Int. J. Nonlinear Anal. Appl. x (xxxx) No. x, xx-xx
ISSN: 2008-6822 (electronic)
http://www.ijnaa.semnan.ac.ir

Fractals of Generalized Θ-Hutchinson Operator

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Abstract

The aim of this paper is to construct a fractal of generalized Θ-Hutchinson Operator with the help of a finite family of Θ-contraction mappings, a class of mappings more general than contractions, defined on a complete metric space. Our results unify, generalize and extend various results in the existing literature.

Keywords: Fixed point, Θ-Hutchinson Operator, metric space.
2010 MSC: Primary 26A25; Secondary 39B62.

1. Introduction and preliminaries

Banach's contraction principle [6] is one of the pivotal results of nonlinear analysis and its applications, which establishes that, if g is a mapping from a complete metric space (X, d) into itself and there exists a constant $k \in [0, 1)$ such that

$$d(gx, gy) \leq kd(x, y)$$

for all $x, y \in X$, then g has a unique fixed point in X. Furthermore, for any initial guess $x_0 \in X$ the sequence of simple iterates $\{x_0, gx_0, g^2x_0, g^3x_0, ...\}$ converges to a fixed point of g.

Due to its importance and simplicity, many authors have obtained a lot of interesting extensions and generalizations of Banach's contraction principle (see [1-12] and references therein).

Nadler [23] was the first who combined the ideas of multivalued mappings and contractions and hence initiated the study of metric fixed point theory of multivalued operators, see also [5, 11, 14]. The fixed point theory of multivalued operators provides important tools and techniques to solve the problems of pure, applied and computational mathematics which can be restructured as an inclusion equation for an appropriate multivalued operator.

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Received: January 19, 2019 *Revised:* January 19, 2019

Iterated function systems are based on the mathematical foundations laid by Hutchinson [17]. He showed that the Hutchinson operator constructed with the help of a finite system of contraction mappings defined on an Euclidean space $\mathbb{R}^n$ has closed and bounded subset of $\mathbb{R}^n$ as its fixed point, called attractor of iterated function system (see also in [7]). In this context, fixed point theory plays significant and vital role to help in construction of fractals.

Fixed point theory is studied in an environment created with appropriate mappings satisfying certain conditions. Recently, many researchers have obtained fixed point results for single and multi-valued mappings defined on metric spaces.

Let (X, d) be a metric space and $\mathcal{H}(X)$ denotes the set of all non-empty compact subsets of X. For $A, B \in \mathcal{H}(X)$, let

$$H(A, B) = \max\{\sup_{b\in B} d(b, A), \sup_{a\in A} d(a, B)\},$$

where $d(x, B) = \inf\{d(x, b) : b \in B\}$ is the distance of a point x from the set B. The mapping H is said to be the Pompeiu-Hausdorff metric induced by d. If (X, d) is a complete metric space, then $(\mathcal{H}(X), H)$ is also a complete metric space.

Very recently, Nazir et al. [24, 25] proved the following lemma which is also very useful in the proof of our main results.

Lemma 1.1. *Let (X, d) be a metric space. For all $A, B, C, D \in \mathcal{H}(X)$, the following hold:*

(i) If $B \subseteq C$, then $\sup_{a\in A} d(a, C) \leq \sup_{a\in A} d(a, B)$.

(ii) $\sup_{x\in A\cup B} d(x, C) = \max\{\sup_{a\in A} d(a, C), \sup_{b\in B} d(b, C)\}$.

(iii) $H(A \cup B, C \cup D) \leq \max\{H(A, C), H(B, D)\}$.

A new approach in the theory of fixed points is Θ-contraction which was first introduced by Jleli and Samet [18]. They established some new fixed point theorems for such contraction in the context of generalized metric spaces.

Definition 1.2. *Let $\Theta : (0, \infty) \to (1, \infty)$ be a function satisfying:*

(Θ_1) Θ is nondecreasing;

(Θ_2) for each sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \Theta(\alpha_n) = 1$ if and only if $\lim_{n\to\infty}(\alpha_n) = 0$;

(Θ_3) there exists $0 < h < 1$ and $l \in (0, \infty]$ such that $\lim_{a\to 0^+} \frac{\Theta(\alpha)-1}{\alpha^h} = l$;

A mapping $g : X \to X$ is said to be Θ-contraction if there exist the function Θ satisfying (Θ_1)-(Θ_3) and a constant $k \in (0, 1)$ such that for all $x, y \in X$,

$$d(gx, gy) > 0 \Longrightarrow \Theta(d(gx, gy)) \leq [\Theta(d(x, y))]^k. \tag{1.1}$$

Theorem 1.3. *[18] Let (X, d) be a complete metric space and $g : X \to X$ be a Θ-contraction, then g has a unique fixed point.*

From (1.1), for all $x, y \in X$ with $d(gx, gy) > 0$, we have

$$\Theta(d(gx, gy)) \leq [\Theta(d(x, y))]^k < \Theta(d(x, y))$$

because $k \in (0, 1)$. Since Θ is strickly increasing, so we have

$$d(gx, gy) < d(x, y),$$

for all $x, y \in X$, with $gx \neq gy$. This implies that g is contractive and continuous.

