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Common Fixed Point Theorems with Applications to Theoretical Computer Science

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Abstract

Owing to the notion of L-fuzzy mapping, we establish some common L-fuzzy fixed point results for almost Θ-contraction in the setting of complete metric spaces. An application to theoretical computer science is also provided to show the significance of the investigations.

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1. Introduction and preliminaries

Answering real-world problems becomes evidently uncomplicated with the initiation of fuzzy set theory in 1965 by Zadeh [35], as it helps in making the explanation of obscurity and inaccuracy fair and more accurate. subsequently, Goguen [17] modified this concept to L-fuzzy set theory by replacing the interval $[0,1]$ in 1967. There are fundamentally two perceptive of the meaning of L, one is when L is a complete lattice equipped with a multiplication $*$ operator satisfying certain assumptions as shown in the basic paper [17] and the second perceptive of the meaning of L is that L is a completely distributive complete lattice with an order-reversing involution .

Definition 1.1. *[17] A partially ordered set* $(L, \precsim_L)$ *is called*

$i)$ a lattice, if $a_1 \vee a_2 \in L$, $a_1 \wedge a_2 \in L$ for each $a_1, a_2 \in L$.
$ii)$ a complete lattice, if $\vee A \in L$, $\wedge A \in L$ for any $A \subseteq L$.
$iii)$ distributive lattice if $a_1 \vee (a_2 \wedge a_3) = (a_1 \vee a_2) \wedge (a_1 \vee a_3)$, $a_1 \wedge (a_2 \vee a_3) = (a_1 \wedge a_2) \vee (a_1 \wedge a_3)$ for any $a_1, a_2, a_3 \in L$.

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Definition 1.2. *[17] Let L be a lattice with top element 1_L and bottom element 0_L and let $a_1, a_2 \in L$. Then a_2 is said to be a complement of a_1, if $a_1 \vee a_2 = 1_L$, and $a_1 \wedge a_2 = 0_L$. If $a \in L$ has a complement element, then it is unique. It is denoted by $\acute{a}$.*

Definition 1.3. *[17] A $L-$fuzzy set A on a nonempty set $\mathcal{S}$ is a function $A : \mathcal{S} \to L$, where L is complete distributive lattice with 1_L and 0_L.*

Remark 1.4. *An $L-$fuzzy set is a fuzzy set if $L = [0, 1]$, so the family of $L-$fuzzy sets is larger than the family of fuzzy sets.*

The α_L-level set of $L-$fuzzy set A, is designated by A_{α_L}, and is given in this way.

$$A_{\alpha_L} = \{u : \alpha_L \precsim_L A(u)\} \text{ if } \alpha_L \in L \backslash \{0_L\},$$

$$A_{0_L} = \overline{\{u : 0_L \precsim_L A(u)\}}.$$

Here $cl(B)$ stands for the closure of the set B.

The characteristic function of a L-fuzzy set A is denoted by χ_{L_A} and is defined as follows:

$$\chi_{L_A} := \begin{cases} 0_L & \text{if } u \notin A \\ 1_L & \text{if } u \in A \end{cases}.$$

In 2014, Azam et al. [29] initiated the concept of β_{F_L}-admissible for a pair of L-fuzzy mappings and exploited it to establish a common L-fuzzy fixed point theorem.

Definition 1.5. ***[29]** Let $\mathcal{S}_1$ be an arbitrary set, $\mathcal{S}_2$ be a metric space. A mapping $\mathcal{Q}$ is said to be an $L-$fuzzy mapping if $\mathcal{Q}$ is a mapping from $\mathcal{S}_1$ into $\Im_L(\mathcal{S}_2)$. An $L-$fuzzy mapping $\mathcal{Q}$ is a $L-$fuzzy subset on $\mathcal{S}_1 \times \mathcal{S}_2$ with membership function $\mathcal{Q}(u)(v)$. The function $\mathcal{Q}(u)(v)$ is the grade of membership of v in $\mathcal{Q}(u)$.*

Definition 1.6. *[29] Let $(\mathcal{S}, \sigma)$ be a metric space and $\mathcal{P}, \mathcal{Q}$ be $L-$fuzzy mappings from $\mathcal{S}$ into $\Im_L(\mathcal{S})$. A point $z \in \mathcal{S}$ is called a $L-$fuzzy fixed point of $\mathcal{Q}$ if $u^* \in [\mathcal{Q}u^*]_{\alpha_L}$, where $\alpha_L \in L \backslash \{0_L\}$. The point $u^* \in \mathcal{S}$ is called a common $L-$fuzzy fixed point of $\mathcal{P}$ and $\mathcal{Q}$ if $u^* \in [\mathcal{P}u^*]_{\alpha_L} \cap [\mathcal{Q}u^*]_{\alpha_L}$. When $\alpha_L = 1_L$, it is called a common fixed point of $L-$fuzzy mappings.*

In 2015, Jleli et al. [24]gave the notion of Θ-contractions and proved some new fixed point results for such contractions in the setting of generalized metric spaces.

