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Relative order and type of entire functions represented by Banach valued Dirichlet series in two variables

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Abstract

In this paper, we introduce the idea of relative order and type of entire functions represented by Banach valued Dirichlet series of two complex variables to generalize some earlier results.

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1. Introduction

For entire function f, let $F(r) = max\{|f(z)| : |z| = r\}$. If f is non-constant then $F(r)$ is strictly increasing and continuous function of r and its inverse

$$F^{-1} : (|f(0)|, \infty) \to (0, \infty)$$

exists and

$$\lim_{R\to\infty} F^{-1}(R) = \infty.$$

Let f and g be two entire functions. Bernal [3] introduced the definition of relative order of f with respect to g, denoted by $\rho_g(f)$, as follows:

$$\rho_g(f) = inf\{\mu > 0 : F(r) < G(r^{\mu}) \text{ for } \text{ all } r > r_0(\mu) > 0\}$$

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After this several papers on relative order of entire functions have appeared in the literature where growing interest of workers on this topic has been noticed {see for example [1], [2], [4], [5], [6], [7], [8]}.
Let $f(s)$ be an entire function of the complex variable $s = \sigma + it$ defined by everywhere absolutely convergent Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{s\lambda_n}, \tag{1.1}$$

where $0 < \lambda_n < \lambda_{n+1}$ $(n \geq 1), \lambda_n \to \infty$ as $n \to \infty$ and $a_n^{,}s$ are complex constants.

Let

$$F(\sigma) = l.u.b._{-\infty<t<\infty}|f(\sigma + it)|.$$

Then the Ritt order [10] of $f(s)$, denoted by $\rho(f)$ is given by

$$\rho(f) = \inf\{\mu > 0 : \log F(\sigma) < \exp(\sigma\mu) \text{ for all } \sigma > R(\mu)\}.$$

In other words,

$$\rho(f) = \limsup_{\sigma\to\infty} \frac{\log\log F(\sigma)}{\sigma}.$$

During the past decades, several authors made close investigations on the properties of entire Dirichlet series related to Ritt order. In 2010, Lahiri and Banerjee [9] introduced the idea of relative Ritt order as follows:
Let $f(s)$ be an entire function defined by everywhere absolutely convergent Dirichlet series (1.1) and $g(s)$ be an entire function. Then the relative Ritt order of $f(s)$ with respect to entire $g(s)$ denoted by $\rho_g(f)$ is defined as

$$\rho_g(f) = \inf\{\mu > 0 : \log F(\sigma) < G(\sigma\mu) \text{for all large } \sigma\},$$

where $G(r) = \max\{|g(s)| : |s| = r\}$.

Recently Srivastava [11] defined the growth parameter such as relative order, relative type, relative lower type of entire functions represented by vector valued Dirichlet series of the form (1.1) as follows:
Let $f(s)$ and $g(s)$ be two entire functions defined by everywhere absolutely convergent vector valued Dirichlet series of the form (1.1), where a_n's belong to a Banach space $(E, \|.\|)$ and λ_n' s are non-negative real numbers such that

$$0 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_n \to \infty$$

as $n \to \infty$ and satisfy the conditions

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = D < \infty$$

and

$$\limsup_{n\to\infty} \frac{\log \|a_n\|}{\lambda_n} = -\infty.$$

Also $F(\sigma)$, $G(\sigma)$ denote their respective maximum moduli. The relative order of $f(s)$ with respect to $g(s)$ denoted by $\rho_g(f)$ is defined as

$$\rho_g(f) = \inf\{\mu > 0 : F(\sigma) < G(\sigma\mu) \text{ for all } \sigma > \sigma_0(\mu)\}$$

i.e.,

$$\rho_g(f) = \limsup_{\sigma\to\infty} \frac{G^{-1}F(\sigma)}{\sigma}.$$

The relative type and relative lower type of $f(s)$ with respect to $g(s)$ denoted respectively by $T_g(f)$ and $\tau_g(f)$ when $\rho_g(f) = 1$ (i.e., $\rho(f) = \rho(g) = \rho$) and defined as

$$\begin{aligned} T_g(f) &= \inf\{\mu > 0 : F(\sigma) < G[\frac{1}{\rho}\log(\mu e^{\rho\sigma})] \text{ for all } \sigma > \sigma_0(\mu)\} \\ &= \limsup_{\sigma\to\infty} \frac{\exp[\rho G^{-1}F(\sigma)]}{\exp(\rho\sigma)} \end{aligned}$$

and

$$\tau_g(f) = \liminf_{\sigma\to\infty} \frac{\exp[\rho G^{-1}F(\sigma)]}{\exp(\rho\sigma)}$$

If $T_g(f) = \tau_g(f)$ then f is said to be of regular type with respect to g.

Srivastava and Sharma [12] introduced the idea of order and type of an entire function represented by vector valued Dirichlet series of two complex variables. Consider

$$f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{mn} e^{(s_1\lambda_m + s_2\mu_n)}, \quad (s_j = \sigma_j + it_j, \quad j = 1, 2) \tag{1.2}$$

where a_{mn}'s belong to the Banach space $(E, ||.||)$; $0 \le \lambda_1 < \lambda_2 < \ldots < \lambda_m \to \infty$ as $m \to \infty$; $0 \le \mu_1 < \mu_2 < \ldots < \mu_n \to \infty$ as $n \to \infty$ and

$$\limsup_{m+n\to\infty} \frac{\log(m+n)}{\lambda_m + \mu_n} = D < +\infty.$$

Such a series is called a vector valued Dirichlet series in two complex variables.