To be consistent with Samet et al. [18], we denote by Ψ the set of all functions $\Theta : (0, \infty) \to (1, \infty)$ satisfying the above conditions.

Hussain et al. [12] extended and generalized the above result in this way:

Theorem 1.4. *[12] Let (X, d) be a complete metric space and $g : X \to X$ be a self-mapping. If there exist a function $\Theta \in \Omega$ and positive real numbers a_1, a_2, a_3, a_4 with $0 \leq a_1 + a_2 + a_3 + 2a_4 < 1$ such that*

$$\begin{aligned}\Theta(d(gx, gy)) \leq\ & [\Theta(d(x, y))]^{a_1} \cdot [\Theta(d(x, gx))]^{a_2} \\ & \cdot [\Theta(d(y, gy))]^{a_3} \cdot [\Theta((d(x, gy) + d(y, gx))]^{a_4}\end{aligned} \tag{1.2}$$

for all $x, y \in X$, then g has a unique fixed point.

Recently, Ahmad et al. [1] applied the following simple condition on the function Θ :

(Θ_4) Θ is continuous.

They established some new fixed point theorems in the context of complete metric space.

To be consistent with Ahmad et al. [1], we denote by Ω the set of all functions $\Theta : (0, \infty) \to (1, \infty)$ satisfying the conditions (Θ_1)-(Θ_4).

In this paper, we define a fractal of generalized Θ-Hutchinson Operator with the help of a finite family of Θ-contractions in the setting of complete metric space.

2. The main results

In this section we define generalized Θ-contraction and establish some fixed point theorems for these contractions.

Theorem 2.1. *Let (X, d) be a metric space, $g : X \to X$ be a Θ-contraction and $\Theta \in \Omega$. Then*

(i) g maps the elements of $\mathcal{H}(X)$ to elements of $\mathcal{H}(X)$,

(ii) if for any $A \in \mathcal{H}(X)$,

$$g(A) = \{g(x) : x \in A\}.$$

Then $g : \mathcal{H}(X) \to \mathcal{H}(X)$ is also a Θ-contraction on $(\mathcal{H}(X), H)$.

Proof . (i). As $\Theta \in \Omega$, so Θ is continuous. Thus image of a compact subset under $g : X \to X$ is compact, that is,

$$A \in \mathcal{H}(X) \text{ implies } g(A) \in \mathcal{H}(X).$$

(ii). Since $g : X \to X$ is Θ-contraction and Θ is strictly increasing, we obtain that

$$0 < d\left(gx, gy\right) < d\left(x, y\right) \tag{2.1}$$

for all $x, y \in X$ with $gx \neq gy$. Let $A, B \in \mathcal{H}(X)$ with $H\left(g\left(A\right), g\left(B\right)\right) \neq 0$. Using (2.1), we have

$$d\left(gx, g\left(B\right)\right) = \inf_{y \in B} d\left(gx, gy\right) < \inf_{y \in B} d\left(x, y\right) = d\left(x, B\right).$$

Also

$$d\left(gy, g\left(A\right)\right) = \inf_{x \in A} d\left(gy, gx\right) < \inf_{x \in A} d\left(y, x\right) = d\left(y, A\right).$$

Now

$$\begin{aligned} H\left(g\left(A\right), g\left(B\right)\right) &= \max\{\sup_{x \in A} d(gx, g\left(B\right)), \sup_{y \in B} d(gy, g\left(A\right))\} \\ &< \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\} = H\left(A, B\right). \end{aligned}$$

Since Θ is strictly increasing, we have

$$\Theta\left(H\left(g\left(A\right), g\left(B\right)\right)\right) < \Theta\left(H\left(A, B\right)\right).$$

Consequently, there exists some $k^* \in (0, 1)$ such that

$$\Theta\left(H\left(g\left(A\right), g\left(B\right)\right)\right) \leq \left[\Theta\left(H\left(A, B\right)\right)\right]^{k^*}.$$

Hence $g : \mathcal{H}(X) \to \mathcal{H}(X)$ is a Θ-contraction. $\square$

Theorem 2.2. *Let (X, d) be a metric space and $\{g_n : n = 1, 2, ..., N\}$ a finite family of Θ-contraction self-mappings on X. Define $F : \mathcal{H}(X) \to \mathcal{H}(X)$ by*

$$F(A) = \cup_{n=1}^{N} g_n(A),$$

for each $A \in \mathcal{H}(X)$. Then F is Θ-contraction on $\mathcal{H}\left(X\right)$.

