Int. J. Nonlinear Anal. Appl. 8 (2017) No. 2, 209-222
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2017.11313.1552

On generalized Hermite-Hadamard inequality for generalized convex function

Mehmet Zeki Sarikaya, Hüseyin Budak*

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-Turkey

(Communicated by M. Eshaghi)

Abstract

In this paper, a new inequality for generalized convex functions which is related to the left side of generalized Hermite-Hadamard type inequality is obtained. Some applications for some generalized special means are also given.

Keywords: Generalized Hermite-Hadamard inequality; Generalized Hölder inequality; Generalized convex functions.
2010 MSC: Primary 26D07, 26D10; Secondary 26D15, 26A33.

1. Introduction

Definition 1.1. [Convex function] The function $f:[a,b]\subset R\to R$, is said to be convex if the following inequality holds

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)$$

for all $x,y\in[a,b]$ and $t\in[0,1]$. We say that f is concave if $(-f)$ is convex.

Theorem 1.2. [Hermite-Hadamard inequality] Let $f:I\subseteq R\to R$ be a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$. If f is a convex function then the following double inequality, which is well known in the literature as the Hermite-Hadamard inequality, holds [6]

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

*Corresponding author
Email addresses: sarikayamz@gmail.com (Mehmet Zeki Sarikaya), hsyn.budak@gmail.com (Hüseyin Budak)

Received: May 2017 *Revised:* October 2017

In [10], Sarikaya et al. established inequalities for twice differentiable convex mappings which are connected with Hadamard's inequality, and they used the following lemma to prove their results.

Lemma 1.3. Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I°, $a, b \in I^{\circ}$ with $a < b$. If $f'' \in L_1[a, b]$, then

$$\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) = \frac{(b-a)^2}{4}\int_0^1 n(t)\left[f''(ta+(1-t)b)+f''(tb+(1-t)a)\right]dt, \tag{1.2}$$

where

$$n(t) := \begin{cases} t^2 & , t \in [0, \frac{1}{2}) \\ (1-t)^2 & , t \in [\frac{1}{2}, 1]. \end{cases}$$

Also, one of the the main inequalities in [10], pointed out as follows:

Theorem 1.4. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a, b]$ where $a, b \in I$, $a < b$. If $|f''|^q$ is convex on $[a, b]$, $q > 1$, then

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})\right| \le \frac{(b-a)^2}{8(2p+1)^{1/p}}\left[\frac{|f''(a)|^q + |f''(b)|^q}{2}\right]^{1/q} \tag{1.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

2. Preliminaries

Recall the set R^{α} of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [11, 12] and so on.

Recently, the theory of Yang's fractional sets [11] was introduced as follows

For $0 < \alpha \le 1$, we have the following α-type set of element sets:

Z^{α} : The α-type set of integer is defined as the set $\{0^{\alpha}, \pm 1^{\alpha}, \pm 2^{\alpha}, \ldots, \pm n^{\alpha}, \ldots\}$.

Q^{α} : The α-type set of the rational numbers is defined as the set $\{m^{\alpha} = \left(\frac{p}{q}\right)^{\alpha} : p, q \in Z,\ q \neq 0\}$.

J^{α} : The α-type set of the irrational numbers is defined as the set $\{m^{\alpha} \neq \left(\frac{p}{q}\right)^{\alpha} : p, q \in Z,\ q \neq 0\}$.

R^{α} : The α-type set of the real line numbers is defined as the set $R^{\alpha} = Q^{\alpha} \cup J^{\alpha}$.

If a^{α}, b^{α} and c^{α} belongs the set R^{α} of real line numbers, then

(1) $a^{\alpha} + b^{\alpha}$ and $a^{\alpha}b^{\alpha}$ belongs the set R^{α};

(2) $a^{\alpha} + b^{\alpha} = b^{\alpha} + a^{\alpha} = (a+b)^{\alpha} = (b+a)^{\alpha}$;

(3) $a^{\alpha} + (b^{\alpha} + c^{\alpha}) = (a+b)^{\alpha} + c^{\alpha}$;

(4) $a^{\alpha}b^{\alpha} = b^{\alpha}a^{\alpha} = (ab)^{\alpha} = (ba)^{\alpha}$;

(5) $a^{\alpha}(b^{\alpha}c^{\alpha}) = (a^{\alpha}b^{\alpha})c^{\alpha}$;

(6) $a^{\alpha}(b^{\alpha} + c^{\alpha}) = a^{\alpha}b^{\alpha} + a^{\alpha}c^{\alpha}$;

(7) $a^{\alpha} + 0^{\alpha} = 0^{\alpha} + a^{\alpha} = a^{\alpha}$ and $a^{\alpha}1^{\alpha} = 1^{\alpha}a^{\alpha} = a^{\alpha}$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 2.1. [11] A non-differentiable function $f : R \to R^{\alpha}$, $x \to f(x)$ is called to be local fractional continuous at x_0, if for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$|f(x) - f(x_0)| < \varepsilon^{\alpha}$$

holds for $|x - x_0| < \delta$, where $\varepsilon, \delta \in R$. If $f(x)$ is local continuous on the interval (a, b), we denote $f(x) \in C_{\alpha}(a, b)$.