Definition 1.7. *Let $\Theta : (0, \infty) \to (1, \infty)$ be a function satisfying:*

(Θ_1) Θ is nondecreasing;

(Θ_2) for each sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \Theta(\alpha_n) = 1$ if and only if $\lim_{n\to\infty}(\alpha_n) = 0$;

(Θ_3) there exists $0 < r < 1$ and $l \in (0, \infty]$ such that $\lim_{\alpha\to 0^+} \frac{\Theta(\alpha)-1}{\alpha^r} = l$.

A mapping $\mathcal{P} : \mathcal{S} \to \mathcal{S}$ is said to be Θ-contraction if there exist the function Θ satisfying (Θ_1)-(Θ_3) and a constant $k \in (0, 1)$ such that for all $u, v \in \mathcal{S}$,

$$\sigma(\mathcal{P}u, \mathcal{P}v) > 0 \Longrightarrow \Theta(\sigma(\mathcal{P}u, \mathcal{P}v)) \leq [\Theta(\sigma(u, v))]^k. \tag{1.1}$$

Theorem 1.8. *[24] Let $(\mathcal{S},\sigma)$ be a complete metric space and $\mathcal{P}:\mathcal{S}\to\mathcal{S}$ be a Θ-contraction, then $\mathcal{P}$ has a unique fixed point.*

They demonstrated that any Banach contraction is a specific case of Θ-contraction while there are Θ-contractions which are not Banach contractions. We express by Ψ the set of all functions $\Theta:(0,\infty)\to(1,\infty)$ satisfying the above assertions (Θ_1)-(Θ_3), consistent with Jleli et al. [24],.

Later on Altune et al.[18] modified the above definitions by adding a general condition (Θ_4) which is given as follows.

(Θ_4) $\Theta(\inf A)=\inf\Theta(A)$ for all $A\subset(0,\infty)$ with $\inf A>0$.

Following Altune et al.[18], we represent the set of all continuous functions $\Theta:\mathbb{R}^+\to\mathbb{R}$ satisfying $(\Theta_1)-(\Theta_4)$ conditions by $\digamma$.

For more details on Θ-contraction, we refer the reader to [3, 27].

For the sake of convenience, we first state the following lemma for subsequent use in the next section.

Let $(\mathcal{S},\sigma)$ be a metric space and $CB(\mathcal{S})$ be the family of nonempty, closed and bounded subsets of $\mathcal{S}$. For$A,B\in CB(\mathcal{S})$, define

$$\mathcal{H}(A,B)=\max\left\{\sup_{a\in A}\sigma(a,B),\sup_{b\in B}\sigma(b,A)\right\}$$

where

$$\sigma(u,A)=\inf_{v\in A}\sigma(u,v).$$

Lemma 1.9. *[29]Let $(\mathcal{S},\sigma)$ be a metric space and $A,B\in CB(\mathcal{S})$, then for each $a\in A$,*

$$\sigma(a,B)\le\mathcal{H}(A,B).$$

In this paper, we obtain common L-fuzzy fixed point theorems for almost Θ-contraction in the setting of complete metric spaces. A significant example is also given to illustrate the validity of main result.

2. Main Results

In this way, we state and prove a common fixed point theorem for L-fuzzy mappings.

Theorem 2.1. *Let $(\mathcal{S},\sigma)$ be a complete metric space and $\{\mathcal{P},\mathcal{Q}\}$ be a pair of L-fuzzy mappings from $\mathcal{S}$ into $\Im_L(\mathcal{S})$ and for each $\alpha_L\in L\backslash\{0_L\}$, $[\mathcal{P}u]_{\alpha_L(u)}$, $[\mathcal{Q}v]_{\alpha_L(v)}$ are nonempty closed bounded subsets of $\mathcal{S}$. If there exist some $\Theta\in\digamma$, $k\in(0,1)$ and $L\ge 0$ such that*

$$\mathcal{H}\left([\mathcal{P}u]_{\alpha_L(u)},[\mathcal{Q}v]_{\alpha_L(v)}\right)>0\Longrightarrow\Theta\left(\mathcal{H}\left([\mathcal{P}u]_{\alpha_L(u)},[\mathcal{Q}v]_{\alpha_L(v)}\right)\right)\le\Theta(\sigma(u,v))^k+Lm(u,v)\tag{2.1}$$

for all $u,v\in\mathcal{S}$, where

$$m(u,v)=\min\left\{\sigma\left(u,[\mathcal{P}u]_{\alpha_L(u)}\right),\sigma\left(v,[\mathcal{Q}v]_{\alpha_L(v)}\right),\sigma\left(u,[\mathcal{Q}v]_{\alpha_L(v)}\right),\sigma\left(v,[\mathcal{P}u]_{\alpha_L(u)}\right)\right\}.\tag{2.2}$$

Then $\mathcal{P}$ and $\mathcal{Q}$ have a common L-fuzzy fixed point.

