Common Fixed Point Theorems with Applications to Theoretical Computer Science

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Abstract

Owing to the notion of $L$-fuzzy mapping, we establish some common $L$-fuzzy fixed point results for almost $\Theta$-contraction in the setting of complete metric spaces. An application to theoretical computer science is also provided to show the significance of the investigations.

Keywords: Fixed point, $\Theta$-contraction, metric space, $L$-fuzzy mappings.

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1. Introduction and preliminaries

Answering real-world problems becomes evidently uncomplicated with the initiation of fuzzy set theory in 1965 by Zadeh [35], as it helps in making the explanation of obscurity and inaccuracy fair and more accurate. subsequently, Goguen [17] modified this concept to $L$-fuzzy set theory by replacing the interval $[0, 1]$ in 1967. There are fundamentally two perceptive of the meaning of $L$, one is when $L$ is a complete lattice equipped with a multiplication $\ast$ operator satisfying certain assumptions as shown in the basic paper [17] and the second perceptive of the meaning of $L$ is that $L$ is a completely distributive complete lattice with an order-reversing involution.

Definition 1.1. [17] A partially ordered set $(L, \preceq_L)$ is called

i) a lattice, if $a_1 \lor a_2 \in L$, $a_1 \land a_2 \in L$ for each $a_1, a_2 \in L$.

ii) a complete lattice, if $\lor A \in L$, $\land A \in L$ for any $A \subseteq L$.

iii) distributive lattice if $a_1 \lor (a_2 \land a_3) = (a_1 \lor a_2) \land (a_1 \lor a_3)$, $a_1 \land (a_2 \lor a_3) = (a_1 \land a_2) \lor (a_1 \land a_3)$ for any $a_1, a_2, a_3 \in L$.

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Definition 1.2. \cite{[17]} Let $L$ be a lattice with top element $1_L$ and bottom element $0_L$ and let $a_1, a_2 \in L$. Then $a_2$ is said to be a complement of $a_1$, if $a_1 \lor a_2 = 1_L$, and $a_1 \land a_2 = 0_L$. If $a \in L$ has a complement element, then it is unique. It is denoted by $\check{a}$.

Definition 1.3. \cite{[17]} A $L$-fuzzy set $A$ on a nonempty set $S$ is a function $A : S \to L$, where $L$ is complete distributive lattice with $1_L$ and $0_L$.

Remark 1.4. An $L$-fuzzy set is a fuzzy set if $L = [0, 1]$, so the family of $L$-fuzzy sets is larger than the family of fuzzy sets.

The $\alpha_L$-level set of $L$-fuzzy set $A$, is designated by $A_{\alpha_L}$, and is given in this way.

$$A_{\alpha_L} = \{ u : \alpha_L \preceq_L A(u) \} \text{ if } \alpha_L \in L \setminus \{ 0_L \},$$

$$A_{0_L} = \{ u : 0_L \preceq_L A(u) \}.$$  

Here $cl(B)$ stands for the closure of the set $B$.

The characteristic function of a $L$-fuzzy set $A$ is denoted by $\chi_{La}$ and is defined as follows:

$$\chi_{La} := \begin{cases} 0_L & \text{if } u \notin A \\ 1_L & \text{if } u \in A \end{cases}. $$

In 2014, Azam et al. \cite{[29]} initiated the concept of $\beta_{F_L}$-admissible for a pair of $L$-fuzzy mappings and exploited it to establish a common $L$-fuzzy fixed point theorem.

Definition 1.5. \cite{[29]} Let $S_1$ be an arbitrary set, $S_2$ be a metric space. A mapping $Q$ is said to be an $L$-fuzzy mapping if $Q$ is a mapping from $S_1$ into $\Im_{L}(S_2)$. An $L$-fuzzy mapping $Q$ is a $L$-fuzzy subset on $S_1 \times S_2$ with membership function $Q(u)(v)$. The function $Q(u)(v)$ is the grade of membership of $v$ in $Q(u)$.

Definition 1.6. \cite{[29]} Let $(S, \sigma)$ be a metric space and $P, Q$ be $L$-fuzzy mappings from $S$ into $\Im_{L}(S)$. A point $z \in S$ is called a $L$-fuzzy fixed point of $Q$ if $u^* \in [Q u^*]_{\alpha_L}$, where $\alpha_L \in L \setminus \{ 0_L \}$. The point $u^* \in S$ is called a common $L$-fuzzy fixed point of $P$ and $Q$ if $u^* \in [P u^*]_{\alpha_L} \cap [Q u^*]_{\alpha_L}$. When $\alpha_L = 1_L$, it is called a common fixed point of $L$-fuzzy mappings.