Proof . We prove the above claim for $N = 2$. Let $g_1, g_2 : X \to X$ be two Θ-contractions. Take $A, B \in \mathcal{H}\left(X\right)$ with $H(F(A), F(B)) \neq 0$. By Lemma 1.1 (iii), it follows that

$$\begin{aligned} \Theta\left(H(F(A), F(B))\right) &= \Theta\left(H(g_1(A) \cup g_2(A), g_1(B) \cup g_2(B))\right) \\ &\leq \Theta\left(\max\{H(g_1(A), g_1(B)), H(g_2(A), g_2(B))\}\right) \\ &\leq \left[\Theta\left(H(A, B)\right)\right]^k \end{aligned}$$

for some $k \in (0, 1)$ $\square$

Theorem 2.3. *Let (X, d) be a complete metric space and $\{g_n : n = 1, 2, ..., N\}$ a finite family of Θ-contractions on X. Define $F : \mathcal{H}(X) \to \mathcal{H}(X)$ by*

$$F(A) = \cup_{n=1}^{N} g_n(A),$$

for each $A \in \mathcal{H}(X)$. Then

(i) $F : \mathcal{H}(X) \to \mathcal{H}(X)$ is a Θ-contraction;

(ii) F has a unique fixed point $U \in \mathcal{H}(X)$, that is $U = F(U) = \cup_{n=1}^{k} g_n(U)$. Moreover, for any initial set $A_0 \in \mathcal{H}(X)$, the sequence of compact sets $\{A_0, F(A_0), F^2(A_0), ...\}$ converges to a fixed point of F.

Proof . (i) Since each g_i is Θ-contraction and $F : \mathcal{H}(X) \to \mathcal{H}(X)$ is defined as

$$F(A) = \cup_{n=1}^{N} g_n(A),$$

for each $A \in \mathcal{H}(X)$. It follows from Theorem 2.1 that F is a Θ-contraction

(ii) The completeness of (X, d) implies that $(\mathcal{H}(X), H)$ is complete. Thus (ii) follows directly from Theorem 1.3. □

Definition 2.4. *Let (X, d) be a metric space. A mapping $F : \mathcal{H}(X) \to \mathcal{H}(X)$ is said to be a generalized Θ-contraction if there exists some $\Theta \in \Omega$ and $k \in (0, 1)$ such that for any $A, B \in \mathcal{H}(X)$ with $H(F(A), F(B)) > 0$, the following condition holds:*

$$\Theta(H(F(A), F(B))) \leq [\Theta(R_F(A, B))]^k, \tag{2.2}$$

where

$$\begin{aligned} R_F(A, B) \;=\; & \max\{H(A, B), H(A, F(A)), H(B, F(B)), \frac{H(A, F(B)) + H(B, F(A))}{2}, \\ & H(F^2(A), F(A)), H(F^2(A), B), H(F^2(A), F(B))\}. \end{aligned} \tag{2.3}$$

The operator F defined above is also called generalized Θ-Hutchinson operator. Note that if F defined in Theorem 2.2 is Θ-contraction, then it is trivially generalized Θ-contraction and so F is generalized Θ-Hutchinson operator. The converse does not hold, see [22].

Definition 2.5. *Let (X, d) be a metric space. If $g_n : X \to X$, $n = 1, 2, ..., N$ are Θ-contraction mappings, then $(X; g_1, g_2, ..., g_N)$ is called generalized (Θ-contractive) iterated function system (IFS).*

Thus the generalized iterated function system consists of a metric space and finite family of Θ-contraction mappings on X.

Definition 2.6. *A nonempty compact set $A \subseteq X$ is said to be an attractor of the generalized Θ-contractive IFS if there exists $F : \mathcal{H}(X) \to \mathcal{H}(X)$ such that*

(a) $F(A) = A$ and

(b) there is an open set $V \subseteq X$ such that $A \subseteq V$ and $\lim_{k\to\infty} F^k(B) = A$ for any compact set $B \subseteq V$, where the limit is taken with respect to the Hausdorff metric.

Now we state a fixed point theorem for generalized Θ- Hutchinson operator.