Proof . Let u_0 be an arbitrary point in $\mathcal{S}$, then by hypotheses there exists $\alpha_L(u_0) \in L\backslash\{0_L\}$ such that $[\mathcal{P}u_0]_{\alpha_L(u_0)}$ is a nonempty closed bounded subset of $\mathcal{S}$ and let $u_1 \in [\mathcal{P}u_0]_{\alpha_L(u_0)}$. For this u_1 there exists $\alpha_L(u_1) \in L\backslash\{0_L\}$ such that $[\mathcal{Q}u_1]_{\alpha_L(u_1)}$ is a nonempty, closed and bounded subset of $\mathcal{S}$. By Lemma 1.9, (Θ_1) and (2.1), we have

$$\Theta(\sigma\left(u_1, [\mathcal{Q}u_1]_{\alpha_L(u_1)}\right) \leq \Theta\left(\mathcal{H}\left([\mathcal{P}u_0]_{\alpha_L(u_0)}, [\mathcal{Q}u_1]_{\alpha_L(u_1)}\right)\right) \leq \Theta(\sigma(u_0,u_1))^k + Lm(u_0,u_1) \tag{2.3}$$

where

$$m(u_0,u_1) = \min\left\{\sigma\left(u_0, [\mathcal{P}u_0]_{\alpha_L(u_0)}\right), \sigma\left(u_1, [\mathcal{Q}u_1]_{\alpha_L(u_1)}\right), \sigma\left(u_0, [\mathcal{Q}u_1]_{\alpha_L(u_1)}\right), \sigma\left(u_1, [\mathcal{P}u_0]_{\alpha_L(u_0)}\right)\right\}.$$

From (Θ_4), we know that

$$\Theta\left(\sigma\left(u_1, [\mathcal{Q}u_1]_{\alpha_L(u_1)}\right)\right) = \inf_{v\in[\mathcal{Q}u_1]_{\alpha_L(u_1)}} \Theta(\sigma(u_1,v)).$$

Thus from (2.3), we get

$$\inf_{v\in[\mathcal{Q}u_1]_{\alpha_L(u_1)}} \Theta(\sigma(u_1,v)) \leq \Theta(\sigma(u_0,u_1)^k + \min\left\{\begin{array}{c}\sigma\left(u_0, [\mathcal{P}u_0]_{\alpha_L(u_0)}\right), \sigma\left(u_1, [\mathcal{Q}u_1]_{\alpha_L(u_1)}\right), \\ \sigma\left(u_0, [\mathcal{Q}u_1]_{\alpha_L(u_1)}\right), \sigma\left(u_1, [\mathcal{P}u_0]_{\alpha_L(u_0)}\right)\end{array}\right\} \tag{2.4}$$

Then, from (2.4), there exists $u_2 \in [\mathcal{Q}u_1]_{\alpha_L(u_1)}$ such that

$$\begin{aligned}\Theta(\sigma(u_1,u_2)) &\leq [\Theta(\sigma(u_0,u_1)]^k \\ &\quad + \min\left\{\sigma\left(u_0,u_1\right), \sigma\left(u_1,u_2\right), \sigma\left(u_0,u_2\right), \sigma\left(u_1,u_1\right)\right\}.\end{aligned}$$

Thus we have

$$\Theta(\sigma(u_1,u_2)) \leq [\Theta(\sigma(u_0,u_1)]^k. \tag{2.5}$$

For this u_2 there exists $\alpha_L(u_2) \in L\backslash\{0_L\}$ such that $[\mathcal{P}u_2]_{\alpha_L(u_2)}$ is a nonempty closed bounded subset of $\mathcal{S}$. By Lemma 1.9, (Θ_1) and (2.1), we have

$$\begin{aligned}\Theta\left(\sigma\left(u_2, [\mathcal{P}u_2]_{\alpha_L(u_2)}\right)\right) &\leq \Theta(\mathcal{H}\left([\mathcal{Q}u_1]_{\alpha_L(u_1)}, [\mathcal{P}u_2]_{\alpha_L(u_2)}\right) = \Theta(\mathcal{H}\left([\mathcal{P}u_2]_{\alpha_L(u_2)}, [\mathcal{Q}u_1]_{\alpha_L(u_1)}\right) \\ &\leq [\Theta(\sigma(u_2,u_1)]^k + Lm(u_2,u_1)\end{aligned}$$

thus we get

$$\Theta\left(\sigma\left(u_2, [\mathcal{P}u_2]_{\alpha_L(u_2)}\right)\right) \leq\leq [\Theta(\sigma(u_2,u_1)]^k + Lm(u_2,u_1) \tag{2.6}$$

