



New Hermite-Hadamard type inequalities on fractal set

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Abstract

In this study, we present the new Hermite-Hadamard type inequality for functions which are h -convex on fractal set \mathbb{R}^α ($0 < \alpha \leq 1$) of real line numbers. Then we provide the special cases of the result using different type of convex mappings.

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1. Introduction

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. If f is a convex function then the following double inequality holds [3]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \quad (1.1)$$

The above inequality (1.1) which is well known in the literature as the Hermite-Hadamard inequality, is the most fundamental and interesting inequality for classical convex functions. This inequality provides a lower and an upper estimations for the integral average of any convex functions defined on a compact interval. For numerous interesting results which generalize, improve and extend the classical Hermite-Hadamard inequality see for instance [3], [10] and references therein.

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2. The preliminaries

The concepts of fractional calculus [6] and local fractional calculus (also called fractal calculus) (see, for details, [18], [19] and [20]) are becoming increasingly useful in a wide variety of problems in mathematical, physical and engineering sciences (see, for example, [21] to [24]). We need the following notations and preliminaries to define the local fractional derivative and the local fractional integral.

Recall the set \mathbb{R}^α of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, (see [18], [19], [20]) and so on. Recently, the theory of Yang's fractional sets [19] was introduced as follows:

For $0 < \alpha \leq 1$, we have the following α -type set of element sets:

\mathbb{Z}^α : The α -type set of integer is defined as the set $\{0^\alpha, \pm 1^\alpha, \pm 2^\alpha, \dots, \pm n^\alpha, \dots\}$.

\mathbb{Q}^α : The α -type set of the rational numbers is defined as the set $\{m^\alpha = \left(\frac{p}{q}\right)^\alpha : p, q \in \mathbb{Z}, q \neq 0\}$.

\mathbb{J}^α : The α -type set of the irrational numbers is defined as the set $\{m^\alpha \neq \left(\frac{p}{q}\right)^\alpha : p, q \in \mathbb{Z}, q \neq 0\}$.

\mathbb{R}^α : The α -type set of the real line numbers is defined as the set $\mathbb{R}^\alpha = \mathbb{Q}^\alpha \cup \mathbb{J}^\alpha$.

If a^α, b^α and c^α belongs the set \mathbb{R}^α of real line numbers, then

- (1) $a^\alpha + b^\alpha$ and $a^\alpha b^\alpha$ belongs the set \mathbb{R}^α ;
- (2) $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a + b)^\alpha = (b + a)^\alpha$;
- (3) $a^\alpha + (b^\alpha + c^\alpha) = (a + b)^\alpha + c^\alpha$;
- (4) $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$;
- (5) $a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha) c^\alpha$;
- (6) $a^\alpha (b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$;
- (7) $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ and $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$.

The definition of the local fractional derivative and local fractional integral can be given as follows:

Definition 2.1. (Yang [19]) A non-differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}^\alpha$, $x \rightarrow f(x)$ is called to be local fractional continuous at x_0 , if for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$|f(x) - f(x_0)| < \varepsilon^\alpha$$

holds for $|x - x_0| < \delta$, where $\varepsilon, \delta \in \mathbb{R}$. If $f(x)$ is local continuous on the interval (a, b) , we denote $f(x) \in C_\alpha(a, b)$.

Definition 2.2. (Yang [19]) The local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined by

$$f^{(\alpha)}(x_0) = \left. \frac{d^\alpha f(x)}{dx^\alpha} \right|_{x=x_0} = \lim_{x \rightarrow x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha},$$

where $\Delta^\alpha (f(x) - f(x_0)) \cong \Gamma(1 + \alpha) (f(x) - f(x_0))$. If there exists $f^{(k+1)\alpha}(x) = \overbrace{D_x^\alpha \dots D_x^\alpha}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq \mathbb{R}$, then we denoted $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, \dots$

Definition 2.3. (Yang [19]) Let $f(x) \in C_\alpha [a, b]$. Then the local fractional integral is defined by,

$${}_a I_b^\alpha f(x) = \frac{1}{\Gamma(1 + \alpha)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(1 + \alpha)} \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha,$$

with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max \{ \Delta t_1, \Delta t_2, \dots, \Delta t_{N-1} \}$, where $[t_j, t_{j+1}]$, $j = 0, \dots, N - 1$ and $a = t_0 < t_1 < \dots < t_{N-1} < t_N = b$ is partition of interval $[a, b]$.