Theorem 2.7. *Let (X, d) be a complete metric space and $\{X : g_n, n = 1, 2, \cdots, k\}$ a generalized iterated function system. Let $F : \mathcal{H}(X) \to \mathcal{H}(X)$ be defined by*

$$F(A) = g_1(A) \cup g_2(A) \cup \cdots \cup g_N(A) = \cup_{n=1}^{N} g_n(A)$$

for each $A \in \mathcal{H}(X)$. If F is a generalized Θ- Hutchinson operator, then F has a unique fixed point $U \in \mathcal{H}(X)$, that is

$$U = F(U) = \cup_{n=1}^{k} g_n(U).$$

Moreover, for any initial set $A_0 \in \mathcal{H}(X)$, the sequence of compact sets $\{A_0, F(A_0), F^2(A_0), ...\}$ converges to a fixed point of F.

Proof . Let A_0 be an arbitrary element in $\mathcal{H}(X)$. If $A_0 = F(A_0)$, then A_0 is a fixed point of F and we have nothing to prove more. So we assume that $A_0 \neq F(A_0)$. Define

$$A_1 = F(A_0),\ A_2 = F(A_1), ..., A_{m+1} = F(A_m)$$

for $m \in \mathbb{N}$. We may assume that $A_m \neq A_{m+1}$ for all $m \in \mathbb{N}$. If not, then $A_k = A_{k+1}$ for some k implies $A_k = F(A_k)$ that is A_k is fixed point of F and this completes the proof. Take $A_m \neq A_{m+1}$ for all $m \in \mathbb{N}$. From (2.2), we have

$$\begin{aligned}
\Theta(H(A_{m+1}, A_{m+2})) &= \Theta(H(F(A_m), F(A_{m+1}))) \\
&\leq [\Theta(R_F(A_m, A_{m+1}))]^k
\end{aligned}$$

where

$$\begin{aligned}
R_F(A_m, A_{m+1}) &= \max\{H(A_m, A_{m+1}), H(A_m, F(A_m)), H(A_{m+1}, F(A_{m+1})), \\
&\quad \frac{H(A_m, F(A_{m+1})) + H(A_{m+1}, F(A_m))}{2}, \\
&\quad H(F^2(A_m), F(A_m)), H\left(F^2(A_m), A_{m+1}\right), H\left(F^2(A_m), F(A_{m+1})\right)\} \\
&= \max\{H(A_m, A_{m+1}), H(A_m, A_{m+1}), H(A_{m+1}, A_{m+2}), \\
&\quad \frac{H(A_m, A_{m+2}) + H(A_{m+1}, A_{m+1})}{2}, \\
&\quad H(A_{m+2}, A_{m+1}), H(A_{m+2}, A_{m+1}), H(A_{m+2}, A_{m+2})\} \\
&\leq \max\{H(A_m, A_{m+1}), H(A_{m+1}, A_{m+2}), \frac{H(A_m, A_{m+1}) + H(A_{m+1}, A_{m+2})}{2}\} \\
&= \max\{H(A_m, A_{m+1}), H(A_{m+1}, A_{m+2})\}.
\end{aligned}$$

Thus, we have

$$\Theta(H(A_{m+1}, A_{m+2})) \leq [\Theta(\max\{H(A_m, A_{m+1}), H(A_{m+1}, A_{m+2})\})]^k.$$

If $\max\{H(A_m, A_{m+1}), H(A_{m+1}, A_{m+2})\} = H(A_{m+1}, A_{m+2})$, then

$$\Theta(H(A_{m+1}, A_{m+2})) \leq [\Theta(H(A_{m+1}, A_{m+2}))]^k$$

a contradiction because $k \in (0, 1)$. Thus we have

$$\Theta(H(A_{m+1}, A_{m+2})) \leq [\Theta(H(A_m, A_{m+1}))]^k$$

for all $m \in \mathbb{N}$. Therefore

$$\begin{aligned}\Theta\left(H(A_n, A_{n+1})\right) &\leq [\Theta\left(H(A_{n-1}, A_n)\right)]^k \\ &\leq [\Theta\left(H(A_{n-2}, A_{n-1})\right)]^{k^2} \\ &\leq ... \leq [\Theta\left(H(A_0, A_1)\right)]^{k^n}. \end{aligned} \tag{2.4}$$

So by taking limit as $n \to \infty$ in (2.4), we have

$$\lim_{n\to\infty} \Theta(H(A_n, A_{n+1})) = 1$$

which implies that

$$\lim_{n\to\infty} H(A_n, A_{n+1}) = 0 \tag{2.5}$$

by (Θ_2). From the condition (Θ_3), there exist $0 < h < 1$ and $l \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\Theta(H(A_n, A_{n+1})) - 1}{H(A_n, A_{n+1})^h} = l. \tag{2.6}$$

