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Homomorphism weak amenability of certain Banach algebras

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Abstract

In this paper we introduce the notion of φ–commutativity for a Banach algebra A, where φ is a continuous homomorphism on A and study the concept of φ–weak amenability for φ–commutative Banach algebras. We give an example to show that the class of φ–weakly amenable Banach algebras is larger than that of weakly amenable commutative Banach algebras. We characterize φ–weak amenability of φ–commutative Banach algebras and prove some hereditary properties. Moreover we verify some of the previous available results about commutative weakly amenable Banach algebras, for φ–commutative φ–weakly amenable Banach algebras.

Keywords: Banach algebra, φ-commutative, φ-derivation, φ-weakly amenability.

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1. Introduction

Let A be a Banach algebra and X be a Banach A–module. A derivation $D : A \longrightarrow X$ is a linear map such that

$$D(ab) = a.D(b) + D(a).b \quad (a, b \in A).$$

The derivation D is inner if it is of the form $a \longmapsto a.x - x.a$ for some $x \in X$. A Banach algebra A is called weakly amenable if every continuous derivation $D : A \longrightarrow A^*$ is inner. The concept of weak amenability was first introduced by Bade, Curtis and Dales in [1] for commutative Banach algebras. Gronbaek in [6], investigated properties of weakly amenable Banach algebras. In particular

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he showed that weakly amenable Banach algebras are essential. A Banach algebra A is called n–weakly amenable ($n \in \mathbb{N}$) if every continuous derivation $D : A \longrightarrow A^{(n)}$ is inner, where $A^{(n)}$ is the n–th dual module of A when $n \geq 1$ and is A itself in the case $n = 0$. The notion of n–weak amenability for Banach algebras, where $n \in \mathbb{N}$, was introduced by Dalse, Ghahramani, and Gronbaek in [3].

Let A and B be two Banach algebras. The set of continuous homomorphisms from A into B is denoted by $\mathtt{Hom}(A, B)$. The case in which $A = B$ we denote the set $\mathtt{Hom}(A, A)$ by $\mathtt{Hom}(A)$. Let X be a Banach A–bimodule and $\varphi \in \mathtt{Hom}(A)$, a linear operator $D : A \longrightarrow X$ is called a φ–derivation if $D(ab) = D(a).\varphi(b) + \varphi(a).D(b)$ $(a, b \in A)$. For every $x \in X$ we define ad_x^{φ} by $ad_x^{\varphi}(a) = \varphi(a).x - x.\varphi(a)$ $(a \in A)$. It is easily seen that ad_x^{φ} is a φ–derivation. Derivations of this form are called inner derivations. A φ–derivation D is called φ–inner if there is $x \in X$ such that $D(a) = ad_x^{\varphi}(a)$ $(a \in A)$. Let $Z^1_{\varphi}(A, X)$ denote the set of all continuous φ-derivations and $N^1_{\varphi}(A, X)$ be the set of all φ–inner derivations from A into X. The first cohomology group $H^1_{\varphi}(A, X)$ is defined to be the quotient space $Z^1_{\varphi}(A, X)/N^1_{\varphi}(A, X)$. A Banach algebra A is called φ–weakly amenable if $H^1_{\varphi}(A, A^*) = 0$. Also A is called n–φ–weakly amenable ($n \in \mathbb{N}$) if $H^1_{\varphi}(A, A^{(n)}) = \{0\}$.

In [2], Bodaghi, Gordji and Medghalchi generalized the concept of weak amenability of Banach algebras and in [8], Mewomo and Akinbo generalized the notion of n–weak amenability of A to that of φ–n–weak amenability for $n \in \mathbb{N}$, whenever φ is a continuous homomorphism on A. Several authors have studied φ–derivations, and φ–amenability of A (see [5, 9]).

This paper is organized as follows. In Section 2, we introduce the concept of φ–commutativity for a Banach algebra A and characterize φ–weak amenability of φ–commutative Banach algebras. We give an example to show that the class of φ–weakly amenable Banach algebras is larger than that of weakly amenable commutative Banach algebras. We investigate relation between homomorphism weak amenability of a φ–commutative Banach algebra A and A/I, where I is a closed ideal of A. Moreover we prove that if A is φ–commutative, then for every $n \in \mathbb{N}$, A is n–φ–weakly amenable if and only if $A^{\#}$ (the unitalization of A) is n–$\varphi^{\#}$–weakly amenable, where $\varphi^{\#}$ is the extension of φ from A to $A^{\#}$. In section 3 for two Banach algebras A and B, we investigate relations between φ–weak amenability of A, ψ–weak amenability of B and $\varphi \otimes \psi$–weak amenability of $A\widehat{\otimes}B$ (resp. $\varphi \oplus \psi$–weak amenability of $A \oplus_1 B$, the l^1–direct sum of A and B), where $\varphi \in \mathtt{Hom}(A)$ and $\psi \in \mathtt{Hom}(B)$.

2. Homomorphism weak amenability

We start this section with the following definition:

Definition 2.1. Let A be a Banach algebra. A Banach A–bimodule X is called φ–symmetric if $\varphi(a).x = x.\varphi(a)$ $(a \in A, x \in X)$. In the case $X = A$, A is called φ–commutative.

The proof of the following proposition is omitted, since it can be proved in the same direction of Proposition 1.3 of [3].

Proposition 2.2. Let A be a Banach algebra, and $\varphi \in \mathtt{Hom}(A)$ be such that $\varphi(a)a = a\varphi(a)$ $(a \in A)$. If A is φ-weakly amenable, then A^2 is dense in A where $A^2 = \mathrm{span}\{a_1a_2 : a_1, a_2 \in A\}$.

Remark 2.3. Let A be a commutative Banach algebra. Then A is weakly amenable if and only if $H^1(A, X) = \{0\}$ for every symmetric Banach A–module X (X is called symmetric if $a.x = x.a$ $(a \in A, x \in X)$) (see [1]).