where

$$m(u_2,u_1) = \min\left\{\sigma\left(u_2, [\mathcal{P}u_2]_{\alpha_L(u_2)}\right), \sigma\left(u_1, [\mathcal{Q}u_1]_{\alpha_L(u_1)}\right), \sigma\left(u_2, [\mathcal{Q}u_1]_{\alpha_L(u_1)}\right), \sigma\left(u_1, [\mathcal{P}u_2]_{\alpha_L(u_2)}\right)\right\}$$

which further implies that

$$\Theta\left(\sigma\left(u_2, [\mathcal{P}u_2]_{\alpha_L(u_2)}\right)\right) \leq \Theta\left[\sigma(u_1,u_2)\right]^k + \min\left\{\begin{array}{c}\sigma\left(u_2, [\mathcal{P}u_2]_{\alpha_L(u_2)}\right), \sigma\left(u_1, [\mathcal{Q}u_1]_{\alpha_L(u_1)}\right), \\ \sigma\left(u_2, [\mathcal{Q}u_1]_{\alpha_L(u_1)}\right), \sigma\left(u_1, [\mathcal{P}u_2]_{\alpha_L(u_2)}\right)\end{array}\right\}. \tag{2.7}$$

From (Θ_4), we know that

$$\Theta\left[\sigma\left(u_2, [\mathcal{P}u_2]_{\alpha_L(u_2)}\right)\right] = \inf_{v_1\in[\mathcal{P}u_2]_{\alpha_L(u_2)}} \Theta(\sigma(u_2,v_1))$$

$$\inf_{v_1\in[\mathcal{P}u_2]_{\alpha_L(u_2)}} \Theta(\sigma(u_2,v_1)) \leq \Theta\left[\sigma(u_1,u_2)\right]^k + \min\left\{\begin{array}{c}\sigma\left(u_2,[\mathcal{P}u_2]_{\alpha_L(u_2)}\right),\sigma\left(u_1,[\mathcal{Q}u_1]_{\alpha_L(u_1)}\right),\\ \sigma\left(u_2,[\mathcal{Q}u_1]_{\alpha_L(u_1)}\right),\sigma\left(u_1,[\mathcal{P}u_2]_{\alpha_L(u_2)}\right)\end{array}\right\}. \tag{2.8}$$

Then, from (2.8), there exists $u_3 \in [\mathcal{P}u_2]_{\alpha_L(u_2)}$ such that

$$\begin{aligned}\Theta(\sigma(u_2,u_3)) \leq\ & [\Theta(\sigma(u_1,u_2))]^k \\ & +\min\left\{\sigma\left(u_2,u_3\right),\sigma\left(u_1,u_2\right),\sigma\left(u_2,u_2\right),\sigma\left(u_1,u_3\right)\right\}.\end{aligned}$$

Thus we have

$$\Theta(\sigma(u_2,u_3)) \leq [\Theta(\sigma(u_1,u_2))]^k. \tag{2.9}$$

So, continuing recursively, we obtain a sequence $\{u_n\}$ in $\mathcal{S}$ such that $u_{2n+1} \in [\mathcal{P}u_{2n}]_{\alpha_L(u_{2n})}$ and $u_{2n+2} \in [\mathcal{Q}u_{2n+1}]_{\alpha_L(u_{2n+1})}$ and

$$\Theta(\sigma(u_{2n+1},u_{2n+2})) \leq [\Theta(\sigma(u_{2n},u_{2n+1})]^k \tag{2.10}$$

and

$$\Theta(\sigma(u_{2n+2},u_{2n+3})) \leq [\Theta(\sigma(u_{2n+1},u_{2n+2})]^k \tag{2.11}$$

for all $n \in \mathbb{N}$. From (2.10) and (2.11), we have

$$\Theta(\sigma(u_n,u_{n+1})) \leq [\Theta(\sigma(u_{n-1},u_n)]^k \tag{2.12}$$

which further implies that

$$\Theta(\sigma(u_n,u_{n+1})) \leq [\Theta(\sigma(u_{n-1},u_n)]^k \leq [\Theta(\sigma(u_{n-2},u_{n-1})]^{k^2} \leq ... \leq [\Theta(\sigma(u_0,u_1)]^{k^n} \tag{2.13}$$

for all $n \in \mathbb{N}$. Since $\Theta \in F$, so by taking limit as $n \to \infty$ in (2.13) we have,

$$\lim_{n\to\infty} \Theta(\sigma(u_n,u_{n+1})) = 1 \tag{2.14}$$

which implies that

$$\lim_{n\to\infty} \sigma(u_n,u_{n+1}) = 0 \tag{2.15}$$

by (Θ_2). From the condition (Θ_3), there exist $0 < r < 1$ and $l \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\Theta(\sigma(u_n,u_{n+1}))-1}{\sigma(u_n,u_{n+1})^r} = l. \tag{2.16}$$

