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# Local higher derivations on $C^*$-algebras are higher derivations

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**Abstract**

Let $\mathfrak{A}$ be a Banach algebra. We say that a sequence $\{D_n\}_{n=0}^{\infty}$ of continuous operators form $\mathfrak{A}$ into $\mathfrak{A}$ is a *local higher derivation* if to each $a \in \mathfrak{A}$ there corresponds a continuous higher derivation $\{d_{a,n}\}_{n=0}^{\infty}$ such that $D_n(a) = d_{a,n}(a)$ for each non-negative integer $n$. We show that if $\mathfrak{A}$ is a $C^*$-algebra then each local higher derivation on $\mathfrak{A}$ is a higher derivation. We also prove that each local higher derivation on a $C^*$-algebra is automatically continuous.

*Keywords:* Higher derivation; local higher derivation; derivation; local derivation.
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## 1. Introduction and preliminaries

Let $\mathfrak{A}$ be a Banach algebra. A continuous operator $\Delta : \mathfrak{A} \to \mathfrak{A}$ is called a *local derivation* if for each $a \in \mathfrak{A}$ there is a derivation $\delta_a : \mathfrak{A} \to \mathfrak{A}$ such that $\Delta(a) = \delta_a(a)$. A celebrated theorem of Johnson [8] states that each local derivation on a $C^*$-algebra is a derivation. Taking idea from this concept, we introduce the notion of a *local higher derivation* and show that each local higher derivation on a $C^*$-algebra is indeed a higher derivation.

Though there is a continuity assumption in the definition of a local derivation, Johnson shows that we can omit this assumption when $\mathfrak{A}$ is a $C^*$-algebra. Similarly, we show that when the domain of a local higher derivation is a $C^*$-algebra, we can remove the continuity assumption from the definition of a local higher derivation and each local higher derivation on a $C^*$-algebra is automatically continuous even if not assumed a priori to be so.

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For a discussion about automatic continuity of derivations and the related subjects, the reader is referred to [3, 4, 13, 9, 16] and [17]. Various works on derivations, higher derivations and their generalizations can be found in [1, 2, 5, 12, 10, 7, 6, 14] and [11].

**Proposition 1.1.** *Let $\{D_n\}_{n=0}^{\infty}$ be a local higher derivation from a Banach algebra $\mathfrak{A}$ into itself with $D_0 = I$. Then there is a sequence $\{\Delta_n\}_{n=1}^{\infty}$ of local derivations on $\mathfrak{A}$ such that*

$$(n+1)D_{n+1} = \sum_{k=0}^{n} \Delta_{k+1} D_{n-k}$$

*for each non-negative integer $n$.*

**Proof .** Let $a$ be an element of $\mathfrak{A}$. Since $\{D_n\}_{n=0}^{\infty}$ is a local higher derivation, there is a continuous higher derivation $\{d_{a,n}\}_{n=0}^{\infty}$ such that $D_n(a) = d_{a,n}(a)$ for each non-negative integer $n$.

We use induction on $n$. For $n = 0$ we have $D_1(a) = d_{a,1}(a) = d_{a,1}(D_0(a)) = d_{a,1}D_0(a)$. Thus if $\Delta_1 : \mathfrak{A} \to \mathfrak{A}$ is defined by $\Delta_1(a) = d_{a,1}$ for each $a \in \mathfrak{A}$, then $\Delta_1$ is a local derivation on $\mathfrak{A}$.

Now suppose that $\Delta_k$ is defined and is a local derivation for $k \leqslant n$. We can inductively assume that for each $a \in \mathfrak{A}$ and each $k \leqslant n$ there is a derivation $\delta_{a,k} : \mathfrak{A} \to \mathfrak{A}$, defined by $\delta_{a,k} = kd_{a,k} - \sum_{i=0}^{k-2} \delta_{a,i+1} d_{a,k-1-i}$, such that $\Delta_k(a) = \delta_{a,k}(a)$.