The following proposition characterize the concept of φ–weak amenability for φ–commutative (not necessarily commutative) Banach algebras.

Proposition 2.4. *Let $\varphi \in \mathtt{Hom}(A)$, and A be a φ–commutative Banach algebra. Then the following two conditions are equivalent:*

(i) *A is φ–weakly amenable.*

(ii) *For every φ–symmetric Banach A–bimodule X, each continuous φ–derivation from A into X is zero.*

Proof . (ii) $\Longrightarrow$ (i) is trivial.
(i) $\Longrightarrow$ (ii): Suppose $D \in Z^1_\varphi(A, X)$. We show that $D = 0$. Assume towards a contradiction that $D \neq 0$. Thus by Proposition 2.2, since A^2 is dense in A, there are $a_1, a_2 \in A$ and $f \in X^*$ such that $\langle D(a_1a_2), f\rangle \neq 0$. Let $R_x : A \longrightarrow \mathbb{C}$ be defined by $R_x(a) = \langle a.x, f\rangle$ $(a \in A)$. Clearly, $R_x \in A^*$. Define $R : X \longrightarrow A^*$ by $R(x) = R_x$. Since X is a φ–symmetric Banach A–bimodule, it follows that

$$R(x.\varphi(a)) = R(x).\varphi(a), R(\varphi(a).x) = \varphi(a).R(x) \ (x \in X, a \in A). \tag{2.1}$$

Define $\tilde{D} : A \longrightarrow A^*$ by $\tilde{D}(a) = R \circ D(a)$. Obviously, $\tilde{D}$ is continuous and by (2.1), one can easily prove that $\tilde{D}$ is a φ–derivation. Now from the fact that A is φ–commutative and φ–weakly amenable, it follows that $\tilde{D} = 0$. Therefore

$$0 = \langle \varphi(a_1), \tilde{D}(a_2)\rangle = \langle \varphi(a_1), R \circ D(a_2)\rangle = \langle \varphi(a_1).D(a_2), f\rangle,$$

and

$$0 = \langle \varphi(a_2), \tilde{D}(a_1)\rangle = \langle \varphi(a_2), R \circ D(a_1)\rangle = \langle \varphi(a_2).D(a_1), f\rangle = \langle D(a_1).\varphi(a_2), f\rangle.$$

Consequently, $\langle D(a_1a_2), f\rangle = \langle D(a_1).\varphi(a_2) + \varphi(a_1).D(a_2), f\rangle = 0$. This contradicts the fact that $\langle D(a_1a_2), f\rangle \neq 0$. Therefore $D = 0$. □

Example 2.5. Let A be a commutative weakly amenable Banach algebra and $\varphi \in \mathtt{Hom}(A)$. Then A is φ–weakly amenable [2].

We need to recall the following remark for give the next example:

Remark 2.6. Let A be a Banach algebra and X be a Banach A–bimodule. Then $A \oplus_1 X$, the l^1–direct sum of A and X becomes a Banach algebra when equipped with the algebra product

$$(a, x).(b, y) = (ab, a.y + x.b) \qquad (a, b \in A, x, y \in X).$$

This Banach algebra is called module extension Banach algebras of A and X (see [11]). Let G be a non Abelian locally compact group, $A = L^1(G)$ and $X = L^1(G)^*(= L^\infty(G))$. Then by Proposition 5.1 of [11], $L^1(G) \oplus_1 L^1(G)^*$ is not weakly amenable. It is obviously $L^1(G) \oplus_1 L^1(G)^*$ is not commutative. Let $(e_\alpha)_\alpha$ be a bounded approximate identity for $L^1(G)$, then it is easy to check that $(e_\alpha, 0)_\alpha$ is a bounded approximate identity for $L^1(G) \oplus_1 L^1(G)^*$.

The following example give a non–weakly amenable non–commutative Banach algebra which is φ–weakly amenable and φ–commutative.

Example 2.7. Let A be a non–weakly amenable non–commutative Banach algebra with a bounded approximate identity (for example let $A = L^1(G) \oplus_1 L^1(G)^*$). Then by Corollary 2.2 of [6], $A^\#$(the unitization of A) is not weakly amenable. Define $\varphi : A^\# \longrightarrow A^\#$ by $\varphi(a + \lambda) = \lambda$ $(a \in A, \lambda \in \mathbb{C})$.

Clearly, φ defines a continuous homomorphism on $A^{\#}$ for which $A^{\#}$ is φ–commutative. For every continuous φ–derivation $D : A^{\#} \longrightarrow (A^{\#})^*$ and $a, b \in A, \lambda_1, \lambda_2 \in \mathbb{C}$ we have,

$$\begin{aligned} D((a+\lambda_1)(b+\lambda_2)) &= D(a+\lambda_1)\varphi(b+\lambda_2) + \varphi(a+\lambda_1)D(b+\lambda_2) \\ &= \lambda_2 D(a+\lambda_1) + \lambda_1 D(b+\lambda_2). \end{aligned}$$

Let $(e_i)_{i\in I}$ be a bounded approximate identity for A. Therefore,

$$D(a+0) = \lim_i D(ae_i+0) = \lim_i D((a+0)(e_i+0)) = 0 \ (a \in A).$$

Also, $D(0+\lambda) = D((0+\lambda)(0+1)) = 2D(0+\lambda)(\lambda \in \mathbb{C})$. Thus $D(0+\lambda) = 0$. So $D(a+\lambda) = D(a+0) + D(0+\lambda) = 0(a \in A, \lambda \in \mathbb{C})$. Therefore $A^{\#}$ is φ–weakly amenable.

It follows from the above example that if $A = L^1(G) \oplus_1 L^1(G)^*$, where G is an Abelian locally compact group, then $A^{\#}$ is a commutative φ–weakly amenable Banach algebra but is not weakly amenable. So the class of φ–weakly amenable Banach algebras is larger than that of weakly amenable commutative Banach algebras.