Suppose that $l < \infty$. In this case, let $\beta = \frac{l}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$|\frac{\Theta(\sigma(u_n,u_{n+1}))-1}{\sigma(u_n,u_{n+1})^r} - l| \leq \beta$$

for all $n > n_0$. This implies that

$$\frac{\Theta(\sigma(u_n,u_{n+1}))-1}{\sigma(u_n,u_{n+1})^r} \geq l-\beta = \frac{l}{2} = \beta$$

for all $n > n_0$. Then

$$n\sigma(u_n,u_{n+1})^r \leq \alpha n[\Theta(\sigma(u_n,u_{n+1}))-1] \tag{2.17}$$

for all $n > n_0$, where $\alpha = \frac{1}{\beta}$. Now we suppose that $l = \infty$. Let $\beta > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\beta \leq \frac{\Theta(\sigma(u_n, u_{n+1})) - 1}{\sigma(u_n, u_{n+1})^r}$$

for all $n > n_0$. This implies that

$$n\sigma(u_n, u_{n+1})^r \leq \alpha n[\Theta(\sigma(u_n, u_{n+1})) - 1]$$

for all $n > n_0$, where $\alpha = \frac{1}{\beta}$. Thus, in all cases, there exist $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that

$$n\sigma(u_n, u_{n+1})^r \leq \alpha n[\Theta(\sigma(u_n, u_{n+1})) - 1] \tag{2.18}$$

for all $n > n_0$. Thus by (2.13) and (2.18), we get

$$n\sigma(u_n, u_{n+1})^r \leq \alpha n([(\Theta\sigma(u_0, u_1))]^{r^n} - 1). \tag{2.19}$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} n\sigma(u_n, u_{n+1})^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$\sigma(u_n, u_{n+1}) \leq \frac{1}{n^{1/r}} \tag{2.20}$$

for all $n > n_1$. Now we prove that $\{u_n\}$ is a Cauchy sequence. For $m > n > n_1$ we have,

$$\sigma(u_n, u_m) \leq \sum_{i=n}^{m-1} \sigma(u_i, u_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/r}} \leq \sum_{i=1}^{\infty} \frac{1}{i^{1/r}}. \tag{2.21}$$

Since, $0 < r < 1$, then $\sum_{i=1}^{\infty} \frac{1}{i^{1/r}}$ converges. Therefore, $\sigma(u_n, u_m) \to 0$ as $m, n \to \infty$. Thus we proved that $\{u_n\}$ is a Cauchy sequence in $(\mathcal{S}, \sigma)$. The completeness of $(\mathcal{S}, \sigma)$ ensures that there exists $u^* \in \mathcal{S}$ such that, $\lim_{n\to\infty} u_n \to u^*$. Now, we prove that $u^* \in [\mathcal{Q}u^*]_{\alpha_L(u^*)}$. We suppose on the contrary that $u^* \notin [\mathcal{Q}u^*]_{\alpha_L(u)}$, then there exist a $n_0 \in \mathbb{N}$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\sigma(u_{2n_k+1}, [\mathcal{Q}u^*]_{\alpha_L(u^*)}) > 0$ for all $n_k \geq n_0$. Since $\sigma(u_{2n_k+1}, [\mathcal{Q}u^*]_{\alpha_L(u)}) > 0$ for all $n_k \geq n_0$, so by (Θ_1), we have

$$\begin{aligned} 1 &< \Theta\left[\sigma(u_{2n_k+1}, [\mathcal{Q}u^*]_{\alpha_L(u^*)})\right] \leq \Theta\left[\mathcal{H}([\mathcal{P}u_{2n_k}]_{\alpha_L(u_{2n_k})}, [\mathcal{Q}u^*]_{\alpha_L(u^*)})\right] \\ &\leq [\Theta(\sigma(u_{2n_k}, u^*))]^k \\ &\quad + \min\left\{ \begin{array}{c} \sigma\left(u_{2n_k}, [\mathcal{P}u_{2n_k}]_{\alpha_L(u_{2n_k})}\right), \sigma\left(u^*, [\mathcal{Q}u^*]_{\alpha_L(u^*)}\right), \\ \sigma\left(u_{2n_k}, [\mathcal{Q}u^*]_{\alpha_L(u^*)}\right), \sigma\left(u^*, [\mathcal{P}u_{2n_k}]_{\alpha_L(u_{2n_k})}\right) \end{array} \right\} \\ &\leq [\Theta(\sigma(u_{2n_k}, u^*))]^k \\ &\quad + L\min\left\{ \begin{array}{c} \sigma\left(u_{2n_k}, u_{2n_k+1}\right), \sigma\left(u^*, [\mathcal{Q}u^*]_{\alpha_L(u^*)}\right), \\ \sigma\left(u_{2n_k}, [\mathcal{Q}u^*]_{\alpha_L(u^*)}\right), \sigma\left(u^*, u_{2n_k+1}\right) \end{array} \right\}. \end{aligned}$$

Letting $n \to \infty$, in above inequality and using the continuity of Θ, we have

$$1 < \Theta\left[\sigma(u^*, [\mathcal{Q}u^*]_{\alpha_L(u^*)})\right] \leq 1$$

which is a conradiction. Hence $u^* \in [\mathcal{Q}u^*]_{\alpha_L(u^*)}$. Similarly, one can easily prove that $u^* \in [\mathcal{P}u^*]_{\alpha_L(u^*)}$. Thus $u^* \in [\mathcal{P}u^*]_{\alpha_L(u^*)} \cap [\mathcal{Q}u^*]_{\alpha_L(u^*)}$. □

The following result is a direct consequence of above theorem by taking$L = 0$.