Definition 2.2. [11] The local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined by

$$f^{(\alpha)}(x_0) = \left.\frac{d^{\alpha} f(x)}{dx^{\alpha}}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^{\alpha}\left(f(x) - f(x_0)\right)}{(x - x_0)^{\alpha}},$$

where $\Delta^{\alpha}\left(f(x) - f(x_0)\right) \cong \Gamma(\alpha + 1)\left(f(x) - f(x_0)\right)$.

If there exists $f^{(k+1)\alpha}(x) = \overbrace{D_x^{\alpha} \ldots D_x^{\alpha}}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq R$, then we denoted $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, \ldots$

Definition 2.3. [11] Let $f(x) \in C_{\alpha}[a, b]$. Then the local fractional integral is defined by,

$${}_aI_b^{\alpha} f(x) = \frac{1}{\Gamma(\alpha + 1)} \int_a^b f(t)(dt)^{\alpha} = \frac{1}{\Gamma(\alpha + 1)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^{\alpha},$$

with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_1, \Delta t_2, \ldots, \Delta t_{N-1}\}$, where $[t_j, t_{j+1}]$, $j = 0, \ldots, N - 1$ and $a = t_0 < t_1 < \ldots < t_{N-1} < t_N = b$ is partition of interval $[a, b]$. Here, it follows that ${}_aI_b^{\alpha} f(x) = 0$ if $a = b$ and ${}_aI_b^{\alpha} f(x) = -{}_bI_a^{\alpha} f(x)$ if $a < b$. If for any $x \in [a, b]$, there exists ${}_aI_x^{\alpha} f(x)$, then we denoted by $f(x) \in I_x^{\alpha}[a, b]$.

Definition 2.4. [Generalized convex function] [11] Let $f : I \subseteq R \to R^{\alpha}$. For any $x_1, x_2 \in I$ and $\lambda \in [0, 1]$, if the following inequality

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda^{\alpha} f(x_1) + (1 - \lambda)^{\alpha} f(x_2)$$

holds, then f is called a generalized convex function on I.

Here are two basic examples of generalized convex functions:
(1) $f(x) = x^{\alpha p}$, $x \geq 0$, $p > 1$;
(2) $f(x) = E_{\alpha}(x^{\alpha})$, $x \in R$ where $E_{\alpha}(x^{\alpha}) = \sum\limits_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$ is the Mittag-Leffer function.

Theorem 2.5. Let $f \in D_{\alpha}(I)$, then the following conditions are equivalent
a) f is a generalized convex function on I
b) $f^{(\alpha)}$ is an increasing function on I
c) for any $x_1, x_2 \in I$,

$$f(x_2) - f(x_1) \geq \frac{f^{(\alpha)}(x_1)}{\Gamma(1 + \alpha)} (x_2 - x_1)^{\alpha}.$$

Corollary 2.6. Let $f \in D_{2\alpha}(a,b)$. Then f is a generalized convex function (or a generalized concave function) if and only if

$$f^{(2\alpha)}(x) \geq 0 \left(\text{or} f^{(2\alpha)}(x) \leq 0\right)$$

for all $x \in (a,b)$.

Lemma 2.7. [11]

(1) (Local fractional integration is anti-differentiation) Suppose that $f(x) = g^{(\alpha)}(x) \in C_\alpha [a,b]$, then we have

$$_aI_b^\alpha f(x) = g(b) - g(a).$$

(2) (Local fractional integration by parts) Suppose that $f(x), g(x) \in D_\alpha [a,b]$ and $f^{(\alpha)}(x), g^{(\alpha)}(x) \in C_\alpha [a,b]$, then we have

$$_aI_b^\alpha f(x) g^{(\alpha)}(x) = f(x)g(x)|_a^b -_a I_b^\alpha f^{(\alpha)}(x) g(x).$$

Lemma 2.8. [11]

$$\frac{d^\alpha x^{k\alpha}}{dx^\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} x^{(k-1)\alpha};$$

$$\frac{1}{\Gamma(\alpha+1)} \int_a^b x^{k\alpha} (dx)^\alpha = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \left(b^{(k+1)\alpha} - a^{(k+1)\alpha}\right), \ k \in R.$$

Lemma 2.9. [Generalized Hölder's inequality] [11] Let $f, g \in C_\alpha [a,b]$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)g(x)| (dx)^\alpha \leq \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)|^p (dx)^\alpha \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |g(x)|^q (dx)^\alpha \right)^{\frac{1}{q}}.$$

In [3], Mo et al. proved the following generalized Hermite-Hadamard inequality for generalized convex function:

Theorem 2.10. [Generalized Hermite-Hadamard's inequality] Let $f(x) \in I_x^\alpha [a,b]$ be generalized convex function on $[a,b]$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_aI_b^\alpha f(x) \leq \frac{f(a)+f(b)}{2^\alpha}.$$

The interested reader is refer to [1],[2], [3]-[5],[7]-[9], [11]-[15] for local fractional theory and theory of inequalities.

The aim of the paper is to establish some new inequality for generalized convex functions which is related to the left side of generalized Hermite- Hadamard type inequality and apply them for some generalized special means.

3. Main results

We will start the generalized identity for local fractional integrals as follow.