Proposition 2.8. *Let $\varphi \in \mathtt{Hom}(A), \psi \in \mathtt{Hom}(B)$ and let A and B be φ–commutative and ψ–commutative Banach algebras, respectively. Let $h : A \longrightarrow B$ be a continuous homomorphism with dense range such that $\psi \circ h = h \circ \varphi$. If A is φ–weakly amenable, then B is ψ–weakly amenable.*

Proof . Let $D : B \longrightarrow B^*$ be a continuous ψ–derivation. Define $\tilde{D} : A \longrightarrow A^*$ by $\tilde{D}(a) = h^* \circ D \circ h(a)$ $(a \in A)$. Using the fact that $\psi \circ h = h \circ \varphi$, one can easily show that $\tilde{D}$ is a continuous φ–derivation. Since A is φ–weakly amenable and φ–commutative, it follows that $\tilde{D} = 0$. By density of range of h and continuity of D, we conclude that $D = 0$. So B is ψ–weakly amenable. □ Before we turn to our next results we note that if for every $\varphi \in \mathtt{Hom}(A)$ and an ideal I with $\varphi(I) \subset I$, one defines

$$\tilde{\varphi} : A/I \longrightarrow A/I, \quad (a+I) \longmapsto \varphi(a) + I, \tag{2.2}$$

then $\tilde{\varphi} \in \mathtt{Hom}(A/I)$.

Corollary 2.9. *Let $\varphi \in \mathtt{Hom}(A)$ and A be a φ–commutative Banach algebra with a closed ideal I such that $\varphi(I) \subset I$. If A is φ–weakly amenable, then A/I is $\tilde{\varphi}$–weakly amenable.*

Proof . Suppose A is φ–weakly amenable and $\pi : A \longrightarrow A/I$ is the quotient map. Since π is a continuous epimorphism and $\tilde{\varphi} \circ \pi = \pi \circ \varphi$, Proposition 2.8, implies that A/I is $\tilde{\varphi}$–weakly amenable. □

Proposition 2.10. *Let $\varphi \in \mathtt{Hom}(A)$ and A be a φ–commutative Banach algebra with a closed ideal I such that $\varphi(I)$ is dense in I. Suppose I is φ–weakly amenable and A/I is $\tilde{\varphi}$–weakly amenable. Then A is φ–weakly amenable.*

Proof . Let $i : I \longrightarrow A$ be the natural embedding, $i^* : A^* \longrightarrow I^*$ be the adjoint of i, and $\pi : A \longrightarrow A/I$ be the quotient map. Let $D : A \longrightarrow A^*$ be a continuous φ–derivation. Then $i^* \circ D \circ i : I \longrightarrow I^*$ is a continuous φ–derivation. Since I is φ–weakly amenable and φ–commutative, it follows that $i^* \circ D \circ i = 0$. For every $a, b \in I$ and $c \in A$, we have

$$\langle c, D(ab)\rangle = \langle c\varphi(a), i^* \circ D \circ i(b)\rangle + \langle \varphi(b)c, i^* \circ D \circ i(a)\rangle = 0.$$

That is $D\mid_{I^2}=0$. By Proposition 2.2, $\overline{I^2}=I$, and therefore $D\mid_I=0$. For every $a\in A$ and $b\in I$, we have $\varphi(b).D(a)=D(ba)-D(b).\varphi(a)=0$. Consequently, for every $b_1,b_2\in I$, we obtain

$$\langle\varphi(b_1)\varphi(b_2),D(a)\rangle=\langle\varphi(b_1),\varphi(b_2).D(a)\rangle=0.$$

This means that $D(a)\mid_{\varphi(I^2)}=0$. Thus $D(a)\mid_{\varphi(I)}=0$, and so $D(a)\mid_I=0$ by assumption. Therefore $D(A)\subseteq I^\perp\cong(A/I)^*$, and $\tilde{D}:A/I\longrightarrow(A/I)^*$ given by $\tilde{D}(a+I)=D(a)$ defines a continuous $\tilde{\varphi}$–derivation. From the $\tilde{\varphi}$–weak amenability of A/I, and the facts that A/I is $\tilde{\varphi}$–commutative, it follows that $\tilde{D}=0$. Hence $D=0$. Therefore A is φ–weakly amenable. □

Proposition 2.11. *Let $\varphi\in\mathtt{Hom}(A)$, A be a φ–commutative Banach algebra, and I be a closed ideal of A such that $\varphi(I)\subset I$. Let $D:I\longrightarrow X$ be a continuous φ-derivation for some φ–symmetric Banach A–bimodule X. Then there is a bilinear map $\tilde{D}:I\times A\longrightarrow X$ satisfying:*

(i) *$\tilde{D}(x,.)$ extends $\varphi(x).D(.)$ for every $i\in I$ (i.e. $\tilde{D}(x,.)|_I=\varphi(x).D(.)$);*
(ii) *For every $x\in I^2$, $\tilde{D}(x,.)$ is a continuous φ–derivation.*

Proof . Define $\tilde{D}:I\times A\longrightarrow X$ by $\tilde{D}(x,a)=D(xa)-\varphi(a).D(x)$. From the fact that D is a φ–derivation, it follows that $\tilde{D}(x,y)=\varphi(x).D(y)$ $(x,y\in I)$, and so (i) holds.
(ii) Clearly, $\tilde{D}(x,0)$ $(x\in I^2)$ is continuous. For every $x,y\in I$ and $a\in A$, we have

$$\begin{aligned}D(xya)&=D(x).\varphi(ya)+\varphi(x).D(ya)=\varphi(ya).D(x)+D(ya).\varphi(x)=D(yax)\\&=D(y).\varphi(ax)+\varphi(y).D(ax)=\varphi(ax).D(y)+D(ax).\varphi(y)=D(axy).\end{aligned}$$