Here, it follows that ${}_a I_b^\alpha f(x) = 0$ if $a = b$ and ${}_a I_b^\alpha f(x) = -{}_b I_a^\alpha f(x)$ if $a < b$. If for any $x \in [a, b]$, there exists ${}_a I_x^\alpha f(x)$, then we denoted by $f(x) \in I_x^\alpha [a, b]$.

Lemma 2.4. (Yang [19])

(i) (Local fractional integration is anti-differentiation) Suppose that $f(x) = g^{(\alpha)}(x) \in C_\alpha [a, b]$, then we have

$${}_a I_b^\alpha f(x) = g(b) - g(a).$$

(ii) (Local fractional integration by parts) Suppose that $f(x), g(x) \in D_\alpha [a, b]$ and $f^{(\alpha)}(x), g^{(\alpha)}(x) \in C_\alpha [a, b]$, then we have

$${}_a I_b^\alpha f(x)g^{(\alpha)}(x) = f(x)g(x)|_a^b - {}_a I_b^\alpha f^{(\alpha)}(x)g(x).$$

Lemma 2.5. (Yang [19])

$$(i) \frac{d^\alpha x^{k\alpha}}{dx^\alpha} = \frac{\Gamma(1 + k\alpha)}{\Gamma(1 + (k - 1)\alpha)} x^{(k-1)\alpha};$$

$$(ii) \frac{1}{\Gamma(1 + \alpha)} \int_a^b x^{k\alpha}(dx)^\alpha = \frac{\Gamma(1 + k\alpha)}{\Gamma(1 + (k + 1)\alpha)} (b^{(k+1)\alpha} - a^{(k+1)\alpha}), k \in \mathbb{R}.$$

Now, we give some definitions which are used in our results:

Definition 2.6. (Mo, Sui, Yu [7]) Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}^\alpha$. For any $x_1, x_2 \in I$ and $\lambda \in [0, 1]$, if the following inequality

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda^\alpha f(x_1) + (1 - \lambda)^\alpha f(x_2)$$

holds, then f is called a generalized convex function on I . If this inequality reversed, then f is called a generalized concave function.

Here are two basic examples of generalized convex functions:

$$(i) f(x) = x^{\alpha p}, x \geq 0, p > 1;$$

$$(ii) f(x) = E_\alpha(x^\alpha), x \in \mathbb{R} \text{ where } E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)} \text{ is the Mittag-Leffer function.}$$

In [7], Mo et al. proved the following generalized Hermite-Hadamard inequality for generalized convex function:

Theorem 2.7. Let $f(x) \in I_x^\alpha [a, b]$ be generalized convex function on $[a, b]$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1 + \alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(x) \leq \frac{f(a) + f(b)}{2^\alpha}.$$

In [17], the definition of h -convex functions on fractal sets was established by Vivas et al., as follows:

Definition 2.8. Let $h : J \rightarrow \mathbb{R}^\alpha$ be a non-negative function and $h \neq 0$, defined over an interval $J \subset \mathbb{R}$ and such that $(0, 1) \subset J$. We say that $f : I \rightarrow \mathbb{R}^\alpha$ defined over an interval $I \subset \mathbb{R}$, is h -convex if f is non-negative and we have

$$f(tx_1 + (1 - t)x_2) \leq h(t)f(x_1) + h(1 - t)f(x_2)$$

for all $t \in (0, 1)$ and $x_1, x_2 \in I$.