In 2015, Jleli et al. \cite{[24]} gave the notion of $\Theta$-contractions and proved some new fixed point results for such contractions in the setting of generalized metric spaces.

Definition 1.7. Let $\Theta : (0, \infty) \to (1, \infty)$ be a function satisfying:

$(\Theta_1)$ $\Theta$ is nondecreasing;

$(\Theta_2)$ for each sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n \to \infty} \Theta(\alpha_n) = 1$ if and only if $\lim_{n \to \infty} (\alpha_n) = 0;

$(\Theta_3)$ there exists $0 < r < 1$ and $l \in (0, \infty]$ such that $\lim_{n \to 0^+} \frac{\Theta(\alpha) - 1}{\alpha^r} = l$.

A mapping $P : S \to S$ is said to be $\Theta$-contraction if there exist the function $\Theta$ satisfying $(\Theta_1)$-$(\Theta_3)$ and a constant $k \in (0, 1)$ such that for all $u, v \in S$,

$$\sigma(P u, P v) > 0 \implies \Theta(\sigma(P u, P v)) \leq [\Theta(\sigma(u, v))]^k.$$  \hspace{1cm} (1.1)
Theorem 1.8. \[24\] Let \((S, \sigma)\) be a complete metric space and \(\mathcal{P} : S \to S\) be a \(\Theta\)-contraction, then \(\mathcal{P}\) has a unique fixed point.

They demonstrated that any Banach contraction is a specific case of \(\Theta\)-contraction while there are \(\Theta\)-contractions which are not Banach contractions. We express by \(\Psi\) the set of all functions \(\Theta : (0, \infty) \to (1, \infty)\) satisfying the above assertions \((\Theta_1)-(\Theta_3)\), consistent with Jleli et al. \[24\].

Later on Altune et al.\[18\] modified the above definitions by adding a general condition \((\Theta_4)\) which is given as follows.

\((\Theta_4)\) \(\Theta(\inf A) = \inf \Theta(A)\) for all \(A \subset (0, \infty)\) with \(\inf A > 0\).

Following Altune et al.\[18\], we represent the set of all continuous functions \(\Theta : \mathbb{R}^+ \to \mathbb{R}\) satisfying \((\Theta_1) - (\Theta_4)\) conditions by \(\mathcal{F}\).

For more details on \(\Theta\)-contraction, we refer the reader to \[3, 27\].

For the sake of convenience, we first state the following lemma for subsequent use in the next section.

Let \((S, \sigma)\) be a metric space and \(CB(S)\) be the family of nonempty, closed and bounded subsets of \(S\). For \(A, B \in CB(S)\), define

\[
\mathcal{H}(A, B) = \max \left\{ \sup_{a \in A} \sigma(a, B), \sup_{b \in B} \sigma(b, A) \right\}
\]

where

\[
\sigma(u, A) = \inf_{v \in A} \sigma(u, v).
\]

Lemma 1.9. \[29\] Let \((S, \sigma)\) be a metric space and \(A, B \in CB(S)\), then for each \(a \in A\),

\[
\sigma(a, B) \leq \mathcal{H}(A, B).
\]

In this paper, we obtain common \(L\)-fuzzy fixed point theorems for almost \(\Theta\)-contraction in the setting of complete metric spaces. A significant example is also given to illustrate the validity of main result.

2. Main Results

In this way, we state and prove a common fixed point theorem for \(L\)-fuzzy mappings.

Theorem 2.1. Let \((S, \sigma)\) be a complete metric space and \(\{\mathcal{P}, \mathcal{Q}\}\) be a pair of \(L\)-fuzzy mappings from \(S\) into \(\mathcal{S}_L(S)\) and for each \(\alpha_L \in L \setminus \{0\}\), \([\mathcal{P}u]_{\alpha_L(u)}\), \([\mathcal{Q}v]_{\alpha_L(v)}\) are nonempty closed bounded subsets of \(S\). If there exist some \(\Theta \in \mathcal{F}\), \(k \in (0, 1)\) and \(L \geq 0\) such that

