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A class of certain properties of approximately n-multiplicative maps between locally multiplicatively convex algebras

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Abstract

We extend the notion of approximately multiplicative to approximately n–multiplicative maps between locally multiplicatively convex algebras and study some properties of these maps. We prove that every approximately n–multiplicative linear functional on a functionally continuous locally multiplicatively convex algebra is continuous. We also study the relationship between approximately multiplicative linear functionals and approximately n–multiplicative linear functionals.

Keywords: Almost multiplicative maps, n–homomorphism maps, Approximately n–multiplicatives, LMC algebras.
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1. Introduction

A locally multiplicatively convex (LMC) algebra is a topological algebra whose topology is defined by a separating family $\mathcal{P} = (p_\alpha)$ of submultiplicative seminorms. A complete metrizable LMC algebra is a Fréchet algebra. The automatic continuity of homomorphisms between different topological algebras, including Fréchet algebras and Banach algebras, have been studied by many mathematicians. It is well–known that every homomorphism $\varphi : A \longrightarrow B$ is automatically continuous,

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when A and B are Banach algebras and B is commutative and semisimple. Let A and B be two complex algebras and $n \geq 2$ be an integer. A map $\varphi : A \to B$ is called an n-multiplicative if $\varphi(a_1a_2 \dots a_n) = \varphi(a_1)\varphi(a_2)\dots\varphi(a_n)$ for all elements $a_1, a_2, \dots, a_n \in A$. Moreover, if φ is a linear mapping, then it is called an n–homomorphism. Clearly, every 2–homomorphism is just a homomorphism, in the usual sense. We recall that a topological algebra A is called functionally continuous if every homomorphism on A is continuous. For the automatic continuity of homomorphisms and n–homomorphisms between Banach algebras and topological algebras one may refer to [2], [3], [4], [5], [6], [7], [8], [12] and [15].

In [10], K. Jarosz introduced the notion of approximately multiplicative function between normed algebras and showed that every approximately multiplicative linear functional on a Banach algebra is bounded.
Let A and B be normed algebras and let $\varphi : A \longrightarrow B$ be a linear map. Then φ is approximately multiplicative linear function if

$$\|\varphi(xy) - \varphi(x)\varphi(y)\| \leqslant \varepsilon\|x\|\|y\| \quad (x, y \in A)$$

for some $\varepsilon > 0$. Many mathematicians have extensively investigated the properties of such maps. See, for example, [1], [9], [11], [13], [14].

In this paper, we define approximately n–multiplicative functions between LMC algebras and investigate some properties of these functions. Let $\varepsilon > 0$ and $n \geq 2$ be an integer. Suppose that $(A, (P_\alpha)_{\alpha\in I})$ and $(B, (q_\alpha)_{\alpha\in J})$ are LMC algebras and let $\varphi : A \longrightarrow B$ be a map. We say that φ is (ε, n)–multiplicative with respect to $(P_\alpha)_{\alpha\in I}$ and $(q_\alpha)_{\alpha\in J}$, if for each $\alpha \in J$ there exists $\beta \in I$ such that

$$q_\alpha(\varphi(x_1 \dots x_n) - \varphi(x_1)\dots\varphi(x_n)) \leqslant \varepsilon p_\beta(x_1)\dots p_\beta(x_n) \quad (x_1, \dots, x_n \in A)$$

and we say φ is approximately n–multiplicative if φ is (ε, n)–multiplicative for some $\varepsilon > 0$. Clearly, every $(\varepsilon, 2)$–multiplicative is just an ε–multiplicative, in the usual sense. In the case where $B = \mathbb{C}$, a map φ on an LMC algebra $(A, (P_\alpha)_{\alpha\in I})$ is (ε, n)–multiplicative with respect to $(P_\alpha)_{\alpha\in I}$, if there exists $\alpha \in I$ such that

$$|\varphi(x_1 \dots x_n) - \varphi(x_1)\dots\varphi(x_n)| \leqslant \varepsilon p_\alpha(x_1)\dots p_\alpha(x_n) \quad (x_1, \dots, x_n \in A).$$

2. The main results

First we give a theorem to show that there exists a relationship between approximately multiplicative linear functionals and approximately n–multiplicative linear functionals.