Example 2.9. Let $0 < s < 1$, $h : (0, 1) \rightarrow \mathbb{R}^\alpha$ defined as $h(t) = t^{s\alpha}$ and $a^\alpha, b^\alpha, c^\alpha \in \mathbb{R}^\alpha$. For $x \in \mathbb{R}_+ = [0, \infty)$, define

$$f(x) = \begin{cases} a^\alpha, & x = 0 \\ b^\alpha x^{s\alpha} + c^\alpha, & x > 0 \end{cases}$$

In [8], Mo and Sui introduced the definitions of two kinds of generalized s -convex functions on fractal sets such as follows:

Definition 2.10. (i) Let $\mathbb{R}_+ = [0, \infty)$. A function $f : \mathbb{R}_+ \rightarrow \mathbb{R}^\alpha$ is said to be generalized s -convex ($0 < s < 1$) in the first sense, if

$$f(\lambda_1 u + \lambda_2 v) \leq \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(v),$$

for all $u, v \in \mathbb{R}_+$ and all $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1^s + \lambda_2^s = 1$. One denotes by $f \in GK_s^1$.

(ii) A function $f : \mathbb{R}_+ \rightarrow \mathbb{R}^\alpha$ is said to be generalized s -convex ($0 < s < 1$) in the second sense, if

$$f(\lambda_1 u + \lambda_2 v) \leq \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(v),$$

for all $u, v \in \mathbb{R}_+$ and all $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$. One denotes by $f \in GK_s^2$.

Note that, if $s = 1$ in Definition 2.10, then we have the generalized convex function.

For more information and recent developments on local fractional theory, please refer to [1],[2], [4]-[9], [11]-[20], [22], [23].

The main goal of this article is to establish new Hermite-Hadamard type inequalities for h -convex.

3. The main results

We start with the following important theorem for our work.

Theorem 3.1. Let $h : [0, 1] \rightarrow \mathbb{R}^\alpha$ be a non-negative function and $f : I \rightarrow \mathbb{R}^\alpha$ be a h -convex function such that $h\left(\frac{1}{2}\right) \neq 0^\alpha$ and ${}_0I_1^\alpha h(t) \geq \left(\frac{1}{2}\right)^\alpha$, then

$$\begin{aligned} \frac{1}{2^{2\alpha} \left[h\left(\frac{1}{2}\right)\right]^2} f\left(\frac{a+b}{2}\right) &\leq \Delta_1 \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_aI_b^\alpha f(x) \\ &\leq \Delta_2 \leq \Gamma(1+\alpha) \left[[f(a) + f(b)] \left\{ h\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^\alpha \right\} \right] {}_0I_1^\alpha h(t), \end{aligned}$$

where

$$\Delta_1 = \frac{1}{2^{2\alpha} h\left(\frac{1}{2}\right)} \left[f\left(\frac{a+3b}{4}\right) + f\left(\frac{3a+b}{4}\right) \right]$$

and

$$\Delta_2 = \Gamma(1+\alpha) \left[\frac{f(a)+f(b)}{2^\alpha} + f\left(\frac{a+b}{2}\right) \right] {}_0I_1^\alpha h(t).$$

Proof . Firstly, we divide interval $[a, b]$ into $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$. Since f function is h -convex function, for $\left[a, \frac{a+b}{2}\right]$ we have

$$\begin{aligned} f\left(\frac{a+\frac{a+b}{2}}{2}\right) &= f\left(\frac{ta+(1-t)\frac{a+b}{2}+(1-t)a+t\frac{a+b}{2}}{2}\right) \\ &\leq h\left(\frac{1}{2}\right) \left[f\left(ta+(1-t)\frac{a+b}{2}\right) + f\left((1-t)a+t\frac{a+b}{2}\right) \right]. \end{aligned}$$