If only a finite number of a_{mn}'s are non zero in (1.2), then we call it as a Banach valued Dirichlet polynomial of two complex variables. Let $f(s_1, s_2)$ defined above represent an entre function and

$$F(\sigma_1, \sigma_2) = \sup\{\|f(\sigma_1 + it_1, \sigma_2 + it_2)\|; -\infty < t_j < \infty; j = 1, 2\}$$

be its maximum modulus. Then the order $\rho(f)$ of $f(s_1, s_2)$ is defined as

$$\rho(f) = \limsup_{\sigma_1,\sigma_2\to\infty} \frac{\log\log F(\sigma_1, \sigma_2)}{\log(e^{\sigma_1} + e^{\sigma_2})}.$$

If $(0 < \rho(f) < \infty)$, then the type $T(f)(0 \le T(f) \le \infty)$ of $f(s_1, s_2)$ is defined as

$$T(f) = \limsup_{\sigma_1,\sigma_2\to\infty} \frac{\log F(\sigma_1, \sigma_2)}{(e^{\rho(f)\sigma_1} + e^{\rho(f)\sigma_2})}.$$

At this stage it therefore seems reasonable to define suitably the relative order of entire functions defined by Banach valued Dirichlet series with respect to an entire function defined by Banach valued Dirichlet series of two complex variables and to enquire its basic properties in the new context. Proving some preliminary theorems on the relative order, we obtain sum and product theorems and we show that the relative order (finite) of an entire function represented by Dirichlet series (1.2) is the same as its partial derivative, under certain restrictions.

The following definitions are now introduced.

Definition 1.1. Let $f(s_1, s_2)$ and $g(s_1, s_2)$ be two entire functions defined by the Banach valued Dirichlet series (1.2). Then the relative order of $f(s_1, s_2)$ with respect to $g(s_1, s_2)$ denoted by $\rho_g(f)$ is defined by

$$\rho_g(f) = inf\{\mu > 0 : F(\sigma_1, \sigma_2) < exp[G(\sigma_1, \sigma_2)]^\mu\}$$

where

$$F(\sigma_1, \sigma_2) = \sup\{\|f(\sigma_1 + it_1, \sigma_2 + it_2)\|; -\infty < t_j < \infty; j = 1, 2\}.$$

If we put $\lambda_n = \mu_n = n - 1$ for $n = 1, 2, 3, \ldots$ and $a_{12} = a_{21} = 1$, and all other a_{mn}'s are zero then $g(s_1, s_2) = e^{s_1} + e^{s_2}$ and consequently

$$\rho_g(f) = \rho(f).$$

Definition 1.2. Let $f(s_1, s_2)$ and $g(s_1, s_2)$ be two entire functions defined by the Banach valued Dirichlet series (1.2) such that $\rho(f) = \rho(g)$. Then the relative type and relative lower type of $f(s_1, s_2)$ with respect to $g(s_1, s_2)$ are denoted respectively by $T_g(f)$ and $\tau_g(f)$ and defined as

$$T_g(f) = \limsup_{\sigma_1, \sigma_2 \to \infty} \frac{\log F(\sigma_1, \sigma_2)}{G(\rho\sigma_1, \rho\sigma_2)}$$

and

$$\tau_g(f) = \liminf_{\sigma_1, \sigma_2 \to \infty} \frac{\log F(\sigma_1, \sigma_2)}{G(\rho\sigma_1, \rho\sigma_2)}$$

where $\rho = \rho(f) = \rho(g)$. Clearly $T_g(f) = T(f)$ if $g(s_1, s_2) = e^{s_1} + e^{s_2}$.

Definition 1.3. Let $f_1(s_1, s_2)$ and $f_2(s_1, s_2)$ be two entire functions defined by the Banach valued Dirichlet series (1.2). Then $f_1(s_1, s_2)$ and $f_2(s_1, s_2)$ are said to be asymptotically equivalent if there exists l, $(0 < l < \infty)$ such that

$$\frac{F_1(\sigma_1, \sigma_2)}{F_2(\sigma_1, \sigma_2)} \to l$$

as $\sigma_1, \sigma_2 \to \infty$ and in this case we write $f_1 \sim f_2$.

If $f_1 \sim f_2$ then clearly $f_2 \sim f_1$.
Throughout the paper we assume that $f(s_1, s_2), f_1(s_1, s_2), g(s_1, s_2), g_1(s_1, s_2)$, etc. are non-constant entire functions defined by Banach valued Dirichlet series (1.2) and $F(\sigma_1, \sigma_2), F_1(\sigma_1, \sigma_2), G(\sigma_1, \sigma_2)$, $G_1(\sigma_1, \sigma_2)$ denote their respective maximum moduli.

The following lemma will be needed in the sequel.

Lemma 1.4. Let $g(s_1, s_2)$ be a non-constant entire function defined by Banach valued Dirichlet series (1.2) and $\gamma > 1$, $0 < \mu < \lambda$. Then

$$\lim_{\sigma_1, \sigma_2 \to \infty} \frac{[G(\sigma_1, \sigma_2)]^\gamma}{G(\sigma_1, \sigma_2)} = \infty$$

and

$$\lim_{\sigma_1, \sigma_2 \to \infty} \frac{[G(\sigma_1, \sigma_2)]^\lambda}{[G(\sigma_1, \sigma_2)]^\mu} = \infty.$$

Proof of the lemma is omitted.

2. Preliminary Theorems

Theorem 2.1. (a) $\rho_g(f) = \limsup_{\sigma_1,\sigma_2\to\infty} \frac{\log\log F(\sigma_1,\sigma_2)}{\log G(\sigma_1,\sigma_2)}$.