Suppose that $l < \infty$. In this case, let $\beta = \frac{l}{2} > 0$. From the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that

$$|\frac{\Theta(H(A_n, A_{n+1})) - 1}{H(A_n, A_{n+1})^h} - l| \leq \beta$$

for all $n > n_1$. This implies that

$$\frac{\Theta(H(A_n, A_{n+1})) - 1}{H(A_n, A_{n+1})^h} \geq l - \beta = \frac{l}{2} = \beta$$

for all $n > n_1$. Then

$$nH(A_n, A_{n+1})^h \leq \alpha n[\Theta(H(A_n, A_{n+1})) - 1] \tag{2.7}$$

for all $n > n_1$, where $\alpha = \frac{1}{\beta}$. Now we suppose that $l = \infty$. Let $\beta > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that

$$\beta \leq \frac{\Theta(H(A_n, A_{n+1})) - 1}{H(A_n, A_{n+1})^h}$$

for all $n > n_1$. This implies that

$$nH(A_n, A_{n+1})^h \leq \alpha n[\Theta(H(A_n, A_{n+1})) - 1]$$

for all $n > n_1$, where $A = \frac{1}{B}$. Thus, in all cases, there exist $A > 0$ and $n_1 \in \mathbb{N}$ such that

$$nH(A_n, A_{n+1})^h \leq \alpha n[\Theta(H(A_n, A_{n+1})) - 1] \tag{2.8}$$

for all $n > n_1$. Thus by (2.4) and (2.8), we get

$$nH(A_n, A_{n+1})^h \leq \alpha n([\Theta\left(H(A_0, A_1)\right)]^{k^n} - 1). \tag{2.9}$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} nH(A_n, A_{n+1})^h = 0.$$

Thus, there exists $n_2 \in \mathbb{N}$ such that

$$H(A_n, A_{n+1})^h \leq \frac{1}{n^{1/h}} \tag{2.10}$$

for all $n > n_2$. Now for $m, n \in \mathbb{N}$ with $m > n \geq n_2$, we have

$$\begin{aligned} H(A_n, A_m) &\leq H(A_n, A_{n+1}) + H(A_{n+1}, A_{n+2}) + \ldots + H(A_{m-1}, A_m) \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/h}}. \end{aligned}$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/h}}$, we get $H(A_n, A_m) \to 0$ as $n, m \to \infty$. Therefore $\{A_n\}$ is a Cauchy sequence in X. Since $(\mathcal{H}(X), H)$ is complete, we have $A_n \to U$ as $n \to \infty$ for some $U \in \mathcal{H}(X)$. In order to show that U is the fixed point of F, we contrary assume that Pompeiu-Hausdorff weight assigned to the U and $F(U)$ is not zero. Now again from (2.2), we have

$$\Theta(H(A_{n+1}, F(U))) = \Theta(H(F(A_n), F(U))) \leq [\Theta(R_F(A_n, U))]^k \tag{2.11}$$

where

$$\begin{aligned} R_F(A_n, U) &= \max\{H(A_n, U), H(A_n, F(A_n)), H(U, F(U)), \frac{H(A_n, F(U)) + H(U, F(A_n))}{2}, \\ &\quad H(F^2(A_n), F(A_n)), H(F^2(A_n), U), H(F^2(A_n), F(U))\} \\ &= \max\{H(A_n, U), H(A_n, A_{n+1}), H(U, F(U)), \frac{H(A_n, F(U)) + H(U, A_{n+1})}{2}, \\ &\quad H(A_{n+2}, A_{n+1}), H(A_{n+2}, U), H(A_{n+2}, F(U))\}. \end{aligned}$$

Now we consider the following cases:
(i) If $R_F(A_n, U) = H(A_n, U)$, then we have

$$\Theta(H(A_{n+1}, F(U))) \leq [\Theta(H(A_n, U))]^k \tag{2.12}$$

Taking limit as $n \to \infty$ in (2.12) and using the continuity of Θ, we have

$$\Theta(H(U, F(U))) \leq [\Theta(H(U, U))]^k,$$

a contradiction.
(ii) When $R_F(A_n, U) = H(A_n, A_{n+1})$, then we have

$$\Theta(H(A_{n+1}, F(U))) \leq [\Theta(H(A_n, A_{n+1}))]^k. \tag{2.13}$$

Taking limit as $n \to \infty$ in (2.13) and using the continuity of Θ, we have

$$\Theta(H(U, F(U))) \leq [\Theta(H(U, U))]^k$$

gives a contradiction.
(iii) In case $R_F(A_n, U) = H(U, F(U))$, then we have

$$\Theta(H(A_{n+1}, F(U))) \leq [\Theta(H(U, F(U)))]^k. \tag{2.14}$$

Taking limit as $n \to \infty$ in (2.14) and using the continuity of Θ, we have

$$\Theta\left(H(U, F(U))\right) \leq \left[\Theta\left(H\left(U, F\left(U\right)\right)\right)\right]^{k} < \Theta\left(H\left(U, F\left(U\right)\right)\right),$$

