exact and approximate solutions for $c = 0, \alpha = \nu = 1$: (a) $N = 10$, (b) $N = 12$, for Example 6.2.

Table 10: The $\|E(y_N)\|^2$ with some N and various values of ν for Example 6.2.

ν	$N=5$		$N=9$	
	$\alpha=1$	$\alpha=\nu$	$\alpha=1$	$\alpha=\nu$
0.6	9.24×10^{-6}	1.04×10^{-3}	2.38×10^{-7}	7.64×10^{-5}
0.7	3.46×10^{-5}	8.19×10^{-5}	4.17×10^{-7}	1.84×10^{-6}
0.8	2.31×10^{-5}	1.61×10^{-6}	1.86×10^{-9}	1.69×10^{-9}
0.9	4.66×10^{-7}	6.19×10^{-6}	5.09×10^{-9}	1.03×10^{-9}

Table 11: The absolute errors for $\nu = 1$ with $N = 12$ and various values of α for Example 6.2.

t	$\alpha=\frac{1}{3}$	$\alpha=\frac{1}{2}$	$\alpha=\frac{2}{3}$	$\alpha=\nu$	$\alpha=2$
0	5.41×10^{-4}	2.66×10^{-5}	2.16×10^{-4}	9.15×10^{-8}	4.16×10^{-2}
0.2	2.12×10^{-2}	2.69×10^{-4}	5.47×10^{-6}	5.48×10^{-9}	3.75×10^{-3}
0.4	2.72×10^{-2}	3.49×10^{-4}	2.07×10^{-5}	1.63×10^{-8}	2.62×10^{-3}
0.6	3.01×10^{-2}	3.79×10^{-4}	3.00×10^{-5}	1.81×10^{-8}	3.46×10^{-3}
0.8	2.86×10^{-2}	3.55×10^{-4}	1.67×10^{-5}	8.18×10^{-9}	1.26×10^{-2}
1	2.34×10^{-2}	2.76×10^{-4}	4.72×10^{-5}	9.15×10^{-8}	2.36×10^{-2}

7. Conclusion

In this study, we use the fractional-order Bernoulli functions and the associated operational matrix of integration $P^{(\nu,\alpha,c)}$ for numerical solution of the nonlinear Riccati differential equation with fractional order. Actually, this matrix and collocation method are translated the initial equation into a system of N nonlinear algebraic equations with N unknown coefficients. The achieved solutions with the suggested method demonstrate that the best case of α and c for this problem is $\alpha = \nu$ and $c = 0$, respectively. The value of parameter c depends on the initial conditions. Comparing with other methods, demonstrate that this method is more accurate than some existing methods. We presented two numerical examples for to demonstrate the powerfulness of the proposed method.

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