(b) *If $f(s_1, s_2)$ be a Dirichlet polynomial and $g(s_1, s_2)$ is not a Dirichlet polynomial, then $\rho_g(f) = 0$.*

(c) *If $F_1(\sigma_1, \sigma_2) \leq F_2(\sigma_1, \sigma_2)$ for all large σ_1, σ_2 then $\rho_g(f_1) \leq \rho_g(f_2)$.*

(d) *If $G_1(\sigma_1, \sigma_2) \leq G_2(\sigma_1, \sigma_2)$ for all large σ_1, σ_2 then $\rho_{g_1}(f) \geq \rho_{g_2}(f)$.*

Proof .

(a) This follows from the definition.

(b) Let f be of the form $f(s_1, s_2) = \sum_{k=1}^{m}\{\sum_{l=1}^{n} a_{kl}e^{s_1\lambda_k+s_2\mu_l}\}$. Then

$$\begin{aligned} F(\sigma_1,\sigma_2) &= \sup_{-\infty<t_1,t_2<\infty} \|\sum_{k=1}^{m}\{\sum_{l=1}^{n} a_{kl}e^{(\sigma_1+it_1)\lambda_k+(\sigma_2+it_2)\mu_l}\}\| \\ &\leq \sum_{k=1}^{m}\{\sum_{l=1}^{n} \|a_{kl}\|e^{\sigma_1\lambda_k+\sigma_2\mu_l}\} \\ &\leq mn \max_{k=1,2,3,\ldots,m;l=1,2,3,\ldots,n} \|a_{kl}\|e^{\sigma_1\lambda_m+\sigma_2\mu_n} = Me^{\sigma_1\lambda_m+\sigma_2\mu_n}, \end{aligned} \tag{2.1}$$

since we may clearly assume that σ_1, σ_2 are positive, where

$$M = mn \max_{k=1,2,3,\ldots,m;l=1,2,3,\ldots,n} ||a_{kl}||$$

is a constant.

On the other hand, since $g(s_1, s_2)$ is not a Dirichlet polynomial, for all large σ_1, σ_2, p and for every $\delta > 0$ and k a constant large at pleasure,

$$[G(\sigma_1,\sigma_2)]^\delta > k^\delta\sigma_1^{\delta p}\sigma_2^{\delta p} > \log M + (\sigma_1\lambda_m + \sigma_2\mu_n) \geq \log F(\sigma_1,\sigma_2)$$

using (2.1). So, for all large σ_1, σ_2 and arbitrary $\delta > 0$

$$\frac{\log\log F(\sigma_1,\sigma_2)}{\log G(\sigma_1,\sigma_2)} < \delta$$

and this gives that $\rho_g(f) = 0$.

(c) Since $F_1(\sigma_1, \sigma_2) \leq F_2(\sigma_1, \sigma_2)$ for all large σ_1, σ_2, so

$$\limsup_{\sigma_1,\sigma_2\to\infty}\frac{\log\log F_1(\sigma_1,\sigma_2)}{\log G(\sigma_1,\sigma_2)} \leq \limsup_{\sigma_1,\sigma_2\to\infty}\frac{\log\log F_2(\sigma_1,\sigma_2)}{\log G(\sigma_1,\sigma_2)}$$

i.e.,

$$\rho_g(f_1) \leq \rho_g(f_2).$$

(d) Proof is similar as that of (c). □

Remark 2.2. Let $f_1(s_1, s_2) = g(s_1, s_2) = e^{s_1+s_2}$ and $f_2(s_1, s_2) = e^{2(s_1+s_2)}$. Then clearly

$$F_1(\sigma_1,\sigma_2) < F_2(\sigma_1,\sigma_2).$$

But $\rho_g(f_1) = \rho_g(f_2) = 0$. Let $f(s_1, s_2) = g_1(s_1, s_2) = e^{s_1+s_2}$ and $g_2(s_1, s_2) = e^{2(s_1+s_2)}$. Then clearly $G_1(\sigma_1, \sigma_2) < G_2(\sigma_1, \sigma_2)$. But $\rho_{g_1}(f) = \rho_{g_2}(f) = 0$.

Theorem 2.3. (a) *If* $\rho(f_1)=\rho(f_2)=\rho(g)$ *and* $F_1(\sigma_1,\sigma_2)\le F_2(\sigma_1,\sigma_2)$ *for all large values of* σ_1,σ_2 *then* $T_g(f_1)\le T_g(f_2)$.

(b) *If* $\rho(f)=\rho(g_1)=\rho(g_2)$ *and* $G_1(\sigma_1,\sigma_2)\le G_2(\sigma_1,\sigma_2)$ *for all large values of* σ_1,σ_2 *then* $T_{g_1}(f)\ge T_{g_2}(f)$.

Proof . This follows from definition. □

Theorem 2.4. *If* $\rho(f_1)=\rho(f_2)=\rho(g)$ *then*

$$\frac{\tau_g(f_1)}{T_g(f_2)}\le\liminf_{\sigma_1,\sigma_2\to\infty}\frac{\log F_1(\sigma_1,\sigma_2)}{\log F_2(\sigma_1,\sigma_2)}\le\frac{\tau_g(f_1)}{\tau_g(f_2)}\le\limsup_{\sigma_1,\sigma_2\to\infty}\frac{\log F_1(\sigma_1,\sigma_2)}{\log F_2(\sigma_1,\sigma_2)}\le\frac{T_g(f_1)}{\tau_g(f_2)}.$$

Proof . Suppose $\rho(f_1)=\rho(f_2)=\rho(g)=\rho$. Then by definition for any $\epsilon>0$ and for all large values of σ_1,σ_2

$$\log F_1(\sigma_1,\sigma_2)>(\tau_g(f_1)-\epsilon)G(\rho\sigma_1,\rho\sigma_2) \tag{2.2}$$

and

$$\log F_2(\sigma_1,\sigma_2)<(T_g(f_2)+\epsilon)G(\rho\sigma_1,\rho\sigma_2). \tag{2.3}$$

