



Simultaneous generalizations of known fixed point theorems for a Meir-Keeler type condition with applications

Wei-Shih Du^a, Themistocles M. Rassias^{*b}

^aDepartment of Mathematics, National Kaohsiung Normal University, Kaohsiung 82444, Taiwan

^bDepartment of Mathematics, National Technical University of Athens, Zografou Campus, 15780 Athens, Greece

Abstract

In this paper, we first establish a new fixed point theorem for a Meir-Keeler type condition. As an application, we devise a simultaneous generalization of Banach contraction principle, Kannan's fixed point theorem, Chatterjea's fixed point theorem and other fixed point theorems. Some new fixed point theorems are also obtained.

Keywords: Simultaneous generalization, Banach's type contraction, Kannan's type contraction, Chatterjea's type contraction, Meir-Keeler's type contraction, Banach contraction principle, Kannan's fixed point theorem, Chatterjea's fixed point theorem, Meir-Keeler's fixed point theorem.
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1. Introduction and preliminaries

Let (X, d) be a metric space and $T : X \rightarrow X$ be a selfmapping. A point x in X is a *fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $\mathcal{F}(T)$. Throughout this paper, we denote by \mathbb{N} and \mathbb{R} , the sets of positive integers and real numbers, respectively.

Recall that a selfmapping $T : X \rightarrow X$ is called

- (i) a *Banach's type contraction*, if there exists a nonnegative number $\gamma < 1$ such that

$$d(Tx, Ty) \leq \gamma d(x, y) \quad \text{for all } x, y \in X.$$

- (ii) a *Kannan's type contraction*, if there exists $\gamma \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \gamma(d(x, Tx) + d(y, Ty)) \quad \text{for all } x, y \in X.$$

*Corresponding author

Email addresses: wshdu@mail.nknu.edu.tw (Wei-Shih Du), trassias@math.ntua.gr (Themistocles M. Rassias)

(iii) a *Chatterjea's type contraction*, if there exists $\gamma \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \gamma(d(x, Ty) + d(y, Tx)) \text{ for all } x, y \in X.$$

(iv) a *Meir-Keeler's type contraction*, if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for $x, y \in X$,

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \quad \text{implies} \quad d(Tx, Ty) < \varepsilon.$$

It is known that every Banach's type contraction is a Meir-Keeler's type contraction. The following examples show that, in a metric space (X, d) , the Banach's type contraction, Kannan's type contraction and Chatterjea's type contraction are independent and different from each other.

Example A. Let $X = [0, 1]$ with the metric $d(x, y) = |x - y|$ for $x, y \in X$. Then (X, d) is a metric space. Define a mapping $T : X \rightarrow X$ by

$$Tx = \frac{1}{2}x \text{ for all } x \in X.$$

Since

$$d(Tx, Ty) = \frac{1}{2}d(x, y) \leq \frac{2}{3}d(x, y) \text{ for all } x, y \in X,$$

T is a Banach's type contraction. Note that

$$d(T(0), T(1)) = \frac{1}{2} = d(0, T(0)) + d(1, T(1)),$$

so T is not a Kannan's type contraction.

Example B. Let $X = [-1, 1]$ with the metric $d(x, y) = |x - y|$ for $x, y \in X$. Then (X, d) is a metric space. Define a mapping $T : X \rightarrow X$ by

$$Tx = -\frac{1}{2}x \text{ for all } x \in X.$$

Then the following hold.

- (a) T is a Banach's type contraction.
- (b) T is a Kannan's type contraction.
- (c) T is not a Chatterjea's type contraction.

Indeed, since

$$d(Tx, Ty) = \frac{1}{2}d(x, y) \leq \frac{2}{3}d(x, y) \text{ for all } x, y \in X,$$

T is a Banach's type contraction. To see (b), for any $x, y \in X$, since $d(x, Tx) = \frac{3}{2}|x|$ and $d(y, Ty) = \frac{3}{2}|y|$, we have

$$d(Tx, Ty) = \frac{1}{2}|x - y| \leq \frac{1}{2}(|x| + |y|) = \frac{1}{3}(d(x, Tx) + d(y, Ty)),$$

which means that T is a Kannan's type contraction. Finally, since $d(-1, T(1)) = \frac{1}{2}$ and $d(1, T(-1)) = \frac{1}{2}$, we get

$$d(T(-1), T(1)) = 1 = d(-1, T(1)) + d(1, T(-1)).$$

Hence T is not a Chatterjea's type contraction.