Theorem 3.1. *Let $I \subseteq R$ be an interval, $f : I^0 \subseteq R \to R^\alpha$ (I^0 is the interior of I) such that $f \in D_{2\alpha}(I^0)$ and $f^{(2\alpha)} \in C_{2\alpha}[a,b]$ for $a, b \in I^0$ with $a < b$. Then, we have the identity*

$$\begin{aligned}
&\frac{\Gamma(1+2\alpha)}{(b-a)^\alpha}\, {}_aI_b^\alpha f(t) - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} f\left(\frac{a+b}{2}\right) \\
&= \frac{(b-a)^{2\alpha}}{2^\alpha \Gamma(1+\alpha)} \int_0^1 m(t)\left[f^{(2\alpha)}(ta+(1-t)b) + f^{(2\alpha)}(tb+(1-t)a)\right](dt)^\alpha
\end{aligned} \tag{3.1}$$

where

$$m(t) = \begin{cases} t^{2\alpha}, & t \in \left[a, \frac{1}{2}\right] \\ (1-t)^{2\alpha}, & t \in \left(\frac{1}{2}, b\right]. \end{cases}$$

Proof . From definition of mapping $m(t)$, we have

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)} \int_0^1 m(t)\left[f^{(2\alpha)}(ta+(1-t)b) + f^{(2\alpha)}(tb+(1-t)a)\right](dt)^\alpha \\
&= \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} t^{2\alpha} f^{(2\alpha)}(ta+(1-t)b)(dt)^\alpha \\
&\quad + \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} t^{2\alpha} f^{(2\alpha)}(tb+(1-t)a)(dt)^\alpha \\
&\quad + \frac{1}{\Gamma(1+\alpha)} \int_{\frac{1}{2}}^1 (1-t)^{2\alpha} f^{(2\alpha)}(ta+(1-t)b)(dt)^\alpha \\
&\quad + \frac{1}{\Gamma(1+\alpha)} \int_{\frac{1}{2}}^1 (1-t)^{2\alpha} f^{(2\alpha)}(tb+(1-t)a)(dt)^\alpha \\
&= K_1 + K_2 + K_3 + K_4.
\end{aligned} \tag{3.2}$$

Using the local fractional integration by parts twice (Lemma 2.7), we have

$$\begin{aligned}
K_1 &= \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} t^{2\alpha} f^{(2\alpha)}(ta+(1-t)b)(dt)^\alpha = \left.\frac{t^{2\alpha} f^{(\alpha)}(ta+(1-t)b)}{(a-b)^\alpha}\right|_0^{\frac{1}{2}} \\
&\quad - \frac{1}{(a-b)^\alpha \Gamma(1+\alpha)} \int_0^{\frac{1}{2}} \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} t^\alpha f^{(\alpha)}(ta+(1-t)b)(dt)^\alpha
\end{aligned}$$

and so

$$\begin{aligned}
K_1 &= \frac{1}{2^{2\alpha}(a-b)^{\alpha}} f^{(\alpha)}\left(\frac{a+b}{2}\right) - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} \frac{t^{\alpha} f(ta+(1-t)b)}{(a-b)^{2\alpha}}\Bigg|_0^{\frac{1}{2}} \\
&\quad + \frac{1}{(a-b)^{2\alpha}\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} \Gamma(1+2\alpha) f(ta+(1-t)b)(dt)^{\alpha} \\
&= -\frac{1}{2^{2\alpha}(b-a)^{\alpha}} f^{(\alpha)}\left(\frac{a+b}{2}\right) - \frac{1}{2^{\alpha}(b-a)^{2\alpha}} \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} f\left(\frac{a+b}{2}\right) \\
&\quad + \frac{\Gamma(1+2\alpha)}{(b-a)^{2\alpha}\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} f(ta+(1-t)b)(dt)^{\alpha}.
\end{aligned} \tag{3.3}$$

Similarly, we have

$$\begin{aligned}
K_2 &= \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} t^{2\alpha} f^{(2\alpha)}(tb+(1-t)a)(dt)^{\alpha} \\
&= \frac{1}{2^{2\alpha}(b-a)^{\alpha}} f^{(\alpha)}\left(\frac{a+b}{2}\right) - \frac{1}{2^{\alpha}(b-a)^{2\alpha}} \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} f\left(\frac{a+b}{2}\right) \\
&\quad + \frac{\Gamma(1+2\alpha)}{(b-a)^{2\alpha}\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} f(tb+(1-t)a)(dt)^{\alpha}.
\end{aligned} \tag{3.4}$$