a contradiction.
(iv) If $R_F\left(A_n, U\right) = \dfrac{H(A_n, F\left(U\right)) + H(U, A_{n+1})}{2}$, then we have

$$\Theta\left(H(A_{n+1}, F\left(U\right))\right) \leq \left[\Theta\left(\frac{H(A_n, F\left(U\right)) + H(U, A_{n+1})}{2}\right)\right]^{k}. \tag{2.15}$$

Taking limit as $n \to \infty$ in (2.15) and using the continuity of Θ, we have

$$\begin{aligned} \Theta\left(H(U, F(U))\right) &\leq \left[\Theta(\frac{H\left(U, F\left(U\right)\right) + H\left(U, U\right)}{2})\right]^{k} \\ &= \left[\Theta(\frac{H\left(U, F\left(U\right)\right)}{2})\right]^{k} < \Theta(\frac{H\left(U, F\left(U\right)\right)}{2}) \end{aligned}$$

a contradiction.
(v) If $R_F\left(A_n, U\right) = H(A_{n+2}, A_{n+1})$, then we have

$$\Theta\left(H(A_{n+1}, F\left(U\right))\right) \leq \left[\Theta\left(H(A_{n+2}, A_{n+1})\right)\right]^{k}. \tag{2.16}$$

Taking limit as $n \to \infty$ in (2.16) and using the continuity of Θ, we have

$$\Theta\left(H(U, F(U))\right) \leq \left[\Theta\left(H\left(U, U\right)\right)\right]^{k}$$

gives a contradiction.
(vi) If $R_F\left(A_n, U\right) = H(A_{n+2}, U)$, then we have

$$\Theta\left(H(A_{n+1}, F\left(U\right))\right) \leq \left[\Theta\left(H(A_{n+2}, U)\right)\right]^{k}. \tag{2.17}$$

Taking limit as $n \to \infty$ in (2.17) and using the continuity of Θ, we have

$$\Theta\left(H(U, F(U))\right) \leq \left[\Theta\left(H\left(U, U\right)\right)\right]^{k}$$

a contradiction.
(vii) Finally if $R_F\left(A_n, U\right) = H(A_{n+2}, F\left(U\right))$, then we have

$$\Theta\left(H(A_{n+1}, F\left(U\right))\right) \leq \left[\Theta\left(H(A_{n+2}, F\left(U\right))\right)\right]^{k}. \tag{2.18}$$

Taking limit as $n \to \infty$ in (2.18) and using the continuity of Θ, we have

$$\Theta\left(H(U, F(U))\right) \leq \left[\Theta(H(U, F\left(U\right)))\right]^{k} < \Theta(H(U, F\left(U\right))),$$

a contradiction. Thus U is the fixed point of F.
To show the uniqueness of fixed point of F, assume that U and V are two fixed points of F with $H\left(U, V\right)$ is not zero. Since F is a Θ-contraction map, we obtain that

$$\begin{aligned} \Theta(H(U, V)) &= \Theta(H(F\left(U\right), F\left(V\right))) \\ &\leq \left[\begin{array}{c} \Theta(\max\{H(U, V), H(U, F\left(U\right)), H(V, F\left(V\right)), \frac{H(U, F(V)) + H(V, F(U))}{2}, \\ H(F^2\left(U\right), U), H(F^2\left(U\right), V), H(F^2\left(U\right), F\left(V\right))\}) \end{array}\right]^{k} \\ &= \left[\begin{array}{c} \Theta(\max\{H\left(U, V\right), H(U, U), H(V, V), \frac{H(U,V) + H(V,U)}{2}, \\ H\left(U, U\right), H(U, V), H(U, V)\}) \end{array}\right]^{k} \\ &= \left[\Theta\left(H(U, V)\right)\right]^{k} < \Theta\left(H(U, V)\right) \end{aligned}$$

a contradiction as $k \in (0, 1)$. Thus F has a unique fixed point U in $\mathcal{H}(X)$. □

Remark 2.8. *In Theorem 2.4, if we take $\mathcal{S}(X)$ the collection of all singleton subsets of X, then clearly $\mathcal{S}(X) \subseteq \mathcal{H}(X)$. Moreover, consider $g_n = g$ for each n, where $g = g_1$ then the mapping F becomes*

$$F(x) = g(x).$$

With this setting we obtain the following fixed point result.