Example C. Let $X = [0, 4]$ with the metric $d(x, y) = |x - y|$ for $x, y \in X$. Then (X, d) is a metric space. Define a mapping $T : X \rightarrow X$ by

$$Tx = \begin{cases} 1 & , \text{ if } 0 \leq x < 3, \\ 0 & , \text{ if } 3 \leq x \leq 4. \end{cases}$$

Then the following hold.

- (a) T is not a Banach's type contraction.
- (b) T is a Kannan's type contraction.
- (c) T is not a Chatterjea's type contraction.

Indeed, since $d(T(2.5), T(3.5)) = 1 = d(2.5, 3.5)$, T is not a Banach's type contraction. It is easy to show that

$$d(Tx, Ty) \leq \frac{1}{3} (d(x, Tx) + d(y, Ty)) \quad \text{for all } x, y \in X.$$

Hence T is a Kannan's type contraction. Since $d(0, T(3)) = 0$ and $d(3, T(0)) = 2$, we obtain

$$d(T(0), T(3)) = 1 > \gamma [d(0, T(3)) + d(3, T(0))] \quad \text{for all } \gamma \in \left[0, \frac{1}{2}\right).$$

So T is not a Chatterjea's type contraction.

In fact, it should be mentioned that Examples A-C do not solve completely the above problem for distinct metrics. For example, a mapping $T : X \rightarrow X$ which is Kannan's type contraction with respect to d may be a Banach's type contraction with respect to some other metric ρ on X , equivalent with d .

The famous Banach contraction principle [2] is one of the best known fixed point theorem in fixed point theory and one of the most powerful tools for nonlinear functional analysis and applied mathematical analysis. A number of generalizations in various different directions of the Banach contraction principle have been investigated by several authors; see [1, 3-25] and references therein.

Theorem 1.1 (Banach contraction principle [2]). *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a Banach's type contraction. Then T has a unique fixed point in X .*

In 1969, Meir and Keeler [19] established an interesting generalization of Banach contraction principle.

Theorem 1.2 (Meir and Keeler [19]). *Let (X, d) be a complete metric space and let $T : X \rightarrow X$ be a Meir-Keeler's type contraction. Then T admits a unique fixed point in X .*

In 1969, Kannan established his interesting fixed point theorem (so-called the Kannan's fixed point theorem [16]) which is different from the Banach contraction principle.

Theorem 1.3 (Kannan [16]). *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a Kannan's type contraction. Then T admits a unique fixed point in X .*

Later, Chatterjea's fixed point theorem [3] was proved in 1972.

Theorem 1.4 (Chatterjea [3]). *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a Chatterjea's type contraction. Then T admits a unique fixed point in X .*

It is worth mentioning that the Banach contraction principle, Kannan's fixed point theorem and Chatterjea's fixed point theorem are different from each other in their mapping conditions. Very recently, Du studied and established some interesting simultaneous generalization of known fixed point theorem; for more detail, one can refer to [10-13].

In this paper, we first establish a new fixed point theorem for a Meir-Keeler type condition. As an application, we derive a simultaneous generalization of Banach contraction principle, Kannan's fixed point theorem, Chatterjea's fixed point theorem and other fixed point theorems. Our new simultaneous generalization is different from these simultaneous generalizations in [10-13]. Some new fixed point theorems are also obtained. Our new results are different from many well known generalizations of these results obtained until now.