\[
\mathcal{H}\left([\mathcal{P}u]_{\alpha_L(u)}, [\mathcal{Q}v]_{\alpha_L(v)}\right) > 0 \implies \Theta\left(\mathcal{H}\left([\mathcal{P}u]_{\alpha_L(u)}, [\mathcal{Q}v]_{\alpha_L(v)}\right)\right) \leq \Theta(\sigma(u, v))^k + Lm(u, v)
\]

for all \(u, v \in S\), where

\[
m(u, v) = \min \left\{ \sigma(u, [\mathcal{P}u]_{\alpha_L(u)}), \sigma(v, [\mathcal{Q}v]_{\alpha_L(v)}), \sigma(u, [\mathcal{Q}v]_{\alpha_L(v)}), \sigma(v, [\mathcal{P}u]_{\alpha_L(u)}) \right\}.
\]

Then \(\mathcal{P}\) and \(\mathcal{Q}\) have a common \(L\)-fuzzy fixed point.
Thus we have

\[
\Theta(\sigma(u_1, [Qu_1]_{\alpha_L(u_1)}) \leq \Theta(\mathcal{H}([Pu_0]_{\alpha_L(u_0)}, [Qu_1]_{\alpha_L(u_1)})) \leq \Theta(\sigma(u_0, u_1))^k + Lm(u_0, u_1)
\]

(2.3)

where

\[
m(u_0, u_1) = \min \left\{ \sigma(u_0, [Pu_0]_{\alpha_L(u_0)}), \sigma(u_1, [Qu_1]_{\alpha_L(u_1)}), \sigma(u_0, [Qu_1]_{\alpha_L(u_1)}), \sigma(u_1, [Pu_0]_{\alpha_L(u_0)}) \right\}.
\]

From (Θ), we know that

\[
\Theta(\sigma(u_1, [Qu_1]_{\alpha_L(u_1)})) = \inf_{v \in [Qu_1]_{\alpha_L(u_1)}} \Theta(\sigma(u_1, v)).
\]

Thus from (2.3), we get

\[
\inf_{v \in [Qu_1]_{\alpha_L(u_1)}} \Theta(\sigma(u_1, v)) \leq \Theta(\sigma(u_0, u_1))^k + \min \left\{ \sigma(u_0, [Pu_0]_{\alpha_L(u_0)}), \sigma(u_1, [Qu_1]_{\alpha_L(u_1)}), \sigma(u_0, [Qu_1]_{\alpha_L(u_1)}), \sigma(u_1, [Pu_0]_{\alpha_L(u_0)}) \right\}
\]

(2.4)

Then, from (2.4), there exists \(u_2 \in [Qu_1]_{\alpha_L(u_1)}\) such that

\[
\Theta(\sigma(u_1, u_2)) \leq \Theta(\sigma(u_0, u_1))^k + \min \{ \sigma(u_0, u_1), \sigma(u_1, u_2), \sigma(u_0, u_2), \sigma(u_1, u_1) \}.
\]

Thus we have

\[
\Theta(\sigma(u_1, u_2)) \leq [\Theta(\sigma(u_0, u_1))]^k.
\]

(2.5)

For this \(u_2\) there exists \(\alpha_L(u_2) \in L \setminus \{\theta_e\}\) such that \([Pu_2]_{\alpha_L(u_2)}\) is a nonempty closed bounded subset of \(S\). By Lemma 1.9 (Θ) and (2.1), we have

\[
\Theta\left(\sigma\left(u_2, [Pu_2]_{\alpha_L(u_2)}\right)\right) \leq \Theta(\mathcal{H}( [Qu_1]_{\alpha_L(u_1)}, [Pu_2]_{\alpha_L(u_2)})) = \Theta(\mathcal{H}( [Pu_2]_{\alpha_L(u_2)}, [Qu_1]_{\alpha_L(u_1)})) \\
\leq [\Theta(\sigma(u_2, u_1))]^k + Lm(u_2, u_1),
\]

thus we get

\[
\Theta\left(\sigma\left(u_2, [Pu_2]_{\alpha_L(u_2)}\right)\right) \leq [\Theta(\sigma(u_2, u_1))]^k + Lm(u_2, u_1)
\]