Corollary 2.2. *Let $(\mathcal{S},\sigma)$ be a complete metric space and $\{\mathcal{P},\mathcal{Q}\}$ be a pair of L-fuzzy mappings from $\mathcal{S}$ into $\Im_L(\mathcal{S})$ and for each $\alpha_L \in L\backslash\{0_L\}$, $[\mathcal{P}u]_{\alpha_L(u)}$, $[\mathcal{Q}v]_{\alpha_L(v)}$ are nonempty closed bounded subsets of $\mathcal{S}$. If there exist some $\Theta \in F$ and $k \in (0,1)$ such that*

$$\mathcal{H}\left([\mathcal{P}u]_{\alpha_L(u)}, [\mathcal{Q}v]_{\alpha_L(v)}\right) > 0 \Longrightarrow \Theta\left(\mathcal{H}\left([\mathcal{P}u]_{\alpha_L(u)}, [\mathcal{Q}v]_{\alpha_L(v)}\right)\right) \leq \Theta(\sigma(u,v))^k$$

for all $u,v \in \mathcal{S}$. Then $\mathcal{P}$ and $\mathcal{Q}$ have a common L-fuzzy fixed point.

If we take a single L-fuzzy mapping, we get the following result.

Corollary 2.3. *Let $(\mathcal{S},\sigma)$ be a complete metric space and let $\mathcal{P}$ be L-fuzzy mapping from $\mathcal{S}$ into $\Im_L(\mathcal{S})$ and for each $\alpha_L \in L\backslash\{0_L\}$, $[\mathcal{P}u]_{\alpha_L(u)}$, $[\mathcal{P}v]_{\alpha_L(v)}$ are nonempty closed bounded subsets of $\mathcal{S}$. If there exist some $\Theta \in F$, $k \in (0,1)$ and $L \geq 0$ such that*

$$\Theta\left(\mathcal{H}\left([\mathcal{P}u]_{\alpha_L(u)}, [\mathcal{P}v]_{\alpha_L(v)}\right)\right) \leq \Theta(\sigma(u,v))^k + Lm(u,v)$$

where

$$m(u,v) = \min\left\{\sigma\left(u,[\mathcal{P}u]_{\alpha_L(u)}\right), \sigma\left(v,[\mathcal{P}v]_{\alpha_L(v)}\right), \sigma\left(u,[\mathcal{P}v]_{\alpha_L(v)}\right), \sigma\left(v,[\mathcal{P}u]_{\alpha_L(u)}\right)\right\}.$$

for all $u,v \in \mathcal{S}$ with $\mathcal{H}\left([\mathcal{P}u]_{\alpha_L(u)}, [\mathcal{P}v]_{\alpha_L(v)}\right) > 0$. Then $\mathcal{P}$ has an L-fuzzy fixed point.

Corollary 2.4. *Let $(\mathcal{S},\sigma)$ be a complete metric space and let $\mathcal{P}$ be L-fuzzy mapping from $\mathcal{S}$ into $\Im_L(\mathcal{S})$ and for each $\alpha_L \in L\backslash\{0_L\}$, $[\mathcal{P}u]_{\alpha_L(u)}$, $[\mathcal{P}v]_{\alpha_L(v)}$ are nonempty closed bounded subsets of $\mathcal{S}$. If there exist some $\Theta \in F$ and $k \in (0,1)$ such that*

$$\Theta\left(\mathcal{H}\left([\mathcal{P}u]_{\alpha_L(u)}, [\mathcal{P}v]_{\alpha_L(v)}\right)\right) \leq \Theta(\sigma(u,v))^k$$

for all $u,v \in \mathcal{S}$ with $\mathcal{H}\left([\mathcal{P}u]_{\alpha_L(u)}, [\mathcal{P}v]_{\alpha_L(v)}\right) > 0$. Then $\mathcal{P}$ has an L-fuzzy fixed point.

L-fuzzy fixed point results are real generalization of fuzzy fixed point theorems. It can be shown in the following Theorem.