Moreover, using the local fractional integration by parts twice (Lemma 2.7), we have

$$\begin{aligned}
K_3 &= \frac{1}{\Gamma(1+\alpha)} \int_{\frac{1}{2}}^{1} (1-t)^{2\alpha} f^{(2\alpha)}(ta+(1-t)b)(dt)^{\alpha} = \frac{(1-t)^{2\alpha} f^{(\alpha)}(ta+(1-t)b)}{(a-b)^{\alpha}}\Bigg|_{\frac{1}{2}}^{1} \\
&\quad - \frac{1}{(a-b)^{\alpha}\Gamma(1+\alpha)} \times \int_{\frac{1}{2}}^{1} (-1)^{\alpha} \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} (1-t)^{\alpha} f^{(\alpha)}(ta+(1-t)b)(dt)^{\alpha} \\
&= -\frac{1}{2^{2\alpha}(a-b)^{\alpha}} f^{(\alpha)}\left(\frac{a+b}{2}\right) - (-1)^{\alpha} \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} \frac{(1-t)^{\alpha} f(ta+(1-t)b)}{(a-b)^{2\alpha}}\Bigg|_{\frac{1}{2}}^{1} \\
&\quad + \frac{1}{(a-b)^{2\alpha}\Gamma(1+\alpha)} \int_{\frac{1}{2}}^{1} (-1)^{2\alpha} \Gamma(1+2\alpha) f(ta+(1-t)b)(dt)^{\alpha} \\
&= \frac{1}{2^{2\alpha}(b-a)^{\alpha}} f^{(\alpha)}\left(\frac{a+b}{2}\right) + \frac{(-1)^{\alpha}}{2^{\alpha}(a-b)^{2\alpha}} \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} f\left(\frac{a+b}{2}\right) \\
&\quad + \frac{\Gamma(1+2\alpha)}{(a-b)^{2\alpha}\Gamma(1+\alpha)} \int_{\frac{1}{2}}^{1} f(ta+(1-t)b)(dt)^{\alpha}
\end{aligned}$$

and so

$$K_3 = \frac{1}{2^{2\alpha}(b-a)^{\alpha}} f^{(\alpha)}\left(\frac{a+b}{2}\right) - \frac{1}{2^{\alpha}(b-a)^{2\alpha}} \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} f\left(\frac{a+b}{2}\right) + \frac{\Gamma(1+2\alpha)}{(b-a)^{2\alpha}\Gamma(1+\alpha)} \int_{\frac{1}{2}}^{1} f(ta+(1-t)b)\,(dt)^{\alpha}. \tag{3.5}$$

Similarly, we have

$$\begin{aligned} K_4 &= \frac{1}{\Gamma(1+\alpha)} \int_{\frac{1}{2}}^{1} (1-t)^{2\alpha} f^{(2\alpha)}(tb+(1-t)a)\,(dt)^{\alpha} \\ &= -\frac{1}{2^{2\alpha}(b-a)^{\alpha}} f^{(\alpha)}\left(\frac{a+b}{2}\right) - \frac{1}{2^{\alpha}(b-a)^{2\alpha}} \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} f\left(\frac{a+b}{2}\right) \\ &\quad + \frac{\Gamma(1+2\alpha)}{(b-a)^{2\alpha}\Gamma(1+\alpha)} \int_{\frac{1}{2}}^{1} f(tb+(1-t)a)\,(dt)^{\alpha}. \end{aligned} \tag{3.6}$$

Putting equality (3.3)-(3.6) in (3.2), we obtain

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_{0}^{1} m(t) \left[f^{(2\alpha)}(ta+(1-t)b) + f^{(2\alpha)}(tb+(1-t)a)\right](dt)^{\alpha} \\ &= K_1+K_2+K_3+K_4 = \frac{\Gamma(1+2\alpha)}{(b-a)^{2\alpha}} \left[\frac{1}{\Gamma(1+\alpha)} \int_{0}^{1} f(ta+(1-t)b)\,(dt)^{\alpha} \right. \\ &\quad \left. + \frac{1}{\Gamma(1+\alpha)} \int_{0}^{1} f(tb+(1-t)a)\,(dt)^{\alpha} \right] - \frac{4^{\alpha}}{2^{\alpha}(b-a)^{2\alpha}} \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} f\left(\frac{a+b}{2}\right) \\ &= \frac{2^{\alpha}\Gamma(1+2\alpha)}{(b-a)^{3\alpha}} {}_aI_b^{\alpha} f(t) - \frac{2^{\alpha}}{(b-a)^{2\alpha}} \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} f\left(\frac{a+b}{2}\right). \end{aligned} \tag{3.7}$$

If we multiply the resulting equality (3.7) with $\frac{(b-a)^{2\alpha}}{2^{\alpha}}$, then we obtain the desired result. □

Remark 3.2. If we assume that $\alpha = 1$, then the identity (3.1) reduces the identity (1.2).

Theorem 3.3. *The assumptions of Theorem* 3.1 *are satisfied. If* $\left|f^{(2\alpha)}\right|$ *is generalized convex on* $[a,b]$*, then we have the inequality*

$$\begin{aligned} &\left|\frac{\Gamma(1+2\alpha)}{(b-a)^{\alpha}} {}_aI_b^{\alpha} f(t) - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} f\left(\frac{a+b}{2}\right)\right| \\ &\qquad \leq \frac{(b-a)^{2\alpha}}{8^{\alpha}} \left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right) \left[\left|f^{(2\alpha)}(a)\right| + \left|f^{(2\alpha)}(b)\right|\right]. \end{aligned} \tag{3.8}$$

Proof . Taking modulus in (3.1), we find that

$$\begin{aligned}&\left|\frac{\Gamma(1+2\alpha)}{(b-a)^{\alpha}}{}_{a}I_{b}^{\alpha}f(t)-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}f\left(\frac{a+b}{2}\right)\right|\\&\leq\frac{(b-a)^{2\alpha}}{2^{\alpha}}\left[\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}|m(t)|\left|f^{(2\alpha)}(ta+(1-t)b)\right|(dt)^{\alpha}\right.\\&\left.+\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}|m(t)|\left|f^{(2\alpha)}(tb+(1-t)a)\right|(dt)^{\alpha}\right].\end{aligned}$$