Integrating both sides of above inequality with respect to t on $[0, 1]$, we obtain

$$\frac{1}{2^{2\alpha} h\left(\frac{1}{2}\right)} f\left(\frac{3a+b}{4}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_aI_{\frac{a+b}{2}}^\alpha f(x). \quad (3.1)$$

Similarly, for $\left[\frac{a+b}{2}, b\right]$ we have

$$\begin{aligned} f\left(\frac{\frac{a+b}{2}+b}{2}\right) &= f\left(\frac{t\frac{a+b}{2}+(1-t)b+(1-t)\frac{a+b}{2}+tb}{2}\right) \\ &\leq h\left(\frac{1}{2}\right) \left[f\left(t\frac{a+b}{2}+(1-t)b\right) + f\left((1-t)\frac{a+b}{2}+tb\right) \right]. \end{aligned}$$

Integrating both sides of above inequality with respect to t on $[0, 1]$, we obtain

$$\frac{1}{2^{2\alpha} h\left(\frac{1}{2}\right)} f\left(\frac{a+3b}{4}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_{\frac{a+b}{2}}I_b^\alpha f(x). \quad (3.2)$$

By adding inequalities (3.1) and (3.2), it yields

$$\begin{aligned} \Delta_1 &= \frac{1}{2^{2\alpha} h\left(\frac{1}{2}\right)} \left[f\left(\frac{a+3b}{4}\right) + f\left(\frac{3a+b}{4}\right) \right] \\ &\leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_aI_b^\alpha f(x) \\ &= \frac{\Gamma(1+\alpha)}{2^\alpha} \left[\frac{2^\alpha}{(b-a)^\alpha} {}_aI_{\frac{a+b}{2}}^\alpha f(x) + \frac{2^\alpha}{(b-a)^\alpha} {}_{\frac{a+b}{2}}I_b^\alpha f(x) \right] \\ &\leq \frac{\Gamma(1+\alpha)}{2^\alpha} \left[\left\{ f(a) + f\left(\frac{a+b}{2}\right) \right\} {}_0I_1^\alpha h(t) \right] + \frac{\Gamma(1+\alpha)}{2^\alpha} \left[\left\{ f\left(\frac{a+b}{2}\right) + f(b) \right\} {}_0I_1^\alpha h(t) \right] \\ &= \frac{\Gamma(1+\alpha)}{2^\alpha} \left[f(a) + f(b) + 2^\alpha f\left(\frac{a+b}{2}\right) \right] {}_0I_1^\alpha h(t) = \Delta_2. \end{aligned}$$

On the other hand, since f is h -convex function and $\Gamma(1 + \alpha) {}_0I_1^\alpha h(t) \geq (\frac{1}{2})^\alpha$, we deduce that

$$\begin{aligned} \frac{1}{2^{2\alpha} [h(\frac{1}{2})]^2} f\left(\frac{a+b}{2}\right) &= \frac{1}{2^{2\alpha} [h(\frac{1}{2})]^2} f\left(\frac{1}{2} \frac{3a+b}{4} + \frac{1}{2} \frac{a+3b}{4}\right) \\ &\leq \frac{1}{2^{2\alpha} [h(\frac{1}{2})]^2} \left[h\left(\frac{1}{2}\right) \left\{ f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right\} \right] \\ &= \frac{1}{2^{2\alpha} [h(\frac{1}{2})]} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] = \Delta_1 \\ &\leq \frac{1}{2^{2\alpha} [h(\frac{1}{2})]} \left[h\left(\frac{1}{2}\right) \left\{ f(a) + f(b) + 2^\alpha f\left(\frac{a+b}{2}\right) \right\} \right] \\ &= \left(\frac{1}{2}\right)^\alpha \left[\frac{f(a) + f(b)}{2^\alpha} + f\left(\frac{a+b}{2}\right) \right] \\ &\leq \Gamma(1 + \alpha) \left[\frac{f(a) + f(b)}{2^\alpha} + f\left(\frac{a+b}{2}\right) \right] {}_0I_1^\alpha h(t) = \Delta_2 \\ &\leq \Gamma(1 + \alpha) \left[\frac{f(a) + f(b)}{2^\alpha} + h\left(\frac{1}{2}\right) [f(a) + f(b)] \right] {}_0I_1^\alpha h(t) \\ &= \Gamma(1 + \alpha) \left[[f(a) + f(b)] \left\{ h\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^\alpha \right\} \right] {}_0I_1^\alpha h(t). \end{aligned}$$