Putting $\Delta_{n+1} = (n+1)D_{n+1} - \sum_{k=0}^{n-1} \Delta_{k+1} D_{n-k}$, we show that the well-defined mapping $\Delta_{n+1}$ is a local derivation on $\mathfrak{A}$. To see this, suppose that $\delta_{a,n+1} = (n+1)d_{a,n+1} - \sum_{k=0}^{n-1} \delta_{a,k+1} d_{a,n-k}$. Clearly, $\Delta_{n+1}(a) = \delta_{a,n+1}(a)$. We show that $\delta_{a,n+1}$ is a derivation. For $x, y \in \mathfrak{A}$ we have

$$\begin{aligned}
\delta_{a,n+1}(xy) &= (n+1)d_{a,n+1}(xy) - \sum_{k=0}^{n-1} \delta_{a,k+1} d_{a,n-k}(xy) \\
&= (n+1)\sum_{k=0}^{n+1} d_{a,k}(x) d_{a,n+1-k}(y) - \sum_{k=0}^{n-1} \delta_{a,k+1} \left( \sum_{\ell=0}^{n-k} d_{a,\ell}(x) d_{a,n-k-\ell}(y) \right).
\end{aligned}$$

Now we have

$$\begin{aligned}
\delta_{a,n+1}(xy) &= \sum_{k=0}^{n+1} (n+1) d_{a,k}(x) d_{a,n+1-k}(y) - \sum_{k=0}^{n-1} \delta_{a,k+1} \left( \sum_{\ell=0}^{n-k} d_{a,\ell}(x) d_{a,n-k-\ell}(y) \right) \\
&= \sum_{k=0}^{n+1} (k+n+1-k) d_{a,k}(x) d_{a,n+1-k}(y) - \sum_{k=0}^{n-1} \delta_{a,k+1} \left( \sum_{\ell=0}^{n-k} d_{a,\ell}(x) d_{a,n-k-\ell}(y) \right).
\end{aligned}$$

Since $\delta_{a,1}, \dots, \delta_{a,n}$ are derivations,

$$\begin{aligned}
\delta_{a,n+1}(xy) &= \sum_{k=0}^{n+1} k d_{a,k}(x) d_{a,n+1-k}(y) + \sum_{k=0}^{n+1} d_{a,k}(x)(n+1-k) d_{a,n+1-k}(y) \\
&\quad - \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-k} \left[ \delta_{a,k+1}(d_{a,\ell}(x)) d_{a,n-k-\ell}(y) + d_{a,\ell}(x) \delta_{a,k+1}(d_{a,n-k-\ell}(y)) \right].
\end{aligned}$$

Writing

$$\begin{aligned}
K &= \sum_{k=0}^{n+1} k d_{a,k}(x) d_{a,n+1-k}(y) - \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-k} \delta_{a,k+1}(d_{a,\ell}(x)) d_{a,n-k-\ell}(y), \\
L &= \sum_{k=0}^{n+1} d_{a,k}(x)(n+1-k) d_{a,n+1-k}(y) - \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-k} d_{a,\ell}(x) \delta_{a,k+1}(d_{a,n-k-\ell}(y))
\end{aligned}$$

we have $\delta_{a,n+1}(xy) = K + L$. Let us compute $K$ and $L$. In the summation $\sum_{k=0}^{n-1}\sum_{\ell=0}^{n-k}$ we have $0 \le k+\ell \le n$ and $k \ne n$. Thus if we put $r = k+\ell$ then we can write it as the form $\sum_{r=0}^{n}\sum_{k+\ell=r,k\ne n}$. Putting $\ell = r-k$ we indeed have

$$\begin{aligned} K &= \sum_{k=0}^{n+1} kd_{a,k}(x)d_{a,n+1-k}(y) - \sum_{r=0}^{n}\sum_{0\le k\le r, k\ne n} \delta_{a,k+1}(d_{a,r-k}(x))d_{a,n-r}(y) \\ &= \sum_{k=0}^{n+1} kd_{a,k}(x)d_{a,n+1-k}(y) - \sum_{r=0}^{n-1}\sum_{k=0}^{r} \delta_{a,k+1}(d_{a,r-k}(x))d_{a,n-r}(y) - \sum_{k=0}^{n-1}\delta_{a,k+1}(d_{a,n-k}(x))y. \end{aligned}$$