Theorem 2.1. *Let $(A, (p_\alpha)_{\alpha\in I})$ be an LMC algebra and let ϕ be an approximately n–multiplicative linear functional. If $\phi(a) = 1$ for some $a \in A$, then the linear functional $\psi : x \longmapsto \phi(ax)$ is an approximately multiplicative linear functional.*

Proof . By the hypothesis, there exist $\varepsilon > 0$ and $\beta \in I$ such that

$$|\phi(x_1 \dots x_n) - \phi(x_1)\dots\phi(x_n)| \leqslant \varepsilon p_\beta(x_1)\dots p_\beta(x_n) \quad (x_1, \dots, x_n \in A).$$

For each $x, y \in A$, we have

$$\begin{aligned}|\psi(xy) - \psi(x)\psi(y)| =& |\phi(axy) - \phi(ax)\phi(ay)|\\ =& |\phi(axy) \pm \phi(a^{n-1}xya) \pm \phi(ax)\phi(ya) \pm \phi(axaya^{n-2}) - \phi(ax)\phi(ay)|\end{aligned}$$

$$\begin{aligned}
&\leq|\phi(axy)-\phi(a^{n-1}xya)|+|\phi(a^{n-1}xya)-\phi(ax)\phi(ya)|\\
&\quad+|\phi(ax)\phi(ya)-\phi(axaya^{n-2})|+|\phi(axaya^{n-2})-\phi(ax)\phi(ay)|\\
&\leq|\phi(a)^{n-2}\phi(axy)\phi(a)-\phi(a^{n-1}xya)|+|\phi(a^{n-1}xya)-\phi(a)^{n-2}\phi(ax)\phi(ya)|\\
&\quad+|\phi(ax)\phi(a)\phi(ya)\phi(a)^{n-3}-\phi(axaya^{n-2})|\\
&\quad+|\phi(axaya^{n-2})-\phi(ax)\phi(ay)\phi(a)^{n-2}|\\
&\leq 4\varepsilon p_\beta^n(a)p_\beta(x)p_\beta(y).
\end{aligned}$$

Then ψ is δ–multiplicative linear functional, where $\delta=4\varepsilon p_\beta^n(a)$. □

A topological space (X,τ) is completely regular if it is Haussdorf and, given every $x\in X$ and every nonempty closed subset K of X such that $x\notin K$, there exists a continuous function $f:X\to[0,1]$ such that $f(x)=0$ and $f(y)=1$ for all $y\in K$.

Example 2.2. Let X be a completely regular topological space. For each non–empty, compact subset K of X, define $p_K(f)=\sup_{x\in K}|f(x)|$, $f\in C(X)$. Then p_K is an algebra seminorm on $C(X)$. The family $\{p_K\}$ of seminorms defines the compact open topology on $C(X)$, where K varying over all non–empty, compact subsets of X. $C(X)$ with respect to this topology is an LMC algebra. Fixed $a\in X$ and $0<\lambda<1$. We define linear functional $\varphi:C(X)\to\mathbb{C}$ by $\varphi(f)=\lambda f(a)$. Then for all $f_1,\dots,f_n\in C(X)$, we have

$$\begin{aligned}
|\varphi(f_1\dots f_n)-\varphi(f_1)\dots\varphi(f_n)|&=|\lambda f_1(a)\dots f_n(a)-\lambda^n f_1(a)\dots f_n(a)|\\
&=|\lambda-\lambda^n||f_1(a)\dots f_n(a)|\\
&\leq|\lambda-\lambda^n|p_{\{a\}}(f_1)\dots p_{\{a\}}(f_n).
\end{aligned}$$

Therefore φ is (ε,n)–homomorphism (with $\varepsilon=|\lambda-\lambda^n|$) but it is not n–homomorphism.