That is

$$D(xya)=D(axy). \tag{2.3}$$

Now for every $x,y\in I$ and $a,b\in A$, we have

$$\begin{aligned}&\tilde{D}(xy,a).\varphi(b)+\varphi(a).\tilde{D}(xy,b)\\&\quad=\big(D(xya)-\varphi(a).D(xy)\big).\varphi(b)+\varphi(a).\big(D(xyb)-\varphi(b).D(xy)\big)\\&\quad=\big(\varphi(x).D(ya)-\varphi(ax).D(y)\big).\varphi(b)+\varphi(a)\big(\varphi(x).D(yb)-\varphi(bx).D(y)\big)\\&\quad=\varphi(xb).D(ya)+\varphi(ax).D(yb)-2\varphi(abx).D(y),\end{aligned}$$

and by (2.3), we obtain

$$\begin{aligned}\tilde{D}(xy,ab)&=D(xyab)-\varphi(ab).D(xy)=D(abxy)-\varphi(ab).D(xy)\\&=D(abx).\varphi(y)-\varphi(a).D(x).\varphi(yb)\\&=D(abx).\varphi(y)-\varphi(a).D(xyb)+\varphi(ax).D(yb)\\&=D(abx).\varphi(y)-\varphi(a).D(bxy)+\varphi(ax).D(yb)\\&=D(abx).\varphi(y)-\varphi(abx).D(y)-\varphi(a).D(bx).\varphi(y)+\varphi(ax).D(yb)\\&=D(abx).\varphi(y)-\varphi(abx).D(y)-D(yabx)\\&\quad+D(ya).\varphi(bx)+\varphi(ax).D(yb)\\&=\varphi(xb).D(ya)+\varphi(ax).D(yb)-2\varphi(abx).D(y).\end{aligned}$$

So $\tilde{D}(x,.)$ is a continuous φ–derivation for every $x\in I^2$. □

Remark 2.12. (i) Let A be a φ–commutative Banach algebra, and I be a closed ideal in A. It is easy to check that $B(I,X)$ (the space of all bounded linear map from I to X) is a φ–commutative Banach A–bimodule with module actions given by $(a.\psi)(i)=\psi(ia)$ and $(\psi.a)(i)=\psi(ai)$ $(i\in I,\ a\in A,\ \psi\in B(I,X))$, for some A–bimodule X.
(ii) The map $J: X\longrightarrow B(I,X)$, defined by $J(x)(i)=i.x$ $(i\in I, x\in X)$, is continuous and if X is a φ–symmetric Banach A–bimodule, then it is clear that

$$J(\varphi(a).x)=\varphi(a).J(x),\ \ J(x.\varphi(a))=J(x).\varphi(a)\ \ (x\in X, a\in A).$$

Proposition 2.13. *Let $\varphi\in \mathtt{Hom}(A)$, A be a φ–commutative Banach algebra, and I be a closed ideal of A such that $\varphi(I)\subset I$. Suppose that A is φ–weakly amenable, then I is φ–weakly amenable if and only if I^2 is dense in I.*

Proof . Suppose that A is φ–weakly amenable and let $D: I\longrightarrow I^*$ be a continuous φ–derivation. Let J be the map defined as in Remark 2.12. Then $J\circ D$ is a continuous φ–derivation from I into $B(I,I^*)$. Let $\tilde{D}$ be the corresponding bilinear map from $I\times A$ into $B(I,I^*)$. By Proposition 2.11, $\tilde{D}(x,.)$ is a φ–derivation from A into $B(I,I^*)$ for all $x\in I^2$. Since $B(I,I^*)$ is a φ–symmetric Banach algebra by Remark 2.12, from the φ–weak amenability of A and Proposition 2.4, we conclude that $\tilde{D}(x,.)=0$. Consequently, by Proposition 2.11, $\varphi(I^2).D(I)=\{0\}$. Now from the fact that I^2 is dense in I and A is φ–commutative, we infer that $D=0$. Therefore I is a φ–weakly amenable Banach algebra.
Conversely, let I be φ–weakly amenable. Then by Proposition 2.2, I^2 is dense in I. □

Let $\varphi\in\mathtt{Hom}(A)$ and define $\varphi^{\#}: A^{\#}\longrightarrow A^{\#}$ by $\varphi^{\#}(a+\lambda)=(\varphi(a)+\lambda)$ $(a\in A,\lambda\in\mathbb{C})$. Then $\varphi^{\#}\in\mathtt{Hom}(A^{\#})$, and $\varphi^{\#}\mid_A=\varphi$. Also if $e=(0,1)$, then $\varphi^{\#}(e)=e$.

Corollary 2.14. *Let $\varphi\in\mathtt{Hom}(A)$, and A be a φ–commutative Banach algebra. If $A^{\#}$ is $\varphi^{\#}$–weakly amenable, then A is φ–weakly amenable.*

Proof . Suppose $A^{\#}$ is $\varphi^{\#}$–weakly amenable, therefore by Proposition 2.2, A^2 is dense in A. By Proposition 2.13, A is φ–weakly amenable. □

To prove our next result we need to quote the following remark from [3].

Remark 2.15. Define $e^*\in(A^{\#})^*$ by requiring that $\langle e,e^*\rangle=1$ and $e^*\mid_A=0$. Then we have the identifications $A^{\#(2n)}=\mathbb{C}e\oplus A^{(2n)}$ $(n\in\mathbb{N})$ and $A^{\#(2n+1)}=\mathbb{C}e^*\oplus A^{(2n+1)}$ $(n\in\mathbb{Z}^+)$. The module operations of $A^{\#}$ on $A^{\#(2n+1)}$ are given by

$$(\alpha e+a).(\gamma e^*+\lambda)=(\alpha\gamma+\langle a,\lambda\rangle)e^*+\alpha\lambda+a.\lambda,$$

$$(\gamma e^*+\lambda).(\alpha e+a)=(\alpha\gamma+\langle a,\lambda\rangle)e^*+\alpha\lambda+\lambda.a.$$

Note that in general $A^{(2n+1)}$ is not a submodule of $A^{\#(2n+1)}$. However, $A^{(2n)}$ is a submodule of $A^{\#(2n)}$.