Since $\left|f^{(2\alpha)}\right|$ is generalized convex on $[a,b]$, then we have

$$\left|f^{(2\alpha)}(ta+(1-t)b)\right|\leq t^{\alpha}\left|f^{(2\alpha)}(a)\right|+(1-t)^{\alpha}\left|f^{(2\alpha)}(b)\right|$$

and

$$\left|f^{(2\alpha)}(tb+(1-t)a)\right|\leq t^{\alpha}\left|f^{(2\alpha)}(b)\right|+(1-t)^{\alpha}\left|f^{(2\alpha)}(a)\right|.$$

Then, we get

$$\begin{aligned}&\left|\frac{\Gamma(1+2\alpha)}{(b-a)^{\alpha}}{}_{a}I_{b}^{\alpha}f(t)-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}f\left(\frac{a+b}{2}\right)\right|\\&\leq\frac{(b-a)^{2\alpha}}{2^{\alpha}}\left[\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\frac{1}{2}}t^{2\alpha}\left[t^{\alpha}\left|f^{(2\alpha)}(a)\right|+(1-t)^{\alpha}\left|f^{(2\alpha)}(b)\right|\right](dt)^{\alpha}\right.\\&+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1}(1-t)^{2\alpha}\left[t^{\alpha}\left|f^{(2\alpha)}(a)\right|+(1-t)^{\alpha}\left|f^{(2\alpha)}(b)\right|\right](dt)^{\alpha}\\&+\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\frac{1}{2}}t^{2\alpha}\left[t^{\alpha}\left|f^{(2\alpha)}(b)\right|+(1-t)^{\alpha}\left|f^{(2\alpha)}(a)\right|\right](dt)^{\alpha}\\&\left.+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1}(1-t)^{2\alpha}\left[t^{\alpha}\left|f^{(2\alpha)}(b)\right|+(1-t)^{\alpha}\left|f^{(2\alpha)}(a)\right|\right](dt)^{\alpha}\right]\\&=\frac{(b-a)^{2\alpha}}{2^{\alpha}}\left[\frac{\left|f^{(2\alpha)}(a)\right|+\left|f^{(2\alpha)}(b)\right|}{\Gamma(1+\alpha)}\left(\int_{0}^{\frac{1}{2}}t^{2\alpha}(dt)^{\alpha}+\int_{\frac{1}{2}}^{1}(1-t)^{2\alpha}t^{\alpha}(dt)^{\alpha}\right.\right.\\&\left.\left.+\int_{0}^{\frac{1}{2}}t^{2\alpha}(1-t)^{\alpha}(dt)^{\alpha}+\int_{\frac{1}{2}}^{1}(1-t)^{3\alpha}(dt)^{\alpha}\right)\right].\end{aligned}\tag{3.9}$$

Using Lemma 2.8, we obtain

$$\int_{0}^{\frac{1}{2}}t^{2\alpha}(dt)^{\alpha}=\int_{\frac{1}{2}}^{1}(1-t)^{3\alpha}(dt)^{\alpha}=\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\left(\frac{1}{16}\right)^{\alpha}\tag{3.10}$$

and

$$\int_{\frac{1}{2}}^{1} (1-t)^{2\alpha} t^{\alpha} (dt)^{\alpha} = \int_{0}^{\frac{1}{2}} t^{2\alpha}(1-t)^{\alpha}(dt)^{\alpha} = \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(\frac{1}{8}\right)^{\alpha} - \frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\left(\frac{1}{16}\right)^{\alpha}. \tag{3.11}$$

Substituting the equalities (3.10) and (3.11) in (3.9), we obtain desired inequality, which completes the proof. □

Remark 3.4. If we assume that $\alpha = 1$, then the inequality (3.8) reduces the following inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{24}\left[\frac{|f''(a)|+|f''(b)|}{2}\right]$$

which was proved by Sarikaya et al. in [10].

Theorem 3.5. *The assumptions of Theorem 3.1 are satisfied. If $\left|f^{(2\alpha)}\right|^q, q>1$ is generalized convex on $[a,b]$, then we have the inequality*

$$\begin{aligned} &\left|\frac{\Gamma(1+2\alpha)}{(b-a)^{\alpha}}\, {}_aI_b^{\alpha} f(t) - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} f\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{(b-a)^{2\alpha}}{4^{\alpha}}\left(\frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{1}{q}}\left[\left|f^{(2\alpha)}(a)\right|+\left|f^{(2\alpha)}(b)\right|\right]^{\frac{1}{q}}, \end{aligned} \tag{3.12}$$

where $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof . Taking modulus in (3.1) and using generalized Hölder's inequality (Lemma 2.9), we have