2. New simultaneous generalizations and their applications

Let (X, d) be a metric space and $T : X \rightarrow X$ be a selfmapping. For any $x, y \in X$, we define

- $K(x, y) = \frac{d(x, Tx) + d(y, Ty)}{2}$ (Kannan type),
- $C(x, y) = \frac{d(x, Ty) + d(y, Tx)}{2}$ (Chatterjea type),
- $I(x, y) = \frac{d(x, Tx) + d(y, Ty)}{2}$,
- $J(x, y) = \frac{d(y, Tx) + d(x, Ty)}{2}$,
- $M(x, y) = \frac{d(x, Tx) + d(y, Ty) + d(y, Tx)}{3}$,
- $P(x, y) = \frac{d(x, Tx) + d(x, Ty) + d(y, Tx)}{3}$,
- $Q(x, y) = \frac{d(y, Ty) + d(x, Ty) + d(y, Tx)}{3}$,
- $U(x, y) = \frac{d(x, Tx) + d(y, Ty) + d(x, Ty) + d(y, Tx)}{4}$,
- $V(x, y) = \frac{d(x, y) + d(x, Tx) + d(y, Ty) + d(x, Ty) + d(y, Tx)}{5}$.

We first establish the following new fixed point theorem for a Meir-Keeler type condition and give its proof in section 3.

Theorem 2.1. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a selfmapping. Define a mapping $S : X \times X \rightarrow [0, \infty)$ by*

$$S(x, y) = \max\{d(x, y), K(x, y), C(x, y), I(x, y), J(x, y), \\ M(x, y), P(x, y), Q(x, y), U(x, y), V(x, y)\}.$$

Suppose that

(DR) for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for $x, y \in X$,

$$\varepsilon \leq S(x, y) < \varepsilon + \delta \quad \text{implies} \quad d(Tx, Ty) < \varepsilon.$$

Then T admits a unique fixed point in X .

By applying Theorem 2.1, we obtain the following new general fixed point theorem which is a simultaneous generalization of Banach contraction principle, Kannan's fixed point theorem, Chatterjea's fixed point theorem and some known results in the literature.

Theorem 2.2. *Let (X, d) be a complete metric space, $T : X \rightarrow X$ be a selfmapping and $S : X \times X \rightarrow [0, \infty)$ be a mapping as in Theorem 2.1. Suppose that there exists a nonnegative real number $\lambda < 1$ such that*

$$d(Tx, Ty) \leq \lambda S(x, y) \quad \text{for all } x, y \in X. \quad (2.1)$$

Then T admits a unique fixed point in X .

Proof. Given $\varepsilon > 0$, choose $\alpha \in (\lambda, 1)$ and take

$$\delta(\varepsilon) = \varepsilon \left(\frac{1}{\alpha} - 1 \right).$$

If $\varepsilon \leq S(x, y) < \varepsilon + \delta(\varepsilon)$, then

$$d(Tx, Ty) \leq \lambda S(x, y) < \alpha(\varepsilon + \delta(\varepsilon)) = \varepsilon.$$

So condition (DR) of Theorem 2.1 is satisfied. Applying Theorem 2.1, the mapping T has a unique fixed point in X . \square

Remark 2.1. Banach contraction principle, Kannan's fixed point theorem and Chatterjea's fixed point theorem are all special cases of Theorems 2.1 and 2.2.

As applications of Theorem 2.2, we can obtain the following new fixed point theorems immediately.

Corollary 2.1. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a selfmapping. Assume that there exists $\gamma \in [0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq \gamma(d(x, Tx) + d(y, Ty)) \quad \text{for all } x, y \in X.$$

Then T admits a unique fixed point in X .

Corollary 2.2. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a selfmapping. Assume that there exists $\gamma \in [0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq \gamma(d(y, Tx) + d(y, Ty)) \quad \text{for all } x, y \in X.$$

Then T admits a unique fixed point in X .

Corollary 2.3. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a selfmapping. Assume that there exists $\gamma \in [0, \frac{1}{3})$ such that*

$$d(Tx, Ty) \leq \gamma(d(x, Tx) + d(y, Ty) + d(y, Tx)) \quad \text{for all } x, y \in X.$$

Then T admits a unique fixed point in X .

Corollary 2.4. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a selfmapping. Assume that there exists $\gamma \in [0, \frac{1}{3})$ such that*

$$d(Tx, Ty) \leq \gamma(d(x, Tx) + d(x, Ty) + d(y, Tx)) \quad \text{for all } x, y \in X.$$

Then T admits a unique fixed point in X .