The following proposition generalizes Proposition 1.4 of [3], with the similar technique of proof.

Proposition 2.16. *Let A be a non–unital Banach algebra, and let $n\in\mathbb{N},\varphi\in\mathtt{Hom}(A)$. Then the following statements are valid:*

(i) *Suppose that $A^{\#}$ is $2n$–$\varphi^{\#}$–weakly amenable. Then A is $2n$–φ–weakly amenable.*
(ii) *Suppose that A is $(2n-1)$–φ–weakly amenable and $\varphi(a)a=a\varphi(a)$ $(a\in A)$. Then $A^{\#}$ is $(2n-1)$–$\varphi^{\#}$–weakly amenable.*

(iii) *Let A be a φ–commutative. Then $A^{\#}$ is n–$\varphi^{\#}$–weakly amenable if and only if A is n–φ–weakly amenable.*

Proof . (i) Suppose that $A^{\#}$ is $2n$–$\varphi^{\#}$–weakly amenable. Since by Remark 2.15, $A^{(2n)}$ is a submodule of $A^{\#(2n)}$, it is easy to check that A is $2n$–φ–weakly amenable.
(ii) Let $D : A^{\#} \longrightarrow A^{\#(2n-1)}$ be a $\varphi^{\#}$–continuous derivation. Since $D(e) = 0$, we can consider D as a map from A into $A^{\#(2n-1)}$. Also by Remark 2.15, since $A^{\#(2n-1)} = \mathbb{C}e^* \oplus A^{(2n-1)}$, it follows that there exist two bounded linear maps $\Lambda : A \longrightarrow \mathbb{C}$ and $\tilde{D} : A \longrightarrow A^{(2n-1)}$ such that $D(a) = \langle a, \Lambda\rangle e^* + \tilde{D}(a)$. It is easy to see that $\tilde{D}$ is a continuous φ–derivation, and from $(2n-1)$–φ–weak amenability of A, it follows that there exists $f \in A^{(2n-1)}$ such that $\tilde{D} = ad_f^{\varphi}$. Since $\varphi^{\#} |_A = \varphi$, we may assume that $\tilde{D} = ad_f^{\varphi^{\#}}$. Since $\langle e, e^*\rangle$=1 and $e^* |_A = 0$, for every $a_1, a_2 \in A$, we have

$$\begin{aligned}\langle a_1a_2, \Lambda\rangle &= \langle \varphi(a_2), \tilde{D}(a_1)\rangle + \langle \varphi(a_1), \tilde{D}(a_2)\rangle \\ &= \langle \varphi(a_2), ad_f^{\varphi}(a_1)\rangle + \langle \varphi(a_1), ad_f^{\varphi}(a_2)\rangle = 0.\end{aligned}$$

This means that $\Lambda |_{A^2} = 0$. By Proposition 2.9 of [4], A is φ–weakly amenable, and hence, by Proposition 2.2, A^2 is dense in A. Thus $\Lambda = 0$, and so $D = \tilde{D}$. Therefore $A^{\#}$ is $(2n-1)$–$\varphi^{\#}$–weakly amenable.
(iii) Suppose that $A^{\#}$ is $2k$–$\varphi^{\#}$–weakly amenable. Then by (i), A is $2k$–φ–weakly amenable. Assume that $A^{\#}$ is $(2k-1)$–$\varphi^{\#}$–weakly amenable. By Proposition 2.9 of [4], $A^{\#}$ is $\varphi^{\#}$–weakly amenable, and by Corollary 2.14, A is φ–weakly amenable, and so A is $(2k-1)$–φ–weakly amenable. Let A be $(2k-1)$–φ–weakly amenable. By (ii), $A^{\#}$ is $(2k-1)$–$\varphi^{\#}$–weakly amenable. Suppose that A is $2k$–φ–weakly amenable, and let $D : A^{\#} \longrightarrow A^{\#(2k)}$ be a continuous $\varphi^{\#}$–derivation. Now as in the proof of (ii), there exist a bounded linear map $\Lambda : A \longrightarrow \mathbb{C}$ with $\Lambda |_{A^2} = 0$ and a continuous φ–derivation $\tilde{D} : A \longrightarrow A^{(2k)}$ such that $D(a) = \langle a, \Lambda\rangle e^* + \tilde{D}(a)$ $(a \in A)$. We now show that $D = 0$. To this end, we first suppose that there exists $\psi \in A^{(2k)} \setminus \{0\}$ with $\varphi(a).\psi = \psi.\varphi(a) = 0$ $(a \in A)$. We claim that A^2 is dense in A. Assume towards a contradiction that A^2 is not dense in A, one can choose a non zero f in A^* with $f |_{A^2} = 0$. Define

$$D_1 : A \longrightarrow A^{(2k)}, \quad a \longmapsto f(a)\psi \ (a \in A).$$

It is easily checked that D_1 is a non zero, continuous φ–derivation. This contradicts the fact that A is $2k$–φ–weakly amenable. Thus A^2 is dense in A and $\Lambda = 0$. Therefore $D = \tilde{D}$, and from the $2k$–φ–weak amenability of A we infer that $D = 0$. Next, assume that, for each $\psi \in A^{(2k)} \setminus \{0\}$, there exists $a \in A$ with $\varphi(a).\psi \neq 0$. For every $a \in A$, we define

$$D_2 : A \longrightarrow A^{(2k)}, \quad b \longmapsto \varphi(a).D(b).$$

It is clear that D_2 is a continuous φ–derivation. From $2k$–φ–weak amenability of A and the fact that A is φ–commutative, we conclude that $\varphi(a).D(b) = 0$ $(a, b \in A)$. Since by our assumption $\varphi(a).\psi \neq 0$, it follows that $D(b) = 0$ $(b \in A)$, and thus $D = 0$. Therefore $A^{\#}$ is $2k$–$\varphi^{\#}$–weakly amenable. □