$$\begin{aligned} &\left|\frac{\Gamma(1+2\alpha)}{(b-a)^{\alpha}}\, {}_aI_b^{\alpha} f(t) - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} f\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{(b-a)^{2\alpha}}{2^{\alpha}}\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1 |m(t)|\left|f^{(2\alpha)}(ta+(1-t)b)\right|(dt)^{\alpha}\right. \\ &\quad \left. + \frac{1}{\Gamma(1+\alpha)}\int_0^1 |m(t)|\left|f^{(2\alpha)}(tb+(1-t)a)\right|(dt)^{\alpha}\right] \\ &\leq \frac{(b-a)^{2\alpha}}{2^{\alpha}}\left[\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1 |m(t)|^p (dt)^{\alpha}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1 \left|f^{(2\alpha)}(ta+(1-t)b)\right|^q (dt)^{\alpha}\right)^{\frac{1}{q}}\right. \\ &\quad \left. + \left(\frac{1}{\Gamma(1+\alpha)}\int_0^1 |m(t)|^p (dt)^{\alpha}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1 \left|f^{(2\alpha)}(tb+(1-t)a)\right|^q (dt)^{\alpha}\right)^{\frac{1}{q}}\right]. \end{aligned} \tag{3.13}$$

Since $\left|f^{(2\alpha)}\right|^q$ is generalized convex on $[a,b]$, then we have

$$\left|f^{(2\alpha)}(ta+(1-t)b)\right|^q \leq t^{\alpha}\left|f^{(2\alpha)}(a)\right|^q + (1-t)^{\alpha}\left|f^{(2\alpha)}(b)\right|^q$$

and

$$\left|f^{(2\alpha)}(tb+(1-t)a)\right|^q \leq t^\alpha \left|f^{(2\alpha)}(b)\right|^q + (1-t)^\alpha \left|f^{(2\alpha)}(a)\right|^q .$$

It follows that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_0^1 \left|f^{(2\alpha)}(ta+(1-t)b)\right|^q (dt)^\alpha \\
&\leq \left|f^{(2\alpha)}(a)\right|^q \frac{1}{\Gamma(1+\alpha)}\int_0^1 t^\alpha (dt)^\alpha + \left|f^{(2\alpha)}(b)\right|^q \int_0^1 (1-t)^\alpha (dt)^\alpha \\
&= \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[\left|f^{(2\alpha)}(a)\right| + \left|f^{(2\alpha)}(b)\right|\right],
\end{aligned} \tag{3.14}$$

and similarly,

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1 \left|f^{(2\alpha)}(tb+(1-t)a)\right|^q (dt)^\alpha = \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[\left|f^{(2\alpha)}(a)\right| + \left|f^{(2\alpha)}(b)\right|\right]. \tag{3.15}$$

Furthermore, we have

$$\begin{aligned}
\frac{1}{\Gamma(1+\alpha)}\int_0^1 |m(t)|^p (dt)^\alpha &= \frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{2p\alpha} (dt)^\alpha + \frac{1}{\Gamma(1+\alpha)}\int_0^1 (1-t)^{2p\alpha} (dt)^\alpha \\
&= \frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\frac{1}{2^{(2p+1)\alpha}} + \frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\frac{1}{2^{(2p+1)\alpha}} \\
&= \frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\frac{1}{4^{p\alpha}}.
\end{aligned} \tag{3.16}$$

Adding (3.14)-(3.16) in (3.13), we obtain the desired result. □

Remark 3.6. If we assume that $\alpha = 1$, then the inequality (3.12) reduces the inequality (1.3).

Theorem 3.7. *The assumptions of Theorem 3.1 are satisfied. If $\left|f^{(2\alpha)}\right|^q, q \geq 1$ is generalized convex on $[a,b]$, then we have the inequality*

$$\begin{aligned}
&\left|\frac{\Gamma(1+2\alpha)}{(b-a)^\alpha}{}_aI_b^\alpha f(t) - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{(b-a)^{2\alpha}}{4^\alpha}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left[\frac{\left|f^{(2\alpha)}(a)\right| + \left|f^{(2\alpha)}(b)\right|}{2^\alpha}\right]^{\frac{1}{q}}.
\end{aligned} \tag{3.17}$$

Proof . Taking modulus in (3.1), we have

$$\begin{aligned}
&\left|\frac{\Gamma(1+2\alpha)}{(b-a)^\alpha}{}_aI_b^\alpha f(t) - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{(b-a)^{2\alpha}}{2^\alpha}\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1 |m(t)|\left|f^{(2\alpha)}(ta+(1-t)b)\right| (dt)^\alpha\right. \\
&\quad \left. + \frac{1}{\Gamma(1+\alpha)}\int_0^1 |m(t)|\left|f^{(2\alpha)}(tb+(1-t)a)\right| (dt)^\alpha\right].
\end{aligned} \tag{3.18}$$

Because of $\frac{1}{p}+\frac{1}{q}=1$, $\alpha\left(\frac{1}{p}+\frac{1}{q}\right)$ can be written instead of α. Using the generalized Holder's inequality, we find that

$$\begin{aligned}&\left|\frac{\Gamma(1+2\alpha)}{(b-a)^{\alpha}}{}_aI_b^{\alpha}f(t)-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}f\left(\frac{a+b}{2}\right)\right|\\&\leq\frac{(b-a)^{2\alpha}}{2^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1|m(t)|(dt)^{\alpha}\right)^{\frac{1}{p}}\\&\times\left[\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1|m(t)|\left|f^{(2\alpha)}(ta+(1-t)b)\right|^q(dt)^{\alpha}\right)^{\frac{1}{q}}\right.\\&\left.+\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1|m(t)|\left|f^{(2\alpha)}(tb+(1-t)a)\right|^q(dt)^{\alpha}\right)^{\frac{1}{q}}\right].\end{aligned}$$