Theorem 2.3. *Let $n\geq 2$ and let $(A,(p_\alpha)_{\alpha\in I})$ and $(B,(q_\alpha)_{\alpha\in J})$ be LMC algebras such that for each $\alpha\in J$ and $x_1,\dots,x_n\in A$,*

$$q_\alpha(x_1\dots x_n)=q_\alpha(x_1)\dots q_\alpha(x_n).$$

If $\varphi:A\longrightarrow B$ is an approximately n–multiplicative, then at least one of the following results holds:

(i) φ is n–multiplicative,

(ii) there exist $\alpha\in J$, $\beta\in I$ and a constant k such that $q_\alpha(\varphi(x))\leqslant kp_\beta(x)$ for each $x\in A$.

Proof . Suppose that φ is not n–multiplicative. Therefore there exist $a_1,\dots,a_n\in A$ such that $\varphi(a_1\dots a_n)-\varphi(a_1)\dots\varphi(a_n)\neq 0$, and so, there exists $\alpha\in J$ such that $q_\alpha(\varphi(a_1\dots a_n)-\varphi(a_1)\dots\varphi(a_n))\neq 0$. On the other hand by the hypothesis, there exist $\varepsilon>0$ and $\beta\in I$ such that

$$q_\alpha(\varphi(x_1\dots x_n)-\varphi(x_1)\dots\varphi(x_n))\leqslant\varepsilon p_\beta(x_1)\dots p_\beta(x_n)\quad(x_1,\dots,x_n\in A).$$

Therefore for each $x\in A$, we have

$$\begin{aligned}
q_\alpha(\varphi(x))^{n-1}q_\alpha(\varphi(a_1\dots a_n)-\varphi(a_1)\dots\varphi(a_n))&=q_\alpha(\varphi(x)^{n-1}\varphi(a_1\dots a_n)-\varphi(x)^{n-1}\varphi(a_1)\dots\varphi(a_n)\\
&\quad\pm\varphi(x^{n-1}a_1\dots a_n)\pm\varphi(x^{n-1}a_1)\varphi(a_2)\dots\varphi(a_n))\\
&\leqslant q_\alpha(\varphi(x)^{n-1}\varphi(a_1\dots a_n)-\varphi(x^{n-1}a_1\dots a_n))\\
&\quad+q_\alpha(\varphi(x^{n-1}a_1\dots a_n)-\varphi(x^{n-1}a_1)\varphi(a_2)\dots\varphi(a_n))\\
&\quad+q_\alpha\left((\varphi(x^{n-1}a_1)-\varphi(x)^{n-1}\varphi(a_1))\varphi(a_2)\dots\varphi(a_n)\right)\\
&\leq\varepsilon p_\beta^{n-1}(x)p_\beta(a_1)\left[2p_\beta(a_2)\dots p_\beta(a_n)+q_\alpha(\varphi(a_2))\dots q_\alpha(\varphi(a_n))\right].
\end{aligned}$$

Thus if

$$k = \left[\frac{\varepsilon p_\beta(a_1)[2p_\beta(a_2)\dots p_\beta(a_n) + q_\alpha(\varphi(a_2))\dots q_\alpha(\varphi(a_n))]}{q_\alpha(\varphi(a_1\dots a_n) - \varphi(a_1)\dots\varphi(a_n))} \right]^{\frac{1}{n-1}},$$

then we have $q_\alpha(\varphi(x)) \leqslant kp_\beta(x)$, as desired. □

Corollary 2.4. Let $(A, (p_\alpha)_{\alpha\in I})$ be an LMC algebra and let $\varphi : A \longrightarrow \mathbb{C}$ be an approximately n–multiplicative map. Then either φ is n–multiplicative or there exist $\alpha \in I$ and a constant k such that $|\varphi(x)| \leqslant kp_\alpha(x)$ for each $x \in A$.