This completes the proof. \square

Corollary 3.2. *If we choose $h(t) = t^\alpha$ in Theorem 3.1, we obtain*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \Delta_1 \leq \frac{\Gamma(1 + \alpha)}{(b-a)^\alpha} {}_aI_b^\alpha f(x) \\ &\leq \Delta_2 \leq [f(a) + f(b)] \frac{[\Gamma(1 + \alpha)]^2}{\Gamma(1 + 2\alpha)}, \end{aligned}$$

where

$$\Delta_1 = \frac{1}{2^\alpha} \left[f\left(\frac{a+3b}{4}\right) + f\left(\frac{3a+b}{4}\right) \right]$$

and

$$\Delta_2 = \left[\frac{f(a) + f(b)}{2^\alpha} + f\left(\frac{a+b}{2}\right) \right] \frac{[\Gamma(1 + \alpha)]^2}{\Gamma(1 + 2\alpha)}.$$

Corollary 3.3. *Let $f : I \rightarrow \mathbb{R}^\alpha$ be a generalized s -convex function in the second sense where $s \in (0, 1]$ such that $\Gamma(1 + \alpha) {}_0I_1^{\alpha t^{s\alpha}} \geq (\frac{1}{2})^\alpha$, then*

$$\begin{aligned} 2^{(2s-2)\alpha} f\left(\frac{a+b}{2}\right) &\leq \Delta_1 \leq \frac{\Gamma(1 + \alpha)}{(b-a)^\alpha} {}_aI_b^\alpha f(x) \\ &\leq \Delta_2 \leq \left[[f(a) + f(b)] \left\{ \left(\frac{1}{2^s}\right)^\alpha + \left(\frac{1}{2}\right)^\alpha \right\} \right] \frac{\Gamma(1 + s\alpha)\Gamma(1 + \alpha)}{\Gamma(1 + (s+1)\alpha)}, \end{aligned}$$

where

$$\Delta_1 = 2^{(s-2)\alpha} \left[f\left(\frac{a+3b}{4}\right) + f\left(\frac{3a+b}{4}\right) \right]$$

and

$$\Delta_2 = \left[\frac{f(a) + f(b)}{2^\alpha} + f\left(\frac{a+b}{2}\right) \right] \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}.$$

Definition 3.4. A function $f : I \rightarrow \mathbb{R}^\alpha$ is said to be generalized P -convex function, if f is non-negative and for all $x, y \in I$ and $t \in [0, 1]$, we have

$$f(tx + (1-t)y) \leq f(x) + f(y). \quad (3.3)$$

Corollary 3.5. Let $f : I \rightarrow \mathbb{R}^\alpha$ be a generalized P -convex function, then

$$\begin{aligned} \frac{1}{2^{2\alpha}} f\left(\frac{a+b}{2}\right) &\leq \Delta_1 \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(x) \\ &\leq \Delta_2 \leq \left(\frac{3}{2}\right)^\alpha [f(a) + f(b)], \end{aligned}$$

where

$$\Delta_1 = \frac{1}{2^{2\alpha}} \left[f\left(\frac{a+3b}{4}\right) + f\left(\frac{3a+b}{4}\right) \right]$$

and

$$\Delta_2 = \left[\frac{f(a) + f(b)}{2^\alpha} + f\left(\frac{a+b}{2}\right) \right].$$

Remark 3.6. If we choose $\alpha = 1$ in the above results, then we obtain the inequalities given by Noor et al. in [9].

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