(2.6)

where

\[
m(u_2, u_1) = \min \left\{ \sigma(u_2, [Pu_2]_{\alpha_L(u_2)}), \sigma(u_1, [Qu_1]_{\alpha_L(u_1)}), \sigma(u_2, [Qu_1]_{\alpha_L(u_1)}), \sigma(u_1, [Pu_2]_{\alpha_L(u_2)}) \right\}
\]

which further implies that

\[
\Theta\left(\sigma\left(u_2, [Pu_2]_{\alpha_L(u_2)}\right)\right) \leq [\Theta(\sigma(u_1, u_2))]^k + \min \left\{ \sigma(u_2, [Pu_2]_{\alpha_L(u_2)}), \sigma(u_1, [Qu_1]_{\alpha_L(u_1)}), \sigma(u_2, [Qu_1]_{\alpha_L(u_1)}), \sigma(u_1, [Pu_2]_{\alpha_L(u_2)}) \right\}.
\]

(2.7)

From (Θ), we know that

\[
\Theta\left[\sigma\left(u_2, [Pu_2]_{\alpha_L(u_2)}\right)\right] = \inf_{v_1 \in [Pu_2]_{\alpha_L(u_2)}} \Theta(\sigma(u_2, v_1)).
\]
\[
\inf_{v_1 \in [\mathcal{P}u_2]_{\alpha_L(u_2)}} \Theta(u_2, v_1) \leq \Theta(u_1, u_2)^k + \min \left\{ \sigma(u_2, [\mathcal{P}u_2]_{\alpha_L(u_2)}), \sigma(u_1, [\mathcal{Q}u_1]_{\alpha_L(u_1)}), \sigma(u_1, [\mathcal{P}u_2]_{\alpha_L(u_2)}) \right\}.
\]

Then, from (2.8), there exists \( u_3 \in [\mathcal{P}u_2]_{\alpha_L(u_2)} \) such that
\[
\Theta(u_2, u_3) \leq [\Theta(u_1, u_2)]^k + \min \{ \sigma(u_2, u_3), \sigma(u_1, u_2), \sigma(u_2, u_2), \sigma(u_1, u_3) \}.
\]

Thus we have
\[
\Theta(u_2, u_3) \leq [\Theta(u_1, u_2)]^k.
\]

So, continuing recursively, we obtain a sequence \( \{u_n\} \) in \( \mathcal{S} \) such that \( u_{2n+1} \in [\mathcal{P}u_{2n}]_{\alpha_L(u_{2n})} \) and \( u_{2n+2} \in [\mathcal{Q}u_{2n+1}]_{\alpha_L(u_{2n+1})} \) and
\[
\Theta(u_{2n+1}, u_{2n+2}) \leq [\Theta(u_{2n}, u_{2n+1})]^k.
\]

and
\[
\Theta(u_{2n+2}, u_{2n+3}) \leq [\Theta(u_{2n+1}, u_{2n+2})]^k.
\]

for all \( n \in \mathbb{N} \). From (2.10) and (2.11), we have
\[
\Theta(u_n, u_{n+1}) \leq [\Theta(u_{n-1}, u_n)]^k
\]

which further implies that
\[
\Theta(u_n, u_{n+1}) \leq [\Theta(u_{n-1}, u_n)]^k \leq [\Theta(u_{n-2}, u_{n-1})]^k \leq \ldots \leq [\Theta(u_0, u_1)]^k
\]

for all \( n \in \mathbb{N} \). Since \( \Theta \in \mathcal{F} \), so by taking limit as \( n \to \infty \) in (2.13) we have,
\[
\lim_{n \to \infty} \Theta(u_n, u_{n+1}) = 1
\]

which implies that
\[
\lim_{n \to \infty} u_n = u_{n+1} = 0
\]

by (\( \Theta_2 \)). From the condition (\( \Theta_3 \)), there exist \( 0 < r < 1 \) and \( l \in (0, \infty] \) such that
\[
\lim_{n \to \infty} \frac{\Theta(u_n, u_{n+1}) - 1}{\sigma(u_n, u_{n+1})^r} = l.
\]

Suppose that \( l < \infty \). In this case, let \( \beta = \frac{l}{2} > 0 \). From the definition of the limit, there exists \( n_0 \in \mathbb{N} \) such that
\[
\left| \frac{\Theta(u_n, u_{n+1}) - 1}{\sigma(u_n, u_{n+1})^r} - l \right| \leq \beta
\]

for all \( n > n_0 \). This implies that
\[
\frac{\Theta(u_n, u_{n+1}) - 1}{\sigma(u_n, u_{n+1})^r} \geq l - \beta = \frac{l}{2} = \beta
\]