Remark 2.5. (Fragoulopoulou, [4, p. 8]) Let $(A, (p_\alpha)_{\alpha\in I})$ and $(B, (q_\alpha)_{\alpha\in J})$ be LMC algebras and let $\varphi : A \longrightarrow B$ be a linear map. Then φ is continuous if and only if for each $\alpha \in J$ there exist $\beta \in I$ and $c_\alpha > 0$ such that

$$q_\alpha(\varphi(x)) \leq c_\alpha p_\beta(x).$$

Corollary 2.6. With the same hypotheses of the Corollary 2.4, if φ is a linear mapping, then it is n–multiplicative or continuous linear functional.

We now have the following result.

Corollary 2.7. Let $(A, (p_\alpha)_{\alpha\in I})$ be a functionally continuous LMC algebra and let φ be an approximately n–multiplicative linear functional on A. Then φ is automatically continuous.

Theorem 2.8. *Let $r \geq 0$ and $(A, (p_\alpha)_{\alpha\in I})$ be an LMC algebra. Suppose that the map $\varphi : A \longrightarrow \mathbb{C}$ satisfies the following conditions:*

(1) $|\varphi(x + y) - \varphi(x) - \varphi(y)| \leqslant \varepsilon(p^r_\beta(x) + p^r_\beta(y))$,

(2) $|\varphi(x_1 \dots x_n) - \varphi(x_1)\dots\varphi(x_n)| \leqslant \varepsilon p^r_\beta(x_1)\dots p^r_\beta(x_n)$,

for each $x, y, x_1, \dots, x_n \in A$ and some $\beta \in I$. Then at least one of the following results holds:

(i) φ is additive and n-multiplicative,

(ii) there exists a constant k such that $|\varphi(x)| \leqslant kp^r_\beta(x)$ for each $x \in A$.

Proof . Suppose that φ is neither n–multiplicative nor additive. If φ is not n-multiplicative, then by Theorem 2.3, the result follows. If φ is not additive, then there exist $a, b \in A$ such that $\varphi(a + b) - \varphi(a) - \varphi(b) \neq 0$. Hence for each $x \in A$, we have

$$\begin{aligned}
|\varphi(x)|^{n-1}|\varphi(a+b) - \varphi(a) - \varphi(b)| =& |\varphi(x)^{n-1}\varphi(a+b) - \varphi(x)^{n-1}\varphi(a) - \varphi(x)^{n-1}\varphi(b) \\
&\pm \varphi(x^{n-1}(a+b)) \pm \varphi(x^{n-1}a) \pm \varphi(x^{n-1}b)| \\
\leqslant& |\varphi(x)^{n-1}\varphi(a+b) - \varphi(x^{n-1}(a+b))| \\
&+ |\varphi(x^{n-1}a) - \varphi(x^{n-1}(a+b)) - \varphi(x^{n-1}b)| \\
&+ |\varphi(x)^{n-1}\varphi(a) - \varphi(x^{n-1}a)| + |\varphi(x^{n-1}b) - \varphi(x)^{n-1}\varphi(b)| \\
\leqslant& \varepsilon p_\beta^{r(n-1)}(x)p^r_\beta(a+b) + \varepsilon\left(p^r_\beta(x^{n-1}a) + p^r_\beta(x^{n-1}b)\right) \\
&+ \varepsilon p_\beta^{r(n-1)}(x)p^r_\beta(a) + \varepsilon p_\beta^{r(n-1)}(x)p^r_\beta(b) \\
\leqslant& \varepsilon p_\beta^{r(n-1)}(x)\left[p^r_\beta(a+b) + 2p^r_\beta(a) + 2p^r_\beta(b)\right],
\end{aligned}$$

which completes the proof. □

Theorem 2.9. *Let $(A,(p_\alpha)_{\alpha\in I})$ be an LMC algebra and $\varphi : A \longrightarrow \mathbb{C}$ be an approximately n–multiplicative linear functional. Then either φ is n–multiplicative or*

$$|\varphi(x)| \leqslant (1+\varepsilon)p_\beta(x) \quad (x \in A),$$

for some $\beta \in I$.