Corollary 2.5. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a selfmapping. Assume that there exists $\gamma \in [0, \frac{1}{3})$ such that*

$$d(Tx, Ty) \leq \gamma(d(y, Ty) + d(x, Ty) + d(y, Tx)) \text{ for all } x, y \in X.$$

Then T admits a unique fixed point in X .

Corollary 2.6. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a selfmapping. Assume that there exists $\gamma \in [0, \frac{1}{4})$ such that*

$$d(Tx, Ty) \leq \gamma(d(x, Tx) + d(y, Ty) + d(x, Ty) + d(y, Tx)) \text{ for all } x, y \in X.$$

Then T admits a unique fixed point in X .

Corollary 2.7. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a selfmapping. Assume that there exists $\gamma \in [0, \frac{1}{5})$ such that*

$$d(Tx, Ty) \leq \gamma(d(x, y) + d(x, Tx) + d(y, Ty) + d(x, Ty) + d(y, Tx)) \text{ for all } x, y \in X.$$

Then T admits a unique fixed point in X .

3. Proof of Theorem 2.1

Let $u \in X$ be given and define a sequence $\{x_n\}_{n \in \mathbb{N}}$ by $x_1 = u$ and $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exists $k \in \mathbb{N}$ such that $x_{k+1} = x_k$, then $x_k \in \mathcal{F}(T)$ and the desired conclusion is proved. For this reason we henceforth will assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, we have

- $K(x_n, x_{n+1}) = \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}$,
- $C(x_n, x_{n+1}) = \frac{d(x_n, x_{n+2})}{2} \leq \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}$,
- $I(x_n, x_{n+1}) = \frac{d(x_n, x_{n+1})}{2}$,
- $J(x_n, x_{n+1}) = \frac{d(x_{n+1}, x_{n+2})}{2}$,
- $M(x_n, x_{n+1}) = \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{3}$,
- $P(x_n, x_{n+1}) = \frac{d(x_n, x_{n+1}) + d(x_n, x_{n+2})}{3} \leq \frac{2d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{3}$,
- $Q(x_n, x_{n+1}) = \frac{d(x_{n+1}, x_{n+2}) + d(x_n, x_{n+2})}{3} \leq \frac{d(x_n, x_{n+1}) + 2d(x_{n+1}, x_{n+2})}{3}$,
- $U(x_n, x_{n+1}) = \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_n, x_{n+2})}{4} \leq \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}$,
- $V(x_n, x_{n+1}) = \frac{2d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_n, x_{n+2})}{5} \leq \frac{3d(x_n, x_{n+1}) + 2d(x_{n+1}, x_{n+2})}{5}$.

Clearly, $S(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. Assume that there exists $j \in \mathbb{N}$ such that $d(x_j, x_{j+1}) \leq d(x_{j+1}, x_{j+2})$. Then, by the definition of S , we have $S(x_j, x_{j+1}) \leq d(x_{j+1}, x_{j+2})$. For $\hat{\varepsilon} := S(x_j, x_{j+1}) > 0$, by condition (DR), we have

$$d(x_{j+1}, x_{j+2}) = d(Tx_j, Tx_{j+1}) < \hat{\varepsilon} = S(x_j, x_{j+1}) \leq d(x_{j+1}, x_{j+2}),$$

which leads to a contradiction. So it must be $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and hence we get $S(x_n, x_{n+1}) = d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Since the sequence $\{d(x_{n+1}, x_n)\}_{n \in \mathbb{N}}$ is strictly decreasing in $[0, \infty)$, we know that

$$\gamma := \lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = \inf_{n \in \mathbb{N}} d(x_{n+1}, x_n) \quad \text{exists.} \quad (3.1)$$

We claim $\gamma = 0$. On the contrary, assume that $\gamma > 0$. For $\delta > 0$, by (3.1), there exists $\ell \in \mathbb{N}$ such that

$$\gamma \leq S(x_{\ell+1}, x_\ell) = d(x_{\ell+1}, x_\ell) < \gamma + \delta.$$

Thus (DS) deduces

$$d(x_{\ell+2}, x_{\ell+1}) < \gamma,$$

a contradiction. Therefore it must be $\gamma = 0$ and we obtain

$$\lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = \inf_{n \in \mathbb{N}} d(x_{n+1}, x_n) = 0. \tag{3.2}$$