Proposition 2.17. *Let $\varphi \in \mathtt{Hom}(A)$, A be a φ–commutative Banach algebra, and let I be a closed ideal of A such that $\varphi(I) \subset I$. Then for every $n \in \mathbb{N}$ the following statements are valid:*

(i) *Suppose that A is $2n$–φ–weakly amenable. Then I is $2n$–φ–weakly amenable if and only if either I^2 is dense in I or $\varphi(I).I^{(2n-1)}$ is dense in $I^{(2n-1)}$.*

(ii) *Let A be φ–weakly amenable, then I is $(2n-1)$–φ–weakly amenable if and only if I^2 is dense in I.*

Proof . (i) The proof is similar to that of Proposition 1.15 of [3].
(ii) Let I be $(2n-1)$–φ–weakly amenable. By Proposition 2.9 of [4], I is φ–weakly amenable and from Proposition 2.13, it follows that I^2 is dense in I. Conversely, let I^2 be dense in I. By Proposition 2.13, I is φ–weakly amenable, and so I is $(2n-1)$–φ–weakly amenable. □

3. Homomorphism weak amenability of $A \oplus_1 B$ and $A\widehat{\otimes}B$

We commence this section with the following:

Let A and B be Banach algebras, it is well known that $A \oplus_1 B$, the l^1–direct sum of A and B, is a Banach algebra with respect to the canonical multiplication defined by $(a,b)(c,d) := (ac,bd)$. Since $(A \oplus B)^* = (0 \oplus B)^{\perp} \dot{+} (A \oplus 0)^{\perp}$, where $\dot{+}$ denotes the l^{∞}-direct sum, and $(0 \oplus B)^{\perp}$ (resp. $(A \oplus 0)^{\perp}$) is isometrically isomorphic to A^* (resp. B^*), for convenience we write: $(A \oplus_1 B)^* = A^* \dot{+} B^*$. Moreover, $(A \oplus_1 B)^*$ is a $A \oplus_1 B$–bimodule with the module operations given by

$$(f,g).(a,b) = (f.a, g.b) \ \ (a,b).(f,g) = (a.f, b.g)$$

for all $a \in A, b \in B$ and $f \in A^*, g \in B^*$. Before stating the next proposition we note that for every $\varphi \in \mathtt{Hom}(A)$, $\psi \in \mathtt{Hom}(B)$, if we define $\varphi \oplus \psi : A \oplus_1 B \longrightarrow A \oplus_1 B$ by $\varphi \oplus \psi(a,b) = \big(\varphi(a), \psi(b)\big)$ $((a,b) \in A \oplus_1 B)$, then $\varphi \oplus \psi \in \mathtt{Hom}(A \oplus_1 B)$.

Theorem 3.1. *Let $\varphi \in \mathtt{Hom}(A)$ and $\psi \in \mathtt{Hom}(B)$. Consider the following statements:*

(i) *$A \oplus_1 B$ is $\varphi \oplus \psi$–weakly amenable.*
(ii) *A is φ-weakly amenable and B is ψ-weakly amenable.*

Then we have: (i) $\Longrightarrow$ (ii)*. If A and B have bounded approximate identities, then* (i) *and* (ii) *are equivalent.*

Proof . Let $A \oplus_1 B$ be $\varphi \oplus \psi$–weakly amenable and let $\pi : A \oplus_1 B \longrightarrow A$ be the natural projection of $A \oplus_1 B$ onto A and $D : A \longrightarrow A^*$ be a continuous φ–derivation. Let $\tilde{D} := \pi^* \circ D \circ \pi : A \oplus_1 B \longrightarrow (A \oplus_1 B)^*$. It is easy to see that $\tilde{D}$ is a continuous $\varphi \oplus \psi$–derivation. From the $\varphi \oplus \psi$–weak amenability of $A \oplus_1 B$ it follows that there exists $H \in (A \oplus_1 B)^*$ such that

$$\tilde{D}(a,b) = \big(\varphi \oplus \psi(a,b)\big).H - H.\big(\varphi \oplus \psi(a,b)\big) \ \big((a,b) \in A \oplus_1 B)\big).$$

Let $F = H\mid_A$. Hence for every $a, a' \in A$, we have

$$\begin{aligned}\langle D(a), a'\rangle &= \langle D\big(\pi(a,0)\big), \pi(a',0)\rangle = \langle \pi^* \circ D \circ \pi(a,0), (a',0)\rangle \\ &= \langle \varphi \oplus \psi(a,0).H - H.\varphi \oplus \psi(a,0), (a',0)\rangle \\ &= \langle H, (a',0).\varphi \oplus \psi(a,0) - \varphi \oplus \psi(a,0).(a',0)\rangle \\ &= \langle F, a'\varphi(a) - \varphi(a)a'\rangle = \langle \varphi(a).F - F.\varphi(a), a'\rangle.\end{aligned}$$

This means that D is a φ–inner derivation and so A is φ–weakly amenable. Similarly, we can show that B is ψ–weakly amenable.