Theorem 2.5. *Let $(\mathcal{S},\sigma)$ be a complete metric space and let $\mathcal{P},\mathcal{Q}$ be fuzzy mappings from $\mathcal{S}$ into $\Im(\mathcal{S})$ and for each $\alpha(u) \in (0,1]$, $[\mathcal{P}u]_{\alpha(u)}$, $[\mathcal{Q}v]_{\alpha(v)}$ are nonempty closed bounded subsets of $\mathcal{S}$. If there exist some $\Theta \in F$, $k \in (0,1)$ and $L \geq 0$ such that*

$$\Theta\left(\mathcal{H}\left([\mathcal{P}u]_{\alpha(u)}, [\mathcal{Q}v]_{\alpha(v)}\right)\right) \leq \Theta(\sigma(u,v))^k + Lm(u,v)$$

where

$$m(u,v) = \min\left\{\sigma\left(u,[\mathcal{P}u]_{\alpha(u)}\right), \sigma\left(v,[\mathcal{Q}v]_{\alpha(v)}\right), \sigma\left(u,[\mathcal{Q}v]_{\alpha(v)}\right), \sigma\left(v,[\mathcal{P}u]_{\alpha(u)}\right)\right\}.$$

for all $u,v \in \mathcal{S}$ with $\mathcal{H}\left([\mathcal{P}u]_{\alpha(u)}, [\mathcal{Q}v]_{\alpha(v)}\right) > 0$. Then $\mathcal{P}$ and $\mathcal{Q}$ have a common fuzzy fixed point.

Proof . Consider an L-fuzzy mapping $\mathcal{J} : \mathcal{S} \to \Im_L(\mathcal{S})$ defined by

$$\mathcal{J}u = \chi_{L_{\mathcal{P}(u)}}.$$

Then for $\alpha_L \in L \backslash \{0_L\}$, we have

$$[\mathcal{J}u]_{\alpha_L(u)} = \mathcal{P}u.$$

Hence by Theorem 2.1 we follow the result. □

Taking $L = 0$ in above result, we have following corollary.

Corollary 2.6. *Let $(\mathcal{S}, \sigma)$ be a complete metric space and let $\mathcal{P}, \mathcal{Q}$ be fuzzy mappings from $\mathcal{S}$ into $\Im(\mathcal{S})$ and for each $\alpha(u) \in (0, 1]$, $[\mathcal{P}u]_{\alpha(u)}$, $[\mathcal{Q}v]_{\alpha(v)}$ are nonempty closed bounded subsets of $\mathcal{S}$. If there exist some $\Theta \in F$ and $k \in (0, 1)$ such that*

$$\Theta\left(\mathcal{H}\left([\mathcal{P}u]_{\alpha(u)}, [\mathcal{Q}v]_{\alpha(v)}\right)\right) \le \Theta(\sigma(u, v))^k$$

for all $u, v \in \mathcal{S}$ with $\mathcal{H}\left([\mathcal{P}u]_{\alpha(u)}, [\mathcal{Q}v]_{\alpha(v)}\right) > 0$. Then $\mathcal{P}$ and $\mathcal{Q}$ have a common fuzzy fixed point.

Example 2.7. *Let $\mathcal{S} = [0, 1]$, $\sigma(u, v) = |u - v|$, whenever $u, v \in \mathcal{S}$. Then $(\mathcal{S}, \sigma)$ is a complete metric space. Let $L = \{\eta, \omega, \tau, \kappa\}$ with $\eta \preceq_L \omega \preceq_L \kappa$ and $\eta \preceq_L \tau \preceq_L \kappa$, where ω and τ are not comparable, then $(L, \preceq_L)$ is a complete distributive lattice. Define $\mathcal{P}, \mathcal{Q} : \mathcal{S} \to \Im_L(\mathcal{S})$ as follows:*

$$\mathcal{P}(u)(t) = \begin{cases} \kappa \text{ if } 0 \le t \le \frac{u}{6} \\ \omega \text{ if } \frac{u}{6} < t \le \frac{u}{3} \\ \tau \text{ if } \frac{u}{3} < t \le \frac{u}{2} \\ \eta \text{ if } \frac{u}{2} < t \le 1 \end{cases},$$

$$\mathcal{Q}(u)(t) = \begin{cases} \kappa \text{ if } 0 \le t \le \frac{u}{12} \\ \eta \text{ if } \frac{u}{12} < t \le \frac{u}{8} \\ \omega \text{ if } \frac{u}{8} < t \le \frac{u}{4} \\ \tau \text{ if } \frac{u}{4} < t \le 1 \end{cases}.$$

Let $\Theta(t) = e^{\sqrt{t}} \in F$ for $t > 0$. And for all $u \in \mathcal{S}$, there exists $\alpha_L(u) = \kappa$, such that

$$[\mathcal{P}u]_{\alpha_L(u)} = \left[0, \frac{u}{6}\right], \qquad [\mathcal{Q}u]_{\alpha_L(u)} = \left[0, \frac{u}{12}\right].$$

and all conditions of Theorem 2.1 are satisfied. And 0 is a common fixed point of $\mathcal{P}$ and $\mathcal{Q}$.