Therefore from Equation (2.2) and (2.3) we get for all large σ_1,σ_2

$$\frac{\log F_1(\sigma_1,\sigma_2)}{\log F_2(\sigma_1,\sigma_2)}>\frac{\tau_g(f_1)-\epsilon}{T_g(f_2)+\epsilon}$$

or,

$$\liminf_{\sigma_1,\sigma_2\to\infty}\frac{\log F_1(\sigma_1,\sigma_2)}{\log F_2(\sigma_1,\sigma_2)}\ge\frac{\tau_g(f_1)}{T_g(f_2)}. \tag{2.4}$$

Again by definition for any $\epsilon>0$ there exist sequences $\{\sigma_{1n}\},\ \sigma_{1n}\to\infty$ and $\{\sigma_{2n}\},\ \sigma_{2n}\to\infty$ as $n\to\infty$ such that

$$\log F_1(\sigma_{1n},\sigma_{2n})<(\tau_g(f_1)+\epsilon)G(\rho\sigma_{1n},\rho\sigma_{2n}). \tag{2.5}$$

Again for all large values of σ_1,σ_2

$$\log F_2(\sigma_1,\sigma_2)>(\tau_g(f_2)-\epsilon)G(\rho\sigma_1,\rho\sigma_2). \tag{2.6}$$

Hence from (2.5) and (2.6) we get

$$\frac{\log F_1(\sigma_{1n},\sigma_{2n})}{\log F_2(\sigma_{1n},\sigma_{2n})}<\frac{\tau_g(f_1)+\epsilon}{\tau_g(f_2)-\epsilon}$$

or

$$\liminf_{\sigma_1,\sigma_2\to\infty}\frac{\log F_1(\sigma_1,\sigma_2)}{\log F_2(\sigma_1,\sigma_2)}\le\frac{\tau_g(f_1)}{\tau_g(f_2)}. \tag{2.7}$$

Also there exist sequences $\{\sigma_{1m}\},\ \sigma_{1m}\to\infty$ and $\{\sigma_{2m}\},\ \sigma_{2m}\to\infty$ as $m\to\infty$ such that

$$\log F_2(\sigma_{1m},\sigma_{2m})<(\tau_g(f_2)+\epsilon)G(\rho\sigma_{1m},\rho\sigma_{2m}). \tag{2.8}$$

Therefore from (2.2) and (2.8) we get,

$$\frac{\log F_1(\sigma_{1m},\sigma_{2m})}{\log F_2(\sigma_{1m},\sigma_{2m})} > \frac{\tau_g(f_1)-\epsilon}{\tau_g(f_2)+\epsilon}$$

or

$$\limsup_{\sigma_1,\sigma_2\to\infty}\frac{\log F_1(\sigma_1,\sigma_2)}{\log F_2(\sigma_1,\sigma_2)} \geq \frac{\tau_g(f_1)}{\tau_g(f_2)}. \tag{2.9}$$

Again for any $\epsilon > 0$ and for all large values of σ_1, σ_2,

$$\log F_1(\sigma_1,\sigma_2) < (T_g(f_1)+\epsilon)G(\rho\sigma_1,\rho\sigma_2) \tag{2.10}$$

and

$$\log F_2(\sigma_1,\sigma_2) > (\tau_g(f_2)-\epsilon)G(\rho\sigma_1,\rho\sigma_2). \tag{2.11}$$

Therefore from (2.10) and (2.11) we get for all large σ_1, σ_2

$$\frac{\log F_1(\sigma_1,\sigma_2)}{\log F_2(\sigma_1,\sigma_2)} < \frac{T_g(f_1)+\epsilon}{\tau_g(f_2)-\epsilon}$$

or

$$\limsup_{\sigma_1,\sigma_2\to\infty}\frac{\log F_1(\sigma_1,\sigma_2)}{\log F_2(\sigma_1,\sigma_2)} \leq \frac{T_g(f_1)}{\tau_g(f_2)}. \tag{2.12}$$

The theorem now follows from (2.4), (2.7), (2.9) and (2.12). □

3. Sum and Product Theorems

Theorem 3.1. *Let $f_1(s_1, s_2)$, $f_2(s_1, s_2)$ and $g(s_1, s_2)$ be three entire functions defined by the Banach valued Dirichlet series* (1.2)*. Then*

$$\rho_g(f_1 \pm f_2) \leq \max\{\rho_g(f_1), \rho_g(f_2)\},$$

sign of equality holds if $\rho_g(f_1) \neq \rho_g(f_2)$.

Proof . We may suppose that $\rho_g(f_1)$ and $\rho_g(f_2)$ both are finite because in the contrary case the inequality follows immediately. We prove the theorem for addition only, because the proof for subtraction is analogous.

Let $f = f_1 + f_2$, $\rho = \rho_g(f)$, $\rho_i = \rho_g(f_i)$, $i = 1, 2$ and $\rho_1 \leq \rho_2$. For arbitrary $\epsilon > 0$ and for all large σ_1, σ_2 we have from Theorem 2.1 (a),

$$F_1(\sigma_1,\sigma_2) < \exp[G(\sigma_1,\sigma_2)]^{\rho_1+\epsilon} \leq \exp[G(\sigma_1,\sigma_2)]^{\rho_2+\epsilon}$$

and

$$F_2(\sigma_1,\sigma_2) < \exp[G(\sigma_1,\sigma_2)]^{\rho_2+\epsilon}.$$

So, for all large σ_1, σ_2

$$F(\sigma_1,\sigma_2) \leq F_1(\sigma_1,\sigma_2) + F_2(\sigma_1,\sigma_2) < 2\exp[G(\sigma_1,\sigma_2)]^{\rho_2+\epsilon} < \exp[G(\sigma_1,\sigma_2)]^{\rho_2+2\epsilon}.$$

Therefore,

$$\frac{\log\log F(\sigma_1,\sigma_2)}{\log G(\sigma_1,\sigma_2)} \le (\rho_2+2\epsilon),$$

for all large σ_1,σ_2. Since $\epsilon>0$ is arbitrary, we obtain

$$\rho \le \rho_2. \tag{3.1}$$

This proves the first part of the theorem.