for all \( n > n_0 \). Then
\[
n\sigma(u_n, u_{n+1})^r \leq \alpha n[\Theta(u_n, u_{n+1}) - 1]
\]
for all $n > n_0$, where $\alpha = \frac{1}{2}$. Now we suppose that $l = \infty$. Let $\beta > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\beta \leq \Theta(\sigma(u_n, u_{n+1})) - 1$$

for all $n > n_0$. This implies that

$$n\sigma(u_n, u_{n+1})^r \leq \omega n[\Theta(\sigma(u_n, u_{n+1})) - 1]$$

for all $n > n_0$, where $\alpha = \frac{1}{2}$. Thus, in all cases, there exist $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that

$$n\sigma(u_n, u_{n+1})^r \leq \omega n[\Theta(\sigma(u_n, u_{n+1})) - 1]$$

(2.18)

for all $n > n_0$. Thus by (2.13) and (2.18), we get

$$n\sigma(u_n, u_{n+1})^r \leq \omega n[|\Theta(u_n, u_1)|]^r - 1).$$

(2.19)

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n \to \infty} n\sigma(u_n, u_{n+1})^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$\sigma(u_n, u_{n+1}) \leq \frac{1}{n^{1/r}}$$

(2.20)

for all $n > n_1$. Now we prove that $\{u_n\}$ is a Cauchy sequence. For $m > n > n_1$ we have

$$\sigma(u_n, u_m) \leq \sum_{i=n}^{m-1} \sigma(u_i, u_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/r}} \leq \sum_{i=1}^{\infty} \frac{1}{i^{1/r}}.$$  

(2.21)

Since, $0 < r < 1$, then $\sum_{i=1}^{\infty} \frac{1}{i^{1/r}}$ converges. Therefore, $\sigma(u_n, u_m) \to 0$ as $m, n \to \infty$. Thus we proved that $\{u_n\}$ is a Cauchy sequence in $(\mathcal{S}, \sigma)$. The completeness of $(\mathcal{S}, \sigma)$ ensures that there exists $u^* \in \mathcal{S}$ such that, $\lim_{n \to \infty} u_n = u^*$. Now, we prove that $u^* \in [Qu^*]_{\alpha_L(u^*)}$. We suppose on the contrary that $u^* \notin [Qu^*]_{\alpha_L(u^*)}$, then there exist a $n_0 \in \mathbb{N}$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\sigma(u_{2n_k+1}, [Qu^*]_{\alpha_L(u^*)}) > 0$ for all $n_k \geq n_0$. Since $\sigma(u_{2n_k+1}, [Qu^*]_{\alpha_L(u^*)}) > 0$ for all $n_k \geq n_0$, so by $(\Theta_1)$, we have

$$1 < \Theta \left[ \sigma(u_{2n_k+1}, [Qu^*]_{\alpha_L(u^*)}) \right] \leq \Theta \left[ \mathcal{H}([Pu_{2n_k}]_{\alpha_L(u_{2n_k})}, [Qu^*]_{\alpha_L(u^*)}) \right]$$

$$\leq \left[ \Theta(\sigma(u_{2n_k}, u^*)) \right]^k + \min \left\{ \sigma(u_{2n_k}, [Pu_{2n_k}]_{\alpha_L(u_{2n_k})}), \sigma(u^*, [Qu^*]_{\alpha_L(u^*)}), \sigma(u^*_{\alpha_L(u^*)}), \sigma(u^*, Pu_{2n_k})_{\alpha_L(u_{2n_k})} \right\}$$

$$\leq \left[ \Theta(\sigma(u_{2n_k}, u^*)) \right]^k + \min \left\{ \sigma(u_{2n_k}, u_{2n_k+1}), \sigma(u^*, [Qu^*]_{\alpha_L(u^*)}), \sigma(u^*, u_{2n_k+1}) \right\}.$$  

Letting $n \to \infty$, in above inequality and using the continuity of $\Theta$, we have

$$1 < \Theta \left[ \sigma(u^*, [Qu^*]_{\alpha_L(u^*)}) \right] \leq 1$$

which is a contradiction. Hence $u^* \in [Qu^*]_{\alpha_L(u^*)}$. Similarly, one can easily prove that $u^* \in [Pu^*]_{\alpha_L(u^*)}$. Thus $u^* \in [Pu^*]_{\alpha_L(u^*)} \cap [Qu^*]_{\alpha_L(u^*)}$. $\square$

The following result is a direct consequence of above theorem by taking $L = 0$. 