If $\left|f^{(2\alpha)}\right|^q$ is generalized convex on $[a,b]$, then we have

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)}\int_0^1|m(t)|\left|f^{(2\alpha)}(ta+(1-t)b)\right|^q(dt)^{\alpha}\\&\leq\frac{1}{\Gamma(1+\alpha)}\int_0^{\frac{1}{2}}t^{2\alpha}\left[t^{\alpha}\left|f^{(2\alpha)}(a)\right|^q+(1-t)^{\alpha}\left|f^{(2\alpha)}(b)\right|^q\right](dt)^{\alpha}\\&+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^1(1-t)^{2\alpha}\left[t^{\alpha}\left|f^{(2\alpha)}(a)\right|^q+(1-t)^{\alpha}\left|f^{(2\alpha)}(b)\right|^q\right](dt)^{\alpha}\\&=\frac{\left|f^{(2\alpha)}(a)\right|^q}{\Gamma(1+\alpha)}\left[\int_0^{\frac{1}{2}}t^{3\alpha}(dt)^{\alpha}+\int_{\frac{1}{2}}^1(1-t)^{2\alpha}t^{\alpha}(dt)^{\alpha}\right]\\&+\frac{\left|f^{(2\alpha)}(b)\right|^q}{\Gamma(1+\alpha)}\left[\int_0^{\frac{1}{2}}t^{2\alpha}(1-t)^{\alpha}(dt)^{\alpha}+\int_{\frac{1}{2}}^1(1-t)^{3\alpha}(dt)^{\alpha}\right]\\&=\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(\frac{1}{8}\right)^{\alpha}\left[\left|f^{(2\alpha)}(a)\right|^q+\left|f^{(2\alpha)}(b)\right|^q\right].\end{aligned}$$

Similarly, we get

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)}\int_0^1|m(t)|\left|f^{(2\alpha)}(tb+(1-t)a)\right|^q(dt)^{\alpha}\\&\leq\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(\frac{1}{8}\right)^{\alpha}\left[\left|f^{(2\alpha)}(a)\right|^q+\left|f^{(2\alpha)}(b)\right|^q\right].\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\left|\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}{}_aI_b^{\alpha}f(t)-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{(b-a)^{2\alpha}}{2^{\alpha}}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\left(\frac{1}{4}\right)^{\alpha}\right)^{\frac{1}{p}} \\
&\quad\times\left[2^{\alpha}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)^{\frac{1}{q}}\left(\frac{1}{8}\right)^{\frac{\alpha}{q}}\left[\left|f^{(2\alpha)}(a)\right|^{q}+\left|f^{(2\alpha)}(b)\right|^{q}\right]^{\frac{1}{q}}\right] \\
&=\frac{(b-a)^{2\alpha}}{4^{\alpha}}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left[\frac{\left|f^{(2\alpha)}(a)\right|+\left|f^{(2\alpha)}(b)\right|}{2^{\alpha}}\right]^{\frac{1}{q}}.
\end{aligned}$$

Here, we used the fact that

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1|m(t)|\,(dt)^{\alpha}=\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\left(\frac{1}{4}\right)^{\alpha},$$

which completes the proof. □

Remark 3.8. If we assume that $\alpha=1$, then the inequality (3.8) reduces the following inequality.

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{(b-a)^2}{24}\left[\frac{|f''(a)|^q+|f''(b)|^q}{2}\right]$$

which was proved by Sarikaya et al. in [10].

4. Applications to Some Special Means

We consider some generalized means as in [10]:

$$A(a,b)=\frac{a^{\alpha}+b^{\alpha}}{2^{\alpha}};$$

$$L_n(a,b)=\left[\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n+1)\alpha)}\left[\frac{b^{(n+1)\alpha}-a^{(n+1)\alpha}}{(b-a)^{\alpha}}\right]\right]^{\frac{1}{n}},n\in Z\backslash\{-1,0\},a,b\in R,a\neq b.$$

Proposition 4.1. *Let $a,b\in R$, $0<a<b$, $0\notin[a,b]$ and $n\in Z$, $|n(n-1)|\geq 3$. Then, we have the inequality*

$$\begin{aligned}
&\left|\Gamma(1+2\alpha)\left[L_n(a,b)\right]^n-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}A^n(a,b)\right| \\
&\leq\frac{(b-a)^{2\alpha}}{4^{\alpha}}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|A(a^{n-2},b^{n-2}).
\end{aligned}$$

Proof . Let us reconsider the inequality (3.8):

$$\begin{aligned}&\left|\frac{\Gamma(1+2\alpha)}{(b-a)^{\alpha}}\,{}_aI_b^{\alpha}f(t)-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}f\left(\frac{a+b}{2}\right)\right|\\&\leq\frac{(b-a)^{2\alpha}}{8^{\alpha}}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left[\left|f^{(2\alpha)}(a)\right|+\left|f^{(2\alpha)}(b)\right|\right].\end{aligned}$$