Proof . Let φ be an (ε, n)–multiplicative for some $\varepsilon > 0$. Then there exists $\beta \in I$ such that

$$|\varphi(x_1 \cdots x_n) - \varphi(x_1)\cdots\varphi(x_n)| \leq \varepsilon p_\beta(x_1)\cdots p_\beta(x_n)$$

for each $x_1, \ldots, x_n \in A$. If φ is not n–multiplicative, then by Theorem 2.3, there exists $k > 0$ such that

$$|\varphi(x)| \leqslant k p_\beta(x) \quad (x \in A).$$

Suppose that there exists $a \in A$ such that $|\varphi(a)| > (1+\varepsilon)p_\beta(a)$. Since $|\varphi(a)| \leqslant k p_\beta(a)$ and $|\varphi(a)| > (1+\varepsilon)p_\beta(a)$, then we have $p_\beta(a) \neq 0$. Hence, we can write $|\varphi(a)| = (1+\varepsilon+p)p_\beta(a)$ for some $p > 0$. Now by induction on $m \in \mathbb{N}$, we prove that

$$|\varphi(a^{n^m})| \geq (1+\varepsilon+mp)p_\beta^{n^m}(a). \tag{2.1}$$

If $m = 1$, then

$$\begin{aligned} |\varphi(a^n)| \geq& |\varphi(a)|^n - |\varphi(a)^n - \varphi(a^n)| \\ \geq& (1+\varepsilon+p)^n p_\beta^n(a) - \varepsilon p_\beta^n(a) \\ \geq& (1+\varepsilon+p)p_\beta^n(a), \end{aligned}$$

so (2.1) is true for $m = 1$. Now assume that (2.1) is true for m. Then

$$\begin{aligned} |\varphi(a^{n^{m+1}})| \geq& |\varphi(a^{n^m})|^n - |\varphi(a^{n^m})^n - \varphi(a^{n^{m+1}})| \\ \geq& (\varepsilon+1+mp)^n p_\beta^{n^{m+1}}(a) - \varepsilon p_\beta^n(a^{n^m}) \\ \geq& (\varepsilon+1+(m+1)p)p_\beta^{n^{m+1}}(a), \end{aligned}$$

this gives (2.1). For each $x_1, \ldots, x_n \in A$, we have

$$|\varphi(x_{n+1})||\varphi(x_1 \ldots x_n) - \varphi(x_1)\ldots\varphi(x_n)| \leq k\varepsilon p_\beta(x_{n+1})p_\beta(x_1)\ldots p_\beta(x_n). \tag{2.2}$$

By taking $x_{n+1} = a^{n^m}$ in (2.2), it follows from (2.1) that

$$\begin{aligned} |\varphi(x_1 \ldots x_n) - \varphi(x_1)\ldots\varphi(x_n)| \leq& \frac{k\varepsilon p_\beta(a^{n^m})p_\beta(x_1)\ldots p_\beta(x_n)}{|\varphi(a^{n^m})|} \\ \leq& \frac{k\varepsilon p_\beta(x_1)\ldots p_\beta(x_n)}{1+\varepsilon+mp}. \end{aligned}$$

If $p_\beta(x_i) \neq 0$, $(1 \leq i \leq n)$, by letting $m \longrightarrow \infty$, we obtain that $\varphi(x_1 \ldots x_n) = \varphi(x_1)\ldots\varphi(x_n)$. Therefore φ is n–multiplicative, which is a contradiction. □

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