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Corollary 2.2. Let $(S, \sigma)$ be a complete metric space and $\{\mathcal{P}, \mathcal{Q}\}$ be a pair of $L$-fuzzy mappings from $S$ into $\mathcal{S}_L(S)$ and for each $\alpha \in L \setminus \{0_L\}$, $[\mathcal{P}u]_{\alpha_L(u)}$, $[\mathcal{Q}v]_{\alpha_L(v)}$ are nonempty closed bounded subsets of $S$. If there exist some $\Theta \in F$ and $k \in (0, 1)$ such that
\[
\mathcal{H} ([\mathcal{P}u]_{\alpha_L(u)} , [\mathcal{Q}v]_{\alpha_L(v)}) > 0 \implies \Theta (\mathcal{H} ([\mathcal{P}u]_{\alpha_L(u)} , [\mathcal{Q}v]_{\alpha_L(v)})) \leq \Theta (\sigma (u, v))^k
\]
for all $u, v \in S$. Then $\mathcal{P}$ and $\mathcal{Q}$ have a common $L$-fuzzy fixed point.

If we take a single $L$-fuzzy mapping, we get the following result.

Corollary 2.3. Let $(S, \sigma)$ be a complete metric space and let $\mathcal{P}$ be $L$-fuzzy mapping from $S$ into $\mathcal{S}_L(S)$ and for each $\alpha \in L \setminus \{0_L\}$, $[\mathcal{P}u]_{\alpha_L(u)}$, $[\mathcal{P}v]_{\alpha_L(v)}$ are nonempty closed bounded subsets of $S$. If there exist some $\Theta \in F$, $k \in (0, 1)$ and $L \geq 0$ such that
\[
\Theta (\mathcal{H} ([\mathcal{P}u]_{\alpha_L(u)} , [\mathcal{P}v]_{\alpha_L(v)})) \leq \Theta (\sigma (u, v))^k + L m(u, v)
\]
where
\[
m(u, v) = \min \{ \sigma (u, [\mathcal{P}u]_{\alpha_L(u)}), \sigma (v, [\mathcal{P}v]_{\alpha_L(v)}), \sigma (u, [\mathcal{P}v]_{\alpha_L(v)}), \sigma (v, [\mathcal{P}u]_{\alpha_L(u)}) \}
\]
for all $u, v \in S$ with $\mathcal{H} ([\mathcal{P}u]_{\alpha_L(u)} , [\mathcal{P}v]_{\alpha_L(v)}) > 0$. Then $\mathcal{P}$ has an $L$-fuzzy fixed point.

Corollary 2.4. Let $(S, \sigma)$ be a complete metric space and let $\mathcal{P}$ be $L$-fuzzy mapping from $S$ into $\mathcal{S}_L(S)$ and for each $\alpha \in L \setminus \{0_L\}$, $[\mathcal{P}u]_{\alpha_L(u)}$, $[\mathcal{P}v]_{\alpha_L(v)}$ are nonempty closed bounded subsets of $S$. If there exist some $\Theta \in F$ and $k \in (0, 1)$ such that
\[
\Theta (\mathcal{H} ([\mathcal{P}u]_{\alpha_L(u)} , [\mathcal{P}v]_{\alpha_L(v)})) \leq \Theta (\sigma (u, v))^k
\]
for all $u, v \in S$ with $\mathcal{H} ([\mathcal{P}u]_{\alpha_L(u)} , [\mathcal{P}v]_{\alpha_L(v)}) > 0$. Then $\mathcal{P}$ has an $L$-fuzzy fixed point.

$L$-fuzzy fixed point results are real generalization of fuzzy fixed point theorems. It can be shown in the following Theorem.

Theorem 2.5. Let $(S, \sigma)$ be a complete metric space and let $\mathcal{P}, \mathcal{Q}$ be fuzzy mappings from $S$ into $\mathcal{S}(S)$ and for each $\alpha (u) \in (0, 1)$, $[\mathcal{P}u]_{\alpha(u)}$, $[\mathcal{Q}v]_{\alpha(v)}$ are nonempty closed bounded subsets of $S$. If there exist some $\Theta \in F$, $k \in (0, 1)$ and $L \geq 0$ such that
\[
\Theta (\mathcal{H} ([\mathcal{P}u]_{\alpha(u)} , [\mathcal{Q}v]_{\alpha(v)})) \leq \Theta (\sigma (u, v))^k + L m(u, v)
\]
where
\[
m(u, v) = \min \{ \sigma (u, [\mathcal{P}u]_{\alpha(u)}), \sigma (v, [\mathcal{Q}v]_{\alpha(v)}), \sigma (u, [\mathcal{Q}v]_{\alpha(v)}), \sigma (v, [\mathcal{P}u]_{\alpha(u)}) \}
\]
for all $u, v \in S$ with $\mathcal{H} ([\mathcal{P}u]_{\alpha(u)} , [\mathcal{Q}v]_{\alpha(v)}) > 0$. Then $\mathcal{P}$ and $\mathcal{Q}$ have a common fuzzy fixed point.
**Proof.** Consider an $L$-fuzzy mapping $\mathcal{J} : \mathcal{S} \to \mathcal{S}_L(\mathcal{S})$ defined by