Next, we verify that $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X . Let $\epsilon > 0$ be given. Choose $\zeta > 0$ with $3\zeta < \epsilon$. By (DR), there exists $\delta(\zeta) > 0$ such that

$$\zeta \leq S(x, y) < \zeta + \delta(\zeta) \quad \text{implies} \quad d(Tx, Ty) < \zeta. \tag{3.3}$$

Take $\delta' = \min\{1, \zeta, \delta(\zeta)\}$. It is obvious that (3.3) is still true with $\delta(\zeta)$ replaced by δ' . By (3.2), there exists $j_0 \in \mathbb{N}$ such that

$$d(x_{n+1}, x_n) < \frac{\delta'}{8} \quad \text{for all } n \geq j_0. \tag{3.4}$$

Let

$$\mathcal{U} = \left\{ k \in \mathbb{N} : k \geq j_0 \text{ and } d(x_k, x_{j_0}) < \zeta + \frac{\delta'}{2} \right\}.$$

Clearly, $\mathcal{U} \neq \emptyset$, because $j_0 \in \mathcal{U}$. We want to prove that $m \in \mathcal{U}$ implies $m + 1 \in \mathcal{U}$, and then, according to the finite induction principle, we can get

$$\mathcal{U} = \{m \in \mathbb{N} : m \geq j_0\}$$

and

$$d(x_m, x_{j_0}) < \zeta + \frac{\delta'}{2} \quad \text{for all } m \geq j_0.$$

Let $m \in \mathcal{U}$ be given. Then $m \geq j_0$ and

$$d(x_m, x_{j_0}) < \zeta + \frac{\delta'}{2}.$$

If $m = j_0$, then $m + 1 \in \mathcal{U}$, by (3.4). If $m > j_0$, we consider the following two possible cases:

Case 1. Assume that $\zeta \leq d(x_m, x_{j_0}) < \zeta + \frac{\delta'}{2}$. Since

$$\begin{aligned} K(x_m, x_{j_0}) &= \frac{1}{2} (d(x_m, x_{m+1}) + d(x_{j_0}, x_{j_0+1})) \\ &< \frac{1}{2} \left(\frac{\delta'}{8} + \frac{\delta'}{8} \right) \\ &= \frac{\delta'}{8} < \zeta + \delta', \end{aligned}$$

$$\begin{aligned} C(x_m, x_{j_0}) &= \frac{1}{2} (d(x_m, x_{j_0+1}) + d(x_{j_0}, x_{m+1})) \\ &\leq \frac{1}{2} (2d(x_m, x_{j_0}) + d(x_{j_0}, x_{j_0+1}) + d(x_m, x_{m+1})) \\ &< \frac{1}{2} \left(2 \left(\zeta + \frac{\delta'}{2} \right) + \frac{\delta'}{8} + \frac{\delta'}{8} \right) \\ &= \zeta + \frac{5}{8} \delta' < \zeta + \delta', \end{aligned}$$

$$\begin{aligned}
I(x_m, x_{j_0}) &= \frac{1}{2} (d(x_m, x_{m+1}) + d(x_{j_0}, x_{m+1})) \\
&\leq \frac{1}{2} (2d(x_m, x_{m+1}) + d(x_{j_0}, x_m)) \\
&< \frac{1}{2} \left(\frac{\delta'}{4} + \zeta + \frac{\delta'}{2} \right) \\
&= \frac{\zeta}{2} + \frac{3}{8}\delta' < \zeta + \delta',
\end{aligned}$$

$$\begin{aligned}
J(x_m, x_{j_0}) &= \frac{1}{2} (d(x_{j_0}, x_{m+1}) + d(x_{j_0}, x_{j_0+1})) \\
&\leq \frac{1}{2} (d(x_{j_0}, x_m) + d(x_m, x_{m+1}) + d(x_{j_0}, x_{j_0+1})) \\
&< \frac{1}{2} \left(\zeta + \frac{\delta'}{2} + \frac{\delta'}{8} + \frac{\delta'}{8} \right) \\
&= \frac{\zeta}{2} + \frac{3}{8}\delta' < \zeta + \delta',
\end{aligned}$$