Putting $r+1$ instead of $k$ in the first summation we have

$$\begin{aligned} & K + \sum_{k=0}^{n-1}\delta_{a,k+1}(d_{a,n-k}(x))y \\ = & \sum_{r=0}^{n}(r+1)d_{a,r+1}(x)d_{a,n-r}(y) - \sum_{r=0}^{n-1}\sum_{k=0}^{r}\delta_{a,k+1}(d_{a,r-k}(x))d_{a,n-r}(y) \\ = & \sum_{r=0}^{n-1}\left[(r+1)d_{a,r+1}(x) - \sum_{k=0}^{r}\delta_{a,k+1}(d_{a,r-k}(x))\right]d_{a,n-r}(y) + (n+1)d_{a,n+1}(x)y. \end{aligned}$$

By our assumption $(r+1)d_{a,r+1}(x) = \sum_{k=0}^{r}\delta_{a,k+1}(d_{a,r-k}(x))$ for $r = 0,\dots,n-1$. We can therefore deduce that

$$K = \left[(n+1)d_{a,n+1}(x) - \sum_{k=0}^{n-1}\delta_{a,k+1}(d_{a,n-k}(x))\right]y = \delta_{a,n+1}(x)y.$$

By a similar argument we have

$$L = x\left[(n+1)d_{a,n+1}(y) - \sum_{k=0}^{n-1}\delta_{a,k+1}(d_{a,n-k}(y))\right] = x\delta_{a,n+1}(y).$$

Thus

$$\delta_{a,n+1}(xy) = K + L = \delta_{a,n+1}(x)y + x\delta_{a,n+1}(y).$$

Whence $\delta_{a,n+1}$ is a derivation on $\mathfrak{A}$. □

**Theorem 1.2.** *Each local higher derivation $\{D_n\}_{n=0}^{\infty}$, with $D_0 = I$, from a $C^*$-algebra $\mathfrak{A}$ into itself is a higher derivation.*

**Proof .** Proposition 1.1 implies the existence of sequence $\{\Delta_n\}_{n=1}^{\infty}$ of local derivations such that $(n+1)D_{n+1} = \sum_{k=0}^{n}\Delta_{k+1}D_{n-k}$. The famous theorem of Johnson [8] now guarantees that $\Delta_n$ are derivations.

To see that $\{\Delta_n\}_{n=1}^{\infty}$ is a higher derivation, let $a, b \in \mathfrak{A}$ and $n$ be a non-negative integer. We use induction on $n$. For $n = 0$ we have $D_0(ab) = ab = D_0(a)D_0(b)$. Let us assume that

$$D_k(ab) = \sum_{i=0}^{k} D_i(a)D_{k-i}(b)$$

for $k \leq n$. Thus we have

$$
\begin{aligned}
(n+1)D_{n+1}(ab) &= \sum_{k=0}^{n} \Delta_{k+1} D_{n-k}(ab) \\
&= \sum_{k=0}^{n} \Delta_{k+1} \sum_{i=0}^{n-k} D_i(a) D_{n-k-i}(b) \\
&= \sum_{i=0}^{n} \left( \sum_{k=0}^{n-i} \Delta_{k+1} D_{n-k-i}(a) \right) D_i(b) \\
&\quad + \sum_{i=0}^{n} D_i(a) \left( \sum_{k=0}^{n-i} \Delta_{k+1} D_{n-k-i}(b) \right).
\end{aligned}
$$