Conversely, suppose that $D : A \oplus_1 B \longrightarrow A^* \dot{+} B^*$ is a continuous $\varphi \oplus \psi$–derivation. Then $D = (D_1, D_2) = (q_A^* \circ D, q_B^* \circ D)$, where $q_A : A \longrightarrow A \oplus_1 B$ and $q_B : B \longrightarrow A \oplus_1 B$ are defined by $q_A(a) = (a, 0)(a \in A)$ and $q_B(b) = (0, b)(b \in B)$. For every $(a, b), (a', b') \in A \oplus_1 B$, we have

$$\begin{aligned} &D\big((a,b)(a',b')\big) \\ &= \big(D_1(a,b), D_2(a,b)\big).(\varphi(a'), \psi(b')) + (\varphi(a), \psi(b)).\big(D_1(a',b'), D_2(a',b')\big) \\ &= \Big(D_1(a,b).\varphi(a') + \varphi(a).D_1(a',b'), D_2(a,b).\psi(b') + \psi(b).D_2(a',b')\Big). \end{aligned}$$

It follows that

$$D_1\big((a,b)(a',b')\big) = D_1(a,b).\varphi(a') + \varphi(a).D_1(a',b') \tag{3.1}$$

and

$$D_2\big((a,b)(a',b')\big) = D_2(a,b).\psi(b') + \psi(b).D_2(a',b'). \tag{3.2}$$

So $q_A^* \circ D \circ q_A = D_1 \circ q_A : A \longrightarrow A^*$ and $q_B^* \circ D \circ q_B = D_2 \circ q_B : B \longrightarrow B^*$ are continuous φ–derivation and ψ–derivation, respectively. Now from the φ–weak amenability of A and ψ–weak amenability of B, it follows that there exist $f \in A^*$ and $g \in B^*$ such that $q_A^* \circ D \circ q_A = ad_f^{\varphi}$ and $q_B^* \circ D \circ q_B = ad_g^{\psi}$. Hence,

$$D_1(a, 0) = q_A^* \circ D \circ q_A(a) = \varphi(a).f - f.\varphi(a) \ (a \in A)$$

and

$$D_2(0, b) = q_B^* \circ D \circ q_B(b) = \psi(b).g - g.\psi(b) \ (b \in B).$$

Let $(b_\beta)_\beta$ be a bounded approximate identity for B. By (3.1), for every $b \in B$, we have

$$D_1(0,b) = \lim_\beta D_1\big((0,b_\beta)(0,b)\big) = \lim_\beta \Big(D_1(0,b_\beta).\varphi(0) + \varphi(0).D_1(0,b)\Big) = 0.$$

Similarly, we may show that $D_2(a, 0) = 0$ $(a \in A)$. For every $a \in A$ and $b \in B$, we have

$$\begin{aligned} D(a,b) &= \big(D_1(a,b), D_2(a,b)\big) = \big(D_1(a,0) + D_1(0,b), D_2(a,0) + D_2(0,b)\big) \\ &= \big(\varphi(a).f - f.\varphi(a), \psi(b).g - g.\psi(b)\big) \\ &= \big(\varphi(a), \psi(b)\big).(f,g) - (f,g).\big(\varphi(a), \psi(b)\big). \end{aligned}$$

So $D = ad_{(f,g)}^{\varphi \oplus \psi}$. Therefore $A \oplus_1 B$ is $\varphi \oplus \psi$–weakly amenable. $\square$

It is well known that $A\widehat{\otimes}B$, the projective tensor product of A and B is Banach algebra with respect to the canonical multiplication defined by $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1a_2 \otimes b_1b_2)$. Before stating the next results we note that for every $\varphi \in \mathtt{Hom}(A)$, $\psi \in \mathtt{Hom}(B)$, if we define $\varphi \otimes \psi : A \otimes B \longrightarrow A \otimes B$ by $\varphi \otimes \psi(a \otimes b) = \varphi(a) \otimes \psi(b)$, then $\varphi \otimes \psi \in \mathtt{Hom}(A \otimes B)$.

We also note that if A and B are unital Banach algebras and X is a $\varphi \otimes \psi$–symmetric Banach $A\widehat{\otimes}B$–module, then it is easy to check that X is both a φ–symmetric Banach A–module and a ψ–symmetric Banach B–module with module actions given by

$$a \bullet x = \big(a \otimes 1_B\big).x, \ \ x \bullet a = x.\big(a \otimes 1_B\big) \ (a \in A, x \in X) \tag{3.3}$$

and

$$b \bullet x = \big(1_A \otimes b\big).x, \ \ x \bullet b = x.\big(1_A \otimes b\big) \ (b \in B, x \in X). \tag{3.4}$$

Proposition 3.2. *Let $\varphi \in \mathtt{Hom}(A)$, $\psi \in \mathtt{Hom}(B)$ and let A and B be φ–commutative and ψ–commutative Banach algebras, respectively. Let $\mathfrak{A} = \overline{\varphi(A)}$ and $\mathfrak{B} = \overline{\psi(B)}$. If $A\widehat{\otimes}B$ is $\varphi \otimes \psi$–weakly amenable, then $\varphi(A^2)$ is dense in $\mathfrak{A}$ and $\psi(B^2)$ is dense in $\mathfrak{B}$.*

Proof . Suppose that $\varphi(A^2)$ is not dense in $\mathfrak{A}$. By the Hahn–Banach theorem there is a non–zero $f \in \mathfrak{A}^*$ such that $f\mid_{\varphi(A^2)}= 0$. Let g be a non–zero element of B^*. The map $f \otimes g : A\widehat{\otimes}B \longrightarrow \mathbb{C}$ is a non-zero bounded linear functional such that $f \otimes g(a \otimes b) = \langle\varphi(a), f\rangle\langle b, g\rangle$ $(a \in A, b \in B)$. Define $D : A\widehat{\otimes}B \longrightarrow (A\widehat{\otimes}B)^*$ by $D(a \otimes b) = \big(f \otimes g(a \otimes b)\big)f \otimes g$ $(a \in A, b \in B)$. It is immediate that D is a non–zero continuous linear map, and and for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$,

$$\begin{aligned} D\big((a_1 \otimes b_1).(a_2 \otimes b_2)\big) &= D(a_1a_2 \otimes b_1b_2) = \big(f \otimes g(a_1a_2 \otimes b_1b_2)\big)f \otimes g \\ &= \langle\varphi(a_1a_2), f\rangle\langle b_1b_2, g\rangle(f \otimes g) = 0 \end{aligned}$$

and

$$\begin{aligned} D(a_1 \otimes b_1).&\big((\varphi \otimes \psi)(a_2 \otimes b_2)\big) + \big((\varphi \otimes \psi)(a_1 \otimes b_1)\big).D(a_2 \otimes b_2) \\ &= \langle\varphi(a_1), f\rangle\langle b_1, g\rangle(f \otimes g).\big(\varphi(a_2) \otimes \psi(b_2)\big) \\ &\quad + \big(\varphi(a_1) \otimes \psi(b_1)\big).\langle\varphi(a_2), f\rangle\langle b_2, g\rangle(f \otimes g) \\ &= 0. \end{aligned}$$