For the second part, let $\rho_1<\rho_2$ and suppose that $\rho_1<\mu_1<\mu<\lambda<\rho_2$. Then for all large σ_1,σ_2

$$F_1(\sigma_1,\sigma_2) < \exp[G(\sigma_1,\sigma_2)]^{\mu} \tag{3.2}$$

and there exist non decreasing sequences

$$\{\sigma_{1n}\},\ \sigma_{1n}\to\infty$$

and

$$\{\sigma_{2n}\},\ \sigma_{2n}\to\infty \text{ as } n\to\infty$$

such that

$$F_2(\sigma_{1n},\sigma_{2n}) > \exp[G(\sigma_{1n},\sigma_{2n})]^{\lambda}. \tag{3.3}$$

Using Lemma 1.4 we see that

$$[G(\sigma_1,\sigma_2)]^{\lambda} > 2[G(\sigma_1,\sigma_2)]^{\mu}\ for\ all\ large\ \ \sigma_1,\sigma_2. \tag{3.4}$$

So, from (3.2), (3.3) and (3.4),

$$F_2(\sigma_{1n},\sigma_{2n}) > 2F_1(\sigma_{1n},\sigma_{2n}) \text{ for } n=1,2,3,\dots\ \ .$$

Therefore,

$$\begin{aligned} F(\sigma_{1n},\sigma_{2n}) &\ge F_2(\sigma_{1n},\sigma_{2n}) - F_1(\sigma_{1n},\sigma_{2n}) > \frac{1}{2}F_2(\sigma_{1n},\sigma_{2n}) \\ &> \frac{1}{2}\exp[G(\sigma_{1n},\sigma_{2n})]^{\lambda} > \exp[G(\sigma_{1n},\sigma_{2n})]^{\mu_1}. \end{aligned}$$

by (3.3). Therefore,

$$\rho \ge \rho_2. \tag{3.5}$$

So, from (3.1) and (3.5) we get $\rho=\rho_2$ and this proves the theorem. □

Remark 3.2. For Banach valued Dirichlet series to hold the equality, the condition $\rho_g(f_1)\neq\rho_g(f_2)$ is not necessary. Because if we take $f_1(s_1,s_2)=2e^{s_1+s_2}$, $f_2(s_1,s_2)=-e^{s_1+s_2}$ and $g(s_1,s_2)=e^{s_1+s_2}$ then clearly $F_1(\sigma_1,\sigma_2)=2e^{\sigma_1+\sigma_2}, F_2(\sigma_1,\sigma_2)=e^{\sigma_1+\sigma_2}$ and

$$G(\sigma_1,\sigma_2)=e^{\sigma_1+\sigma_2}$$

Therefore

$$\rho_g(f_1)=\rho_g(f_2)=0.$$

On the other hand
$$f_1 + f_2 = e^{s_1+s_2}.$$
Therefore
$$\rho_g(f_1 + f_2) = 0.$$
Thus,
$$\rho_g(f_1 + f_2) = \max\{\rho_g(f_1), \rho_g(f_2)\}.$$

Also if we take $f_1(s_1, s_2) = 2e^{s_1+s_2}$, $f_2(s_1, s_2) = g(s_1, s_2) = e^{s_1+s_2}$. Then clearly $\rho_g(f_1 - f_2) = \max\{\rho_g(f_1), \rho_g(f_2)\}$.

Theorem 3.3. *Let $f_1(s_1, s_2)$ and $g(s_1, s_2)$ be two entire functions defined by the Banach valued Dirichlet series* (1.2) *and $f_2(s_1, s_2)$ be an entire function defined by* (1.2) *where the coefficients $a_{mn} \in \mathbb{C}$. Then $\rho_g(f_1.f_2) \leq \max\{\rho_g(f_1), \rho_g(f_2)\}$.*

Proof . Let $f = f_1.f_2$ and the notations ρ, ρ_1, ρ_2 have the analogous meanings as in Theorem 3.1. Without loss of generality let ρ_1 and ρ_2 both are finite and $\rho_1 \leq \rho_2$. Then for arbitrary $\epsilon > 0$ and for all large σ_1, σ_2
$$\begin{aligned} F(\sigma_1, \sigma_2) &\leq F_1(\sigma_1, \sigma_2)F_2(\sigma_1, \sigma_2) \\ &< \exp[G(\sigma_1, \sigma_2)]^{\rho_1+\epsilon} \exp[G(\sigma_1, \sigma_2)]^{\rho_2+\epsilon} \\ &\leq \exp\{2[G(\sigma_1, \sigma_2)]^{\rho_2+\epsilon}\} \\ &\leq \exp[G(\sigma_1, \sigma_2)]^{\rho_2+2\epsilon}. \end{aligned}$$
Therefore,
$$\log\log F(\sigma_1, \sigma_2) < (\rho_2 + 2\epsilon) \log G(\sigma_1, \sigma_2) \text{ for all large } \sigma_1, \sigma_2.$$
Since $\epsilon > 0$ is arbitrary so $\rho \leq \rho_2$, which proves the theorem. □