Using our assumption, we can write

$$
\begin{aligned}
(n+1)D_{n+1}(ab) &= \sum_{i=0}^{n} (n-i+1) D_{n-i+1}(a) D_i(b) \\
&\quad + \sum_{i=0}^{n} D_i(a)(n-i+1) D_{n-i+1}(b) \\
&= \sum_{i=1}^{n+1} i D_i(a) D_{n+1-i}(b) + \sum_{i=0}^{n} (n+1-i) D_i(a) D_{n+1-i}(b) \\
&= (n+1) \sum_{k=0}^{n+1} D_k(a) D_{n+1-k}(b).
\end{aligned}
$$

□

**Corollary 1.3.** *Each local higher derivation $\{D_n\}_{n=0}^{\infty}$, with $D_0 = I$, from a $C^*$-algebra $\mathfrak{A}$ into itself is automatically continuous.*

**Proof .** We can inductively prove that each $D_n$ is continuous. Clearly, $D_0 = I$ is continuous. Let $D_k$ be continuous for $k \leqslant n$. A beautiful theorem of Sakai [15] states that each derivation on a $C^*$-algebra is automatically continuous. Thus $\Delta_n$'s of Proposition 1.1 are continuous. This implies that $D_{n+1} = \frac{1}{n+1} \sum_{k=0}^{n} \Delta_{k+1} D_{n-k}$ to be continuous as a linear combination of compositions of continuous operators. □

## References


[1] M. Brešar, *On the distance of the compositions of two derivations to the generalized derivations*, Glasgow Math. J., 33 (1991) 89–93.

[2] M. Brešar and A.R. Villena, *The noncommutative Singer-Wermer conjecture and $\phi$-derivations*, J. London Math. Soc., 66 (2002) 710–720.

[3] H.G. Dales, *Automatic continuity: a survey*, Bull. London Math. Soc., 10 (1978) 129–183.

[4] H.G. Dales, *Banach algebras and automatic continuity*, London Mathematical Society Monographs. New Series, 24. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 2000.

[5] J. Hartwig, D. Larsson and S.D. Silvestrov, *Deformations of Lie algebras using $\sigma$-derivations*, J. Algebra, 295 (2006) 314–361.

[6] H. Hasse, *Noch eine Begrüdung der theorie der höheren Differential quotienten in einem algebraischen Funtionenkörper einer Unbestimmeten*, J. Reine Angew. Math., 177 (1937) 215-223.
[7] N.P. Jewell, *Continuity of module and higher derivations*, Pacific J. Math., 68 (1977) 91–98.
[8] B.E. Johnson, *Local derivations on $C^*$-algebras are derivations*, Trans. Amer. Math. Soc., 353 (2001) 313–325.
[9] B.E. Johnson and A.M. Sinclair, *Continuity of derivations and a problem of Kaplansky*, Amer. J. Math., 90 (1968) 1067–1073.
[10] R.J. Loy, *Continuity of higher derivations*, Proc. Amer. Math. Soc., 5 (1973) 505–510.
[11] M. Mirzavaziri, *Characterization of higher derivations on algebras*, Comm. Algebra, 38 (2010) 981–987.
[12] M. Mirzavaziri and M.S. Moslehian, *Automatic continuity of $\sigma$-derivations on $C^*$-algebras*, Proc. Amer. Math. Soc., 134 (2006) 3319–3327.
[13] J.R. Ringrose, *Automatic continuity of derivations of operator algebras*, J. London Math. Soc., 2 (1972) 432–438.
[14] A. Roy and R. Sridharan, *Higher derivations and central simple algebras*, Nagoya Math. J., 32 (1968) 21–30.
[15] S. Sakai, *On a conjecture of Kaplansky*, Tohoku Math. J., 12 (1960) 31–33.
[16] A.M. Sinclair, *Automatic continuity of linear operators, London Mathematical Society Lecture Note Series*, No. 21, Cambridge University Press, Cambridge-New York-Melbourne, 1976.
[17] A.R. Villena, *Automatic continuity in associative and nonassociative context*, Irish Math. Soc. Bull., 46 (2001) 43–76.