3. Applications to domain of words

Suppose Ω be a nonempty alphabet and Ω^∞ be the collection of all finite and infinite sequences ("words") over Ω, where we adopt the convention that the empty sequence $\emptyset$ is an element of Ω^∞. Moreover, on Ω^∞, we consider the prefix order $\precsim$ given by:

$$u \precsim v \quad \text{if and only if } u \text{ is a prefix of } v.$$

For each nonempty $u \in \Omega^\infty$ denote by $l(u)$ the length of u. Then $l(u) \in [0, \infty]$, whenever $u \neq \emptyset$ and $l(\emptyset) = 0$. For each $u, v \in \Omega^\infty$, let $u \sqcap v$ be the common prefix of u and v. Clearly, $u = v$ if and only if $u \precsim v$ and $v \precsim u$ and $l(u) = l(v)$. Then, the the Baire metric $\sigma_\precsim$ is defined on $\Omega^\infty \times \Omega^\infty$ by

$$\begin{cases} \sigma_\precsim(u, v) = 0, \text{ if } u = v \\ \sigma_\precsim(u, v) = 2^{-l(u \sqcap v)}, \text{ otherwise} \end{cases}$$

such that the metric space $(\Omega^\infty, \sigma_{\varoiint})$ is complete. Certainly, we assign to the average case time complexity analysis of the Quicksort divide-and-conquer sorting algorithm in [32].

Exactly, we deal with the following recurrence relation:

$$\mathfrak{R}(1) = 0 \quad \text{and} \quad \mathfrak{R}(n) = \frac{2(n-1)}{n} + \frac{n+1}{n}\mathfrak{R}(n-1), \quad n \geq 2. \tag{3.1}$$

Consider as an alphabet Ω the set of nonnegative real numbers, i.e., $\Omega = \mathbb{R}^+$. We accomplice to $\mathfrak{R}$ the functional $\Phi : \Omega^\infty \to \Omega^\infty$ given by

$$(\Phi(u))_1 = \mathfrak{R}(1)$$

and

$$(\Phi(u))_n = \frac{2(n-1)}{n} + \frac{n+1}{n}u_{n-1}$$

for all $n \geq 2$ (if $u \in \Omega^\infty$ has length $n < \infty$, we write $u := u_1 u_2 ... u_n,$ and if u is an infinite word we write $u := u_1 u_2....$). It follows by the construction that $l(\Phi(u)) = l(u) + 1$ for all $u \in \Omega^\infty$ and $l(\Phi(u)) = +\infty$ whenever $l(u) = +\infty$. We will prove that the functional Φ has an L-fuzzy fixed point by an application of 2.4. Let $\mathcal{P} : \Omega^\infty \to \Im(\Omega^\infty)$ be the L-fuzzy mapping given by

$$\mathcal{P}_u = (\Phi(u))_{\alpha_L} \text{ for all } u \in \Omega^\infty \text{ and } \alpha_L \in L \backslash \{ 0_L \}.$$

and analyze the following two cases:

Case 01: If $u = v$, then we have

$$\mathcal{H}_{\varoiint}((\Phi(u))_{\alpha_L}, (\Phi(u))_{\alpha_L}) = 0 = \sigma_{\varoiint}(u, u).$$

Case 02: If $u \neq v$, then we write

$$\begin{aligned} \mathcal{H}_{\varoiint}((\Phi(u))_{\alpha_L}, (\Phi(v))_{\alpha_L}) &= \sigma_{\varoiint}((\Phi(u))_{\alpha_L}, (\Phi(v))_{\alpha_L}) = 2^{-(l(\Phi(u))_{\alpha_L} \sqcap (\Phi(v))_{\alpha_L})} \\ &\leq 2^{-(l(\Phi(u \sqcap v))_{\alpha_L})} = 2^{-(l(u \sqcap v)+1)} \\ &= \frac{1}{2} 2^{-l(u \sqcap v)} = (\frac{1}{\sqrt{2}})^2 \sigma_{\varoiint}(u, v). \end{aligned}$$

It is immediate to achieve that all the assertions of the Corollary 2.4 are satisfied with $\Theta(t) = e^{\sqrt{t}}$ and $k = \frac{1}{\sqrt{2}}$. Consequently, the L- fuzzy mapping $\mathcal{P}$ has a L- fuzzy fixed point $u = u_1 u_2 ... \in \Omega^\infty$ that is, $u \in (\mathcal{P}_u)_{\alpha_L}$. Also, in the light of the definition of $\mathcal{P}, u$ is a fixed point of Φ, and hence, u solves the recurrence relation (3.1). We have

$$u_1 = 0,$$

$$u_n = \frac{2(n-1)}{n} + \frac{n+1}{n}u_{n-1}, \quad n \geq 2.$$

4. Conclusions

We proved some common L-fuzzy fixed point results for almost Θ-contraction in the setting of complete metric spaces by using the notion of L-fuzzy mappings.We also presented an application to domain of words which shows the significance of the investigation of this paper.

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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