Remark 3.4. For Banach valued Dirichlet series the equality may hold. For example, suppose $f_1(s_1, s_2) = f_2(s_1, s_2) = g(s_1, s_2) = e^{s_1+s_2}$. Then clearly $\rho_g(f_1) = \rho_g(f_2) = 0$. On the other hand $f_1.f_2 = e^{2(s_1+s_2)}$. Therefore, $\rho_g(f_1.f_2) = 0$. Thus
$$\rho_g(f_1.f_2) = \max\{\rho_g(f_1), \rho_g(f_2)\}.$$

Theorem 3.5. *Let $f_1(s_1, s_2)$, $f_2(s_1, s_2)$ and $g(s_1, s_2)$ be three entire functions defined by the Banach valued Dirichlet series* (1.2) *such that $T_g(f_1)$, $T_g(f_2)$ and $T_g(f_1 \pm f_2)$ are defined. Then $T_g(f_1 \pm f_2) \leq \max\{T_g(f_1), T_g(f_2)\}$, the equality holds if $T_g(f_1) \neq T_g(f_2)$.*

Proof . We may suppose that $T_g(f_1)$ and $T_g(f_2)$ both are finite because in the contrary case the inequality follows immediately. We prove the theorem for addition only, because the proof for subtraction is analogous.

Let $f = f_1 + f_2$, $T = T_g(f)$, $T_i = T_g(f_i)$ and $\rho = \rho(f_i) = \rho(f) = \rho(g)$, $i = 1, 2$ and suppose that $T_1 \leq T_2$. For arbitrary $\epsilon > 0$ and for all large σ_1, σ_2 we have by definition,
$$\log F_1(\sigma_1, \sigma_2) < (T_1 + \epsilon)G(\rho\sigma_1, \rho\sigma_2)$$
or
$$F_1(\sigma_1, \sigma_2) < \exp[(T_1 + \epsilon)G(\rho\sigma_1, \rho\sigma_2)] \leq \exp[(T_2 + \epsilon)G(\rho\sigma_1, \rho\sigma_2)]$$

and

$$F_2(\sigma_1, \sigma_2) < \exp[(T_2 + \epsilon)G(\rho\sigma_1, \rho\sigma_2)].$$

So, for all large σ_1, σ_2,

$$F(\sigma_1, \sigma_2) \leq F_1(\sigma_1, \sigma_2) + F_2(\sigma_1, \sigma_2) < 2\exp[(T_2 + \epsilon)G(\rho\sigma_1, \rho\sigma_2)].$$

Therefore,

$$\log F(\sigma_1, \sigma_2) < \log 2 + (T_2 + \epsilon)G(\rho\sigma_1, \rho\sigma_2)$$

or

$$\frac{\log F(\sigma_1, \sigma_2)}{G(\rho\sigma_1, \rho\sigma_2)} < (T_2 + \epsilon) + o(1).$$

Since $\epsilon > 0$ is arbitrary so

$$T \leq T_2. \tag{3.6}$$

This proves the first part of the theorem.

For the second part, let $T_1 < T_2$ and suppose that $T_1 < \mu < \lambda < T_2$. Then for all large σ_1, σ_2

$$F_1(\sigma_1, \sigma_2) < \exp[\mu G(\rho\sigma_1, \rho\sigma_2)] \tag{3.7}$$

and there exist non decreasing sequences $\{\sigma_{1n}\}$, $\sigma_{1n} \to \infty$, and $\{\sigma_{2n}\}$, $\sigma_{2n} \to \infty$ as $n \to \infty$ such that

$$F_2(\sigma_{1n}, \sigma_{2n}) > \exp[\lambda G(\rho\sigma_{1n}, \rho\sigma_{2n})]. \tag{3.8}$$

Now, by using (3.7) and (3.8).

$$\begin{aligned} F(\sigma_{1n}, \sigma_{2n}) &\geq F_2(\sigma_{1n}, \sigma_{2n}) - F_1(\sigma_{1n}, \sigma_{2n}) \\ &> \exp[\lambda G(\rho\sigma_{1n}, \rho\sigma_{2n})] - \exp[\mu G(\rho\sigma_{1n}, \rho\sigma_{2n})] \\ &> 2\exp[\mu G(\rho\sigma_{1n}, \rho\sigma_{2n})] - \exp[\mu G(\rho\sigma_{1n}, \rho\sigma_{2n})] \\ &> \exp[\mu G(\rho\sigma_{1n}, \rho\sigma_{2n})] \end{aligned}$$

or

$$\log F(\sigma_{1n}, \sigma_{2n}) > \mu G(\rho\sigma_{1n}, \rho\sigma_{2n})$$

or

$$\frac{\log F(\sigma_{1n}, \sigma_{2n})}{G(\rho\sigma_{1n}, \rho\sigma_{2n})} > \mu.$$

Therefore,

$$T \geq T_2. \tag{3.9}$$

From (3.6) and (3.9) we get $T = T_2$ and this proves the theorem. □

Theorem 3.6. *Let $f_1(s_1, s_2)$, $g(s_1, s_2)$ be two entire functions defined by the Banach valued Dirichlet series* (1.2) *and $f_2(s_1, s_2)$ be an entire function defined by (1.2) where the coefficients $a_{mn} \in \mathbb{C}$ such that $T_g(f_1)$, $T_g(f_2)$ and $T_g(f_1.f_2)$ are defined. Then $T_g(f_1.f_2) \leq T_g(f_1) + T_g(f_2)$.*