Thus D is a $\varphi \otimes \psi$–derivation. From the $\varphi \otimes \psi$–weak amenability of $A\widehat{\otimes}B$, and the fact that $A\widehat{\otimes}B$ is $\varphi \otimes \psi$–commutative, it follows that $D = 0$. This contradicts the fact that $D \neq 0$. So, $\varphi(A^2)$ is dense in $\mathfrak{A}$. Similarly, we can show that $\psi(B^2)$ is dense in $\mathfrak{B}$. □

Gronbaek in [7], proved that if A and B are commutative weakly amenable Banach algebras, then $A\widehat{\otimes}B$ is weakly amenable and Yazdanpanah in [10], showed that the converse is also valid. In the following theorem, we prove similar results for φ–commutative Banach algebras (not necessarily commutative).

Theorem 3.3. *Let $\varphi \in \mathtt{Hom}(A)$, $\psi \in \mathtt{Hom}(B)$, and let A and B be φ–commutative and ψ–commutative Banach algebras, respectively. Then $A\widehat{\otimes}B$ is $\varphi \otimes \psi$–weakly amenable if and only if A is φ–weakly amenable and B is ψ–weakly amenable.*

Proof . Suppose that A is φ–weakly amenable and B is ψ–weakly amenable. By Proposition 2.2, we conclude that A^2 is dense in A and B^2 is dense in B. Thus $(A\widehat{\otimes}B)^2$ is dense in $A\widehat{\otimes}B$. Also from Proposition 2.16 (iii), it follows that $A^\#$ is $\varphi^\#$–weakly amenable and $B^\#$ is $\psi^\#$–weakly amenable. Let X be a $\varphi^\# \otimes \psi^\#$–symmetric Banach $A^\#\widehat{\otimes}B^\#$-module, and let D be a continuous $\varphi^\# \otimes \psi^\#$–derivation from $A^\#\widehat{\otimes}B^\#$ into X. Let $D_1 = D\mid_{A^\#\widehat{\otimes}1_{B^\#}}$. By (3.3), we may assume that X is a $\varphi^\#$–symmetric Banach $A^\#$–module. Define

$$\tilde{D_1} : A^\# \longrightarrow X^*,\ (a + \lambda) \longmapsto D_1\big((a + \lambda) \otimes 1_{B^\#}\big).$$

It is easy to check that $\tilde{D_1}$ is a continuous $\varphi^\#$–derivation. Since $A^\#$ is $\varphi^\#$–commutative, and $\varphi^\#$–weakly amenable, it follows that $\tilde{D_1} = 0$. So $D_1 = D\mid_{A^\#\widehat{\otimes}1_{B^\#}}= 0$. Similarly, if $D_2 = D\mid_{1_{A^\#}\widehat{\otimes}B^\#}$, we can show that $D_2 = D\mid_{1_{A^\#}\widehat{\otimes}B^\#}= 0$. Therefore, from the fact that

$$A^\#\widehat{\otimes}B^\# = \big((A^\#\widehat{\otimes}1_{B^\#})(1_{A^\#}\widehat{\otimes}B^\#)\big)^-,$$

we conclude that $D = 0$. Consequently, $A^\#\widehat{\otimes}B^\#$ is $\varphi^\# \otimes \psi^\#$–weakly amenable. Since $A\widehat{\otimes}B$ is a closed ideal of $A^\#\widehat{\otimes}B^\#$ and $(A\widehat{\otimes}B)^2$ is dense in $A\widehat{\otimes}B$, by Proposition 2.13, we conclude that $A\widehat{\otimes}B$ is $\varphi \otimes \psi$–weakly amenable.

Conversely, suppose that $A\widehat{\otimes}B$ is $\varphi\otimes\psi$–weakly amenable. Let g be a non–zero element of $\big(\psi(B)\big)^*$. By Hahn–Banach theorem we extend g to a linear functional $\tilde{g}$ on B. By Proposition 3.2, there are $c,d\in B$ such that $\langle\psi(cd),g\rangle=1$. Let $D:A\longrightarrow A^*$ be a continuous φ–derivation and define $\tilde{D}:A\widehat{\otimes}B\longrightarrow(A\widehat{\otimes}B)^*$ by

$$\langle a'\otimes b',\tilde{D}(a\otimes b)\rangle=\langle a',D(a)\rangle\langle b'\psi(b),\tilde{g}\rangle\quad(a',a\in A,\ b',b\in B).$$

Obviously, $\tilde{D}$ is continuous and from the ψ–commutativity of B, we may infer that $\tilde{D}$ is a $\varphi\otimes\psi$–derivation. Since A and B are φ-commutative and ψ–commutative Banach algebras, respectively, it follows that $A\widehat{\otimes}B$ is $\varphi\otimes\psi$–commutative Banach algebra. Now $\varphi\otimes\psi$–weak amenability of $A\widehat{\otimes}B$, implies that $\tilde{D}=0$. Hence, for every $a,a'\in A$, we have

$$\begin{aligned}\langle a',D(a)\rangle&=\langle a',D(a)\rangle\langle\psi(cd),g\rangle=\langle a',D(a)\rangle\langle\psi(c)\psi(d),\tilde{g}\rangle\\&=\langle a'\otimes\psi(c),\tilde{D}(a\otimes d)\rangle=0.\end{aligned}$$

This means that $D=0$. Therefore A is φ–weakly amenable. Similar arguments show that B is ψ-weakly amenable. □

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