Corollary 2.9. *Let (X, d) be a complete metric space and $\{X : g_n, n = 1, 2, \cdots, k\}$ a generalized iterated function system. Let $g : X \to X$ be a mapping defined as in Remark 2.1. If there exists some $\Theta \in \Omega$ and $k \in (0, 1)$ such that for any $x, y \in \mathcal{H}(X)$ with $d(g(x), g(y)) > 0$, the following condition holds:*

$$\Theta\left(d\left(gx, gy\right)\right) \leq [\Theta(R_g(x, y))]^k,$$

where

$$R_g(x, y) = \max\left\{d(x, y), d(x, gx), d(y, gy), \frac{d(x, gy) + d(y, gx)}{2}, d(g^2x, y), d(g^2x, gx), d(g^2x, gy)\right\}.$$

Then g has a unique fixed point $x \in X$, Moreover, for any initial set $x_0 \in X$, the sequence of compact sets $\{x_0, gx_0, g^2x_0, ...\}$ converges to a fixed point of g.

Corollary 2.10. *Let (X, d) be a complete metric space and $(X; g_n, n = 1, 2, \cdots, k)$ be iterated function system where each g_i for $i = 1, 2, ..., k$ is a contraction self-mapping on X. Then $F : \mathcal{H}(X) \to \mathcal{H}(X)$ defined in Theorem 2.3 has a unique fixed point in $\mathcal{H}\left(X\right)$. Furthermore, for any set $A_0 \in \mathcal{H}\left(X\right)$, the sequence of compact sets $\{A_0, F\left(A_0\right), F^2\left(A_0\right), ...\}$ converges to a fixed point of F.*

Proof . It follows from Theorem 1.1 that if each g_i for $i = 1, 2, ..., k$ is a contraction mapping on X, then the mapping $F : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by

$$F(A) = \cup_{n=1}^{k} g_n(A), \text{ for all } A \in \mathcal{H}(X)$$

is contraction on $\mathcal{H}\left(X\right)$. Using Theorem 2.3, the result follows. □

Corollary 2.11. *Let (X, d) be a complete metric space and $(X; g_n, n = 1, 2, \cdots, k)$ an iterated function system where each g_i for $i = 1, 2, ..., k$ is a mapping on X satisfying*

$$d\left(g_i x, g_i y\right) \leq k^2 d\left(x, y\right),$$

for all $x, y \in X$, $g_i x \neq g_i y$, where $k \in (0, 1)$. Then the mapping $F : \mathcal{H}(X) \to \mathcal{H}(X)$ defined in Theorem 2.3 has a unique fixed point in $\mathcal{H}\left(X\right)$. Furthermore, for any set $A_0 \in \mathcal{H}\left(X\right)$, the sequence of compact sets $\{A_0, F\left(A_0\right), F^2\left(A_0\right), ...\}$ converges to a fixed point of F.

Proof . Consider the mapping $\Theta(t) = e^{\sqrt{t}}$, for $t > 0$ in Theorem 2.2. Then obviously Θ satisfies (Θ_1)-(Θ_4). Now each mapping g_i for $i = 1, 2, ..., k$ on X satisfies

$$d\left(g_i x, g_i y\right) \leq k^2 d\left(x, y\right),$$

for all $x, y \in X$, $g_i x \neq g_i y$, where $k \in (0, 1)$. Again from Theorem 2.2, the mapping $F : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by

$$F(A) = \cup_{n=1}^{k} g_n(A),$$

for all $A \in \mathcal{H}(X)$, satisfies

$$H\left(F\left(A\right), F\left(B\right)\right) \leq k^2 H\left(A, B\right),$$

for all $A, B \in \mathcal{H}(X)$, with $H\left(F\left(A\right), F\left(B\right)\right) > 0$. Using Theorem 2.3, the result follows. □

Theorem 2.12. *Let (X,d) be a complete metric space and $(X;g_n,n=1,2,\cdots,k)$ be iterated function system such that each g_i for $i=1,2,...,k$ is a mapping on X satisfying*

$$2-\frac{2}{\pi}\arctan(\frac{1}{d\left(g_i x, g_i y\right)^{\lambda}}) \leq [2-\frac{2}{\pi}\arctan(\frac{1}{d\left(x, y\right)^{\lambda}})]^k$$

for all $x,y\in X$, $g_i x\neq g_i y$, where $k\in(0,1)$. Then the mapping $F:\mathcal{H}(X)\to\mathcal{H}(X)$ defined in Theorem 2.3 has a unique fixed point in $\mathcal{H}(X)$. Furthermore, for any set $A_0\in\mathcal{H}(X)$, the sequence of compact sets $\{A_0,F(A_0),F^2(A_0),...\}$ converges to a fixed point of F.