Proof . Let $f = f_1.f_2$ and the notations ρ, T, T_1 and T_2 have the analogous meanings as in Theorem 3.5. Suppose T_1 and T_2 both are finite because in the contrary case the theorem is obvious. Then for arbitrary $\epsilon > 0$ and for all large σ_1, σ_2

$$\begin{aligned} F(\sigma_1, \sigma_2) &\leq F_1(\sigma_1, \sigma_2)F_2(\sigma_1, \sigma_2) \\ &< \exp[(T_1 + \epsilon)G(\rho\sigma_1, \rho\sigma_2)]\exp[(T_2 + \epsilon)G(\rho\sigma_1, \rho\sigma_2)] \\ &= \exp[(T_1 + T_2 + 2\epsilon)G(\rho\sigma_1, \rho\sigma_2)]. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary we obtain $T \leq T_1 + T_2$ and this proves the theorem. □

4. Relative order and type of the partial derivatives

Theorem 4.1. *Let $f(s_1, s_2)$ and $g(s_1, s_2)$ be two entire functions defined by the Banach valued Dirichlet series* (1.2). *Then $\rho_g(f) = \rho_g(\frac{\partial f}{\partial s_1})$.*

Proof . We write,

$$\bar{F}_{s_1}(\sigma_1, \sigma_2) = \sup\{\|\frac{\partial f(\sigma_1 + it_1, \sigma_2 + it_2)}{\partial(\sigma_1 + it_1)}\|; -\infty < t_j < \infty; j = 1, 2\}.$$

From [[11], p.68], we may write for fixed σ_2 and for all large values of σ_1

$$F(\sigma_1, \sigma_2) < \sigma_1 \bar{F}_{s_1}(\sigma_1, \sigma_2) + O(1)$$

or

$$\log F(\sigma_1, \sigma_2) < \log \bar{F}_{s_1}(\sigma_1, \sigma_2) + \log \sigma_1 + O(1). \tag{4.1}$$

Therefore,

$$\frac{\log \log F(\sigma_1, \sigma_2)}{\log G(\sigma_1, \sigma_2)} < \frac{\log \log \bar{F}_{s_1}(\sigma_1, \sigma_2)}{\log G(\sigma_1, \sigma_2)} + o(1)$$

for fixed σ_2 and for all large σ_1, and so,

$$\rho_g(f) \leq \rho_g\left(\frac{\partial f}{\partial s_1}\right). \tag{4.2}$$

To obtain the reverse inequality we have from [[11], p.68] for fixed σ_2 and for large σ_1

$$\bar{F}_{s_1}(\sigma_1, \sigma_2) - \epsilon \leq \frac{1}{\delta} F(\sigma_1 + \delta, \sigma_2),$$

where $\epsilon > 0$ is arbitrary and $\delta > 0$ is fixed. So,

$$\log \bar{F}_{s_1}(\sigma_1, \sigma_2) \leq \log F(\sigma_1 + \delta, \sigma_2) + O(1) \tag{4.3}$$

or

$$\frac{\log \log \bar{F}_{s_1}(\sigma_1, \sigma_2)}{\log G(\sigma_1, \sigma_2)} \leq \frac{\log \log F(\sigma_1 + \delta, \sigma_2)}{\log G(\sigma_1, \sigma_2)} + o(1).$$

Since σ_2 is any fixed real number, σ_1 is large and δ is any fixed number so,

$$\rho_g(\frac{\partial f}{\partial s_1}) \leq \rho_g(f). \tag{4.4}$$

From (4.2) and (4.4) we get

$$\rho_g(f) = \rho_g\left(\frac{\partial f}{\partial s_1}\right).$$

□

Remark 4.2. In Theorem 4.1 putting $g(s_1, s_2) = e^{s_1} + e^{s_2}$, we get $\rho(f) = \rho(\frac{\partial f}{\partial s_1})$.

Theorem 4.3. *Let $f(s_1, s_2)$ and $g(s_1, s_2)$ be two entire functions defined by the Banach valued Dirichlet series* (1.2) *such that $\rho(f) = \rho(g)$. Then $T_g(f) = T_g(\frac{\partial f}{\partial s_1})$.*

Proof . From Remark 4.2, we have, $\rho(f) = \rho(\frac{\partial f}{\partial s_1})$. Therefore, $\rho(\frac{\partial f}{\partial s_1}) = \rho(g)$ and so $T_g(\frac{\partial f}{\partial s_1})$ exists. Suppose $\rho(f) = \rho(g) = \rho(\frac{\partial f}{\partial s_1}) = \rho$. As in Theorem 4.1 we write

$$\bar{F}_{s_1}(\sigma_1, \sigma_2) = \sup\left\{\left\|\frac{\partial f(\sigma_1 + it_1, \sigma_2 + it_2)}{\partial(\sigma_1 + it_1)}\right\|; -\infty < t_j < \infty; j = 1, 2\right\}.$$

Then from (4.1) we get

$$\log F(\sigma_1, \sigma_2) < \log \bar{F}_{s_1}(\sigma_1, \sigma_2) + \log \sigma_1 + O(1)$$

or

$$\frac{\log F(\sigma_1, \sigma_2)}{G(\rho\sigma_1, \rho\sigma_2)} < \frac{\log \bar{F}_{s_1}(\sigma_1, \sigma_2)}{G(\rho\sigma_1, \rho\sigma_2)} + o(1).$$

So taking $\sigma_1, \sigma_2 \to \infty$ we get

$$T_g(f) \leq T_g(\frac{\partial f}{\partial s_1}). \tag{4.5}$$

Again for a fixed σ_2 and large σ_1 we get from (4.3) for a fixed $\delta > 0$

$$\frac{\log \bar{F}_{s_1}(\sigma_1, \sigma_2)}{G(\rho\sigma_1, \rho\sigma_2)} \leq \frac{\log F(\sigma_1 + \delta, \sigma_2)}{G(\rho\sigma_1, \rho\sigma_2)} + o(1).$$