$$\mathcal{J}u = \chi_{\mathcal{P}(u)}.$$  

Then for $\alpha_L \in L \setminus \{0_L\}$, we have

$$[\mathcal{J}u]_{\alpha_L(u)} = \mathcal{P}u.$$

Hence by Theorem 2.1 we follow the result. $\square$

Taking $L = 0$ in above result, we have following corollary.

**Corollary 2.6.** Let $(\mathcal{S}, \sigma)$ be a complete metric space and let $\mathcal{P}, \mathcal{Q}$ be fuzzy mappings from $\mathcal{S}$ into $\mathcal{S}(\mathcal{S})$ and for each $\alpha(u) \in (0, 1]$, $[\mathcal{P}u]_{\alpha(u)}$, $[\mathcal{Q}v]_{\alpha(v)}$ are nonempty closed bounded subsets of $\mathcal{S}$. If there exist some $\Theta \in F$ and $k \in (0, 1)$ such that

$$\Theta \left( \mathcal{H} \left( [\mathcal{P}u]_{\alpha(u)}, [\mathcal{Q}v]_{\alpha(v)} \right) \right) \leq \Theta(\sigma(u, v))^k$$

for all $u, v \in \mathcal{S}$ with $\mathcal{H} \left( [\mathcal{P}u]_{\alpha(u)}, [\mathcal{Q}v]_{\alpha(v)} \right) > 0$. Then $\mathcal{P}$ and $\mathcal{Q}$ have a common fuzzy fixed point.

**Example 2.7.** Let $\mathcal{S} = [0, 1]$, $\sigma(u, v) = |u - v|$, whenever $u, v \in \mathcal{S}$. Then $(\mathcal{S}, \sigma)$ is a complete metric space. Let $L = \{\eta, \omega, \tau, \kappa\}$ with $\eta \preceq_L \omega \preceq_L \kappa$ and $\eta \preceq_L \tau \preceq_L \kappa$, where $\omega$ and $\tau$ are not comparable, then $(L, \preceq_L)$ is a complete distributive lattice. Define $\mathcal{P}, \mathcal{Q} : \mathcal{S} \to \mathcal{S}_L(\mathcal{S})$ as follows:

$$\mathcal{P}(u)(t) = \begin{cases} 
\kappa & \text{if } 0 \leq t \leq \frac{1}{6}, \\
\omega & \text{if } \frac{1}{6} < t \leq \frac{1}{3}, \\
\tau & \text{if } \frac{1}{3} < t \leq \frac{1}{2}, \\
\eta & \text{if } \frac{1}{2} < t \leq 1 
\end{cases},$$

$$\mathcal{Q}(u)(t) = \begin{cases} 
\kappa & \text{if } 0 \leq t \leq \frac{1}{12}, \\
\eta & \text{if } \frac{1}{12} < t \leq \frac{1}{6}, \\
\omega & \text{if } \frac{1}{6} < t \leq \frac{1}{4}, \\
\tau & \text{if } \frac{1}{4} < t \leq 1 
\end{cases}.$$

Let $\Theta(t) = e^{\sqrt{t}} \in F$ for $t > 0$. And for all $u \in \mathcal{S}$, there exists $\alpha_L(u) = \kappa$, such that

$$[\mathcal{P}u]_{\alpha_L(u)} = \left[0, \frac{u}{6}\right], \quad [\mathcal{Q}u]_{\alpha_L(u)} = \left[0, \frac{u}{12}\right].$$

and all conditions of Theorem 2.1 are satisfied. And 0 is a common fixed point of $\mathcal{P}$ and $\mathcal{Q}$.