Consider the mapping $f:(0,\infty)\to R^{\alpha}$, $f(x)=x^{n\alpha}$, $n\in Z\backslash\{-1,0\}$. Then, $0<a<b$, we have

$$\begin{aligned}f\left(\frac{a+b}{2}\right)=A^n(a,b),\frac{{}_aI_b^{\alpha}f(t)}{(b-a)^{\alpha}}&=[L_n(a,b)]^n,\left|f^{(2\alpha)}(a)\right|\\&=\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|a^{(n-2)\alpha}\end{aligned}$$

and

$$\left|f^{(2\alpha)}(b)\right|=\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|b^{(n-2)\alpha}.$$

Then, we obtain

$$\begin{aligned}&\left|\Gamma(1+2\alpha)[L_n(a,b)]^n-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}A^n(a,b)\right|\\&\leq\frac{(b-a)^{2\alpha}}{4^{\alpha}}\left(\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right)\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|\left[\frac{a^{(n-2)\alpha}+b^{(n-2)\alpha}}{2^{\alpha}}\right]\\&=\frac{(b-a)^{2\alpha}}{4^{\alpha}}\left(\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right)\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|A(a^{n-2},b^{n-2}).\end{aligned}$$

This completes the proof. □

Proposition 4.2. *Let $a,b\in R$, $0<a<b$, $0\notin[a,b]$ and $n\in Z$, $|n(n-1)|\geq 3$. Then, we have the inequality*

$$\begin{aligned}&\left|\Gamma(1+2\alpha)[L_n(a,b)]^n-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}A^n(a,b)\right|\\&\leq\frac{(b-a)^{2\alpha}}{4^{\alpha}}\left(\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right)\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|^{\frac{1}{q}}\left[A(a^{n-2},b^{n-2})\right]^{\frac{1}{q}}.\end{aligned}$$

Proof . From Theorem 3.7 with $f(x)=x^{n\alpha}$, $f:(0,\infty)\to R^{\alpha}$ and the above equalities, we have

$$\begin{aligned}&\left|\Gamma(1+2\alpha)[L_n(a,b)]^n-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}A^n(a,b)\right|\\&\leq\frac{(b-a)^{2\alpha}}{4^{\alpha}}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|^{\frac{1}{q}}\left[\frac{a^{(n-2)\alpha}+b^{(n-2)\alpha}}{2^{\alpha}}\right]^{\frac{1}{q}}\\&=\frac{(b-a)^{2\alpha}}{4^{\alpha}}\left(\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right)\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|^{\frac{1}{q}}\left[A(a^{n-2},b^{n-2})\right]^{\frac{1}{q}}.\end{aligned}$$

This completes the proof. □

References

[1] G.-S. Chen, *Generalizations of Hölder's and some related integral inequalities on fractal space,* J. Function Spaces Appl. (2013), Article ID 198405, 9.

[2] S.S. Dragomir *A companion of Ostrowski's inequality for functions of bounded variation and applications*, Int. J. Nonlinear Anal. Appl. 5 (2014) 89–97.

[3] H. Mo, X Sui and D. Yu, *Generalized convex functions on fractal sets and two related inequalities*, Abst. Appl. Anal. (2014), Article ID 636751, 7 pages.

[4] H. Mo, *Generalized Hermite-Hadamard inequalities involving local fractional integral,* arXiv: 1410.1062.

[5] T. Mohamed, D. Zeglami and S. Kabbaj, *On Hilbert Golab-Schinzel type functional equation*, Int. J. Nonlinear Anal. Appl. 6 (2015) 149–159.

[6] J.E. Pečarić, F. Proschan and Y.L. Tong, *Convex functions, partial orderings and statistical applications*, Academic Press, Boston, 1992.

[7] S. Abbaszadeh and M. Eshaghi, *A Hadamard-type inequality for fuzzy integrals based on r-convex functions*, Soft Comput. 20 (2016) 3117–3124.

[8] S. Abbaszadeh, M. Eshaghi and M. de la Sen, *The Sugeno fuzzy integral of log-convex functions*, J. Inequal. Appl. 2015 (2015): 362.

[9] S. Abbaszadeh and A. Ebadian, *Nonlinear integrals and Hadamard-type inequalities*, Soft Comput, In press.

[10] M.Z. Sarikaya, A. Saglam and H. Yildirim, *New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are convex and quasi-convex*, Int. J. Open Problems Comput. Sci. Math. (IJOPCM) 5 (2012) 1–14.

[11] X.J. Yang, *Advanced Local Fractional Calculus and Its Applications*, World Science Publisher, New York, 2012.

[12] J. Yang, D. Baleanu and X. J. Yang, *Analysis of fractal wave equations by local fractional Fourier series method*, Adv. Math. Phys. 2013 (2013), Article ID 632309.

[13] X.J. Yang, *Local fractional integral equations and their applications*, Adv. Comput. Sci. Appl. (ACSA) 1 (4), 2012.

[14] X.J. Yang, *Generalized local fractional Taylor's formula with local fractional derivative*, arXiv preprint arXiv:1106.2459 (2011).

[15] X.J. Yang, *Local fractional Fourier analysis,*Adv. Mech. Engin. Appl. 1 (2012) 12–16.