Since σ_2 is any fixed real number, σ_1 is large and δ is any fixed number so,

$$T_g(\frac{\partial f}{\partial s_1}) \leq T_g(f). \tag{4.6}$$

From (4.5) and (4.6) we get $T_g(f) = T_g(\frac{\partial f}{\partial s_1})$. $\Box$

5. Asymptotic behavior

Theorem 5.1. *Let $f(s_1, s_2)$, $g_1(s_1, s_2)$ and $g_2(s_1, s_2)$ be three entire functions defined by the Banach valued Dirichlet series* (1.2) *and suppose $g_1 \sim g_2$. Then $\rho_{g_1}(f) = \rho_{g_2}(f)$.*

Proof . Let $\epsilon > 0$. Then by definition, for all large σ_1, σ_2, there exists l $(0 < l < \infty)$ such that

$$G_1(\sigma_1, \sigma_2) < (l + \epsilon)G_2(\sigma_1, \sigma_2). \tag{5.1}$$

Now for all large σ_1, σ_2

$$\log\log F(\sigma_1, \sigma_2) < (\rho_{g_1}(f) + \epsilon)\log G_1(\sigma_1, \sigma_2)$$

or using (5.1),

$$\begin{aligned} F(\sigma_1, \sigma_2) &< \exp[G_1(\sigma_1, \sigma_2)]^{\rho_{g_1}(f)+\epsilon} \\ &< \exp[(l + \epsilon)G_2(\sigma_1, \sigma_2)]^{\rho_{g_1}(f)+\epsilon} \\ &< \exp[G_2(\sigma_1, \sigma_2)]^{\rho_{g_1}(f)+2\epsilon}. \end{aligned}$$

Therefore,

$$\frac{\log\log F(\sigma_1, \sigma_2)}{\log G_2(\sigma_1, \sigma_2)} < \rho_{g_1}(f) + 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary small, so $\rho_{g_2}(f) \leq \rho_{g_1}(f)$. The reverse inequality is clear because $g_2 \sim g_1$ and so $\rho_{g_1}(f) = \rho_{g_2}(f)$. $\Box$

Remark 5.2. For Banach valued Dirichlet series the condition $g_1 \sim g_2$ is not necessary. For example, let $f(s_1, s_2) = g_1(s_1, s_2) = e^{s_1+s_2}$ and $g_2(s_1, s_2) = e^{2(s_1+s_2)}$. Then clearly $F(\sigma_1, \sigma_2) = G_1(\sigma_1, \sigma_2) = e^{\sigma_1+\sigma_2}$ and $G_2(\sigma_1, \sigma_2) = e^{2(\sigma_1+\sigma_2)}$. Therefore, $g_1 \sim g_2$ does not hold but $\rho_{g_1}(f) = \rho_{g_2}(f) = 0$.

Theorem 5.3. *Let $f_1(s_1, s_2)$, $f_2(s_1, s_2)$ and $g(s_1, s_2)$ be three entire functions defined by the Banach valued Dirichlet series* (1.2) *and suppose $f_1 \sim f_2$. Then $\rho_g(f_1) = \rho_g(f_2)$.*

Proof . Let $\epsilon > 0$. Then by definition, for all large σ_1, σ_2, there exists l, $(0 < l < \infty)$ such that

$$F_1(\sigma_1, \sigma_2) < (l + \epsilon)F_2(\sigma_1, \sigma_2).$$

Now for all large σ_1, σ_2

$$\log F_1(\sigma_1, \sigma_2) < \log F_2(\sigma_1, \sigma_2) + \log(l + \epsilon).$$

Therefore,

$$\frac{\log \log F_1(\sigma_1, \sigma_2)}{\log G(\sigma_1, \sigma_2)} < \frac{\log \log F_2(\sigma_1, \sigma_2)}{\log G(\sigma_1, \sigma_2)} + o(1)$$

i.e.

$$\rho_g(f_1) \leq \rho_g(f_2).$$

The reverse inequality is clear because $f_2 \sim f_1$ and so $\rho_g(f_1) = \rho_g(f_2)$. □

Remark 5.4. For Banach valued Dirichlet series the condition $f_1 \sim f_2$ is not necessary which follows from the following example.

Let $f_1(s_1, s_2) = g(s_1, s_2) = e^{s_1+s_2}$ and $f_2(s_1, s_2) = e^{2(s_1+s_2)}$. Then clearly $f_1 \sim f_2$ does not hold but $\rho_g(f_1) = \rho_g(f_2) = 0$.

Theorem 5.5. *Let $f_1(s_1, s_2)$, $f_2(s_1, s_2)$ and $g(s_1, s_2)$ be three entire functions defined by the Banach valued Dirichlet series* (1.2) *such that $T_g(f_1)$ and $T_g(f_2)$ are defined and suppose $f_1 \sim f_2$. Then*

$$T_g(f_1) = T_g(f_2).$$

Proof . Let $\epsilon > 0$ and $\rho(f_1) = \rho(f_2) = \rho(g) = \rho$. Then by definition, for all large σ_1, σ_2, there exists l $(0 < l < \infty)$ such that

$$F_1(\sigma_1, \sigma_2) < (l + \epsilon)F_2(\sigma_1, \sigma_2)$$

or

$$\log F_1(\sigma_1, \sigma_2) < \log F_2(\sigma_1, \sigma_2) + O(1)$$

or

$$\frac{\log F_1(\sigma_1, \sigma_2)}{G(\rho\sigma_1, \rho\sigma_2)} < \frac{\log F_2(\sigma_1, \sigma_2)}{G(\rho\sigma_1, \rho\sigma_2)} + o(1).$$

Since $\epsilon > 0$ is arbitrary small, so $T_g(f_1) \leq T_g(f_2)$. The reverse inequality is clear because $f_2 \sim f_1$ and so $T_g(f_1) = T_g(f_2)$. □

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