3. Applications to domain of words

Suppose $\Omega$ be a nonempty alphabet and $\Omega^\infty$ be the collection of all finite and infinite sequences ("words") over $\Omega$, where we adopt the convention that the empty sequence $\emptyset$ is an element of $\Omega^\infty$. Moreover, on $\Omega^\infty$, we consider the prefix order $\preceq$ given by:

$$u \preceq v \quad \text{if and only if} \quad u \text{ is a prefix of } v.$$  

For each nonempty $u \in \Omega^\infty$ denote by $l(u)$ the length of $u$. Then $l(u) \in [0, \infty]$, whenever $u \neq \emptyset$ and $l(\emptyset) = 0$. For each $u, v \in \Omega^\infty$, let $u \sqcup v$ be the common prefix of $u$ and $v$. Clearly, $u = v$ if and only if $u \preceq v$ and $v \preceq u$ and $l(u) = l(v)$. Then, the the Baire metric $\sigma_\preceq$ is defined on $\Omega^\infty \times \Omega^\infty$ by

$$\sigma_\preceq(u, v) = \begin{cases} 
0, & \text{if } u = v \\
2^{-l(u \sqcup v)}, & \text{otherwise} 
\end{cases}$$
such that the metric space $(Ω^∞, σ_∞)$ is complete. Certainly, we assign to the average case time complexity analysis of the Quicksort divide-and-conquer sorting algorithm in [32].

Exactly, we deal with the following recurrence relation:

$$R(1) = 0 \text{ and } R(n) = \frac{2(n-1)}{n} + \frac{n+1}{n} R(n-1), \quad n \geq 2. \quad (3.1)$$

Consider as an alphabet $Ω$ the set of nonnegative real numbers, i.e., $Ω = \mathbb{R}^+$. We accomplice to $R$ the functional $Φ : Ω^∞ → Ω^∞$ given by

$$Φ(u)_1 = R(1)$$

and

$$Φ(u)_n = \frac{2(n-1)}{n} + \frac{n+1}{n} u_{n-1}$$

for all $n \geq 2$ (if $u ∈ Ω^∞$ has length $n < ∞$, we write $u := u_1u_2...$). It follows by the construction that $l(Φ(u)) = l(u) + 1$ for all $u ∈ Ω^∞$ and $l(Φ(u)) = +∞$ whenever $l(u) = +∞$. We will prove that the functional $Φ$ has an $L$-fuzzy fixed point by an application of [2.4]. Let $P : Ω^∞ → Ω^∞$ be the $L$-fuzzy mapping given by

$$P_u = (Φ(u))_{α_L} \text{ for all } u ∈ Ω^∞ \text{ and } α_L ∈ L\{0\}.$$ and analyze the following two cases:

**Case 01:** If $u = v$, then we have

$$H_Φ((Φ(u))_{α_L}, (Φ(v))_{α_L}) = 0 = σ_∞(u, u).$$

**Case 02:** If $u ≠ v$, then we write

$$H_Φ((Φ(u))_{α_L}, (Φ(v))_{α_L}) = σ_∞((Φ(u))_{α_L}, (Φ(v))_{α_L}) = 2^{-(l(Φ(u))_{α_L} ∨ (Φ(v))_{α_L})}$$

$$≤ 2^{-l(Φ(uv))_{α_L}} = 2^{-l(u^v)+1}$$

$$= \frac{1}{2} 2^{-l(u^v)} = (\frac{1}{√2})^2 σ_∞(u, v).$$

It is immediate to achieve that all the assertions of the Corollary [2.4] are satisfied with $Θ(t) = e^{√t}$ and $k = \frac{1}{√2}$. Consequently, the $L$-fuzzy mapping $P$ has a $L$-fuzzy fixed point $u = u_1u_2... ∈ Ω^∞$ that is, $u ∈ (P_u)_{α_L}$. Also, in the light of the definition of $P$, $u$ is a fixed point of $Φ$, and hence, $u$ solves the recurrence relation (3.1). We have

$$u_1 = 0,$$

$$u_n = \frac{2(n-1)}{n} + \frac{n+1}{n} u_{n-1}, \quad n ≥ 2.$$}

4. Conclusions

We proved some common $L$-fuzzy fixed point results for almost $Θ$-contraction in the setting of complete metric spaces by using the notion of $L$-fuzzy mappings. We also presented an application to domain of words which shows the significance of the investigation of this paper.

**Conflict of Interests**

The authors declare that they have no competing interests.
Authors’ Contribution
All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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