Proof . Taking $\Theta(t)=2-\frac{2}{\pi}\arctan(\frac{1}{t^{\lambda}})$, where $0<\lambda<1$ and $t>0$ in Theorem 2.2, we obtain that each mapping g_i for $i=1,2,...,k$ on X satisfies

$$2-\frac{2}{\pi}\arctan(\frac{1}{d\left(g_i x, g_i y\right)^{\lambda}}) \leq [2-\frac{2}{\pi}\arctan(\frac{1}{d\left(x, y\right)^{\lambda}})]^k$$

for all $x,y\in X$, $g_i x\neq g_i y$, where $k\in(0,1)$. Again it follows from Theorem 2.2 that the mapping $F:\mathcal{H}(X)\to\mathcal{H}(X)$ defined by

$$F(A)=\cup_{n=1}^{k} g_n(A),\ \text{for all } A\in\mathcal{H}(X)$$

satisfies

$$2-\frac{2}{\pi}\arctan(\frac{1}{H\left(F\left(A\right),F\left(B\right)\right)^{\lambda}}) \leq [2-\frac{2}{\pi}\arctan(\frac{1}{H\left(A,B\right)^{\lambda}})]^k$$

for all $A,B\in\mathcal{H}(X)$, $H(F(A),F(B))>0$. Using Theorem 2.3, the result follows. □

Example 2.13. *Let $X=[0,1]\times[0,1]$ and d be a Euclidean metric on X. Define $g_1,g_2:X\to X$ as*

$$\begin{aligned} g_1(x,y) &= (\frac{1}{x+1},\frac{y}{y+1}) \text{ and} \\ g_2(x,y) &= (\frac{\sin x}{\sin x+1},\frac{1}{\sin y+1}). \end{aligned}$$

Note that, for all $\mathbf{x}=(x_1,y_1),\mathbf{y}=(x_2,y_2)\in X$ with $\mathbf{x}\neq\mathbf{y}$,

$$\begin{aligned} & d(g_1(\mathbf{x}),g_1(\mathbf{y})) \\ = & d((\frac{1}{x_1+1},\frac{y_1}{y_1+1}),(\frac{1}{x_2+1},\frac{y_2}{y_2+1})) \\ = & \sqrt{\frac{(x_1-x_2)^2}{(x_1+1)^2(x_2+1)^2}+\frac{(y_1-y_2)^2}{(y_1+1)^2(y_2+1)^2}} \\ < & \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \\ = & d((x_1,y_1),(x_2,y_2)) \\ = & d(\mathbf{x},\mathbf{y}). \end{aligned}$$

Also

$$\begin{aligned} & d(g_2(\mathbf{x}), g_2(\mathbf{y})) \\ = \; & d((\frac{\sin x_1}{\sin x_1+1}, \frac{1}{\sin y_1+1}), (\frac{\sin x_2}{\sin x_2+1}, \frac{1}{\sin y_2+1})) \\ = \; & \sqrt{\frac{(\sin x_1 - \sin x_2)^2}{(\sin x_1+1)^2(\sin x_2+1)^2} + \frac{(\sin y_1 - \sin y_2)^2}{(\sin y_1+1)^2(\sin y_2+1)^2}} \\ < \; & \sqrt{(\sin x_1 - \sin x_2)^2 + (\sin y_1 - \sin y_2)^2} \\ \leq \; & \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} \\ = \; & d((x_1,y_1),(x_2,y_2)) \\ = \; & d(\mathbf{x},\mathbf{y}). \end{aligned}$$

Now there exists $k \in (0,1)$ such that

$$\begin{aligned} d(g_1(\mathbf{x}), g_1(\mathbf{y})) &\leq k^2 d(\mathbf{x},\mathbf{y}) \text{ and} \\ d(g_2(\mathbf{x}), g_2(\mathbf{y})) &\leq k^2 d(\mathbf{x},\mathbf{y}) \end{aligned}$$

are satisfied. Consider the iterated function system $\{\mathbb{R}^2; g_1, g_2\}$ with mapping $F : \mathcal{H}([0,1]^2) \to \mathcal{H}([0,1]^2)$ given as

$$F(A) = g_1(A) \cup g_2(A) \text{ for all } A \in \mathcal{H}([0,1]^2).$$

For all $A, B \in \mathcal{H}([0,1]^2)$ with $H(F(A), F(B)) \neq 0$, by Theorem 2.1, we have

$$H(F(A), F(B)) \leq k^2 H(A, B)$$

holds.

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

Acknowledgement This article was funded by the Deanship of Scientific Research (DSR), University of Jeddah. Therefore, authors acknowledge with thanks DSR, UJ for financial support.

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