$$\begin{aligned}
M(x_m, x_{j_0}) &= \frac{1}{3} (d(x_m, x_{m+1}) + d(x_{j_0}, x_{j_0+1}) + d(x_{j_0}, x_{m+1})) \\
&\leq \frac{1}{3} (2d(x_m, x_{m+1}) + d(x_{j_0}, x_{j_0+1}) + d(x_{j_0}, x_m)) \\
&< \frac{1}{3} \left(\frac{\delta'}{4} + \frac{\delta'}{8} + \zeta + \frac{\delta'}{2} \right) \\
&= \frac{\zeta}{3} + \frac{7}{24}\delta' < \zeta + \delta',
\end{aligned}$$

$$\begin{aligned}
P(x_m, x_{j_0}) &= \frac{1}{3} (d(x_m, x_{m+1}) + d(x_m, x_{j_0+1}) + d(x_{j_0}, x_{m+1})) \\
&\leq \frac{1}{3} (2d(x_m, x_{m+1}) + 2d(x_m, x_{j_0}) + d(x_{j_0}, x_{j_0+1})) \\
&< \frac{1}{3} \left(\frac{\delta'}{4} + 2 \left(\zeta + \frac{\delta'}{2} \right) + \frac{\delta'}{8} \right) \\
&= \frac{2}{3}\zeta + \frac{11}{24}\delta' < \zeta + \delta',
\end{aligned}$$

$$\begin{aligned}
Q(x_m, x_{j_0}) &= \frac{1}{3} (d(x_{j_0}, x_{j_0+1}) + d(x_m, x_{j_0+1}) + d(x_{j_0}, x_{m+1})) \\
&\leq \frac{1}{3} (2d(x_{j_0}, x_{j_0+1}) + 2d(x_m, x_{j_0}) + d(x_m, x_{m+1})) \\
&< \frac{1}{3} \left(\frac{\delta'}{4} + 2 \left(\zeta + \frac{\delta'}{2} \right) + \frac{\delta'}{8} \right) \\
&= \frac{2}{3}\zeta + \frac{11}{24}\delta' < \zeta + \delta',
\end{aligned}$$

$$\begin{aligned}
 U(x_m, x_{j_0}) &= \frac{1}{4} (d(x_m, x_{m+1}) + d(x_{j_0}, x_{j_0+1}) + d(x_m, x_{j_0+1}) + d(x_{j_0}, x_{m+1})) \\
 &\leq \frac{1}{2} (d(x_m, x_{m+1}) + d(x_{j_0}, x_{j_0+1}) + d(x_m, x_{j_0})) \\
 &< \frac{1}{2} \left(\frac{\delta'}{8} + \frac{\delta'}{8} + \zeta + \frac{\delta'}{2} \right) \\
 &= \frac{1}{2} \zeta + \frac{3}{8} \delta' < \zeta + \delta',
 \end{aligned}$$

and

$$\begin{aligned}
 V(x_m, x_{j_0}) &= \frac{1}{5} (d(x_m, x_{j_0}) + d(x_m, x_{m+1}) + d(x_{j_0}, x_{j_0+1}) + d(x_m, x_{j_0+1}) + d(x_{j_0}, x_{m+1})) \\
 &\leq \frac{1}{5} (3d(x_m, x_{j_0}) + 2d(x_m, x_{m+1}) + 2d(x_{j_0}, x_{j_0+1})) \\
 &< \frac{1}{5} \left(3 \left(\zeta + \frac{\delta'}{2} \right) + \frac{\delta'}{4} + \frac{\delta'}{4} \right) \\
 &= \frac{3}{5} \zeta + \frac{2}{5} \delta' < \zeta + \delta',
 \end{aligned}$$

we get

$$\zeta \leq d(x_m, x_{j_0}) \leq S(x_m, x_{j_0}) < \zeta + \delta'.$$

By (DS), we have

$$d(Tx_m, Tx_{j_0}) < \zeta.$$

From the triangle inequality, we get

$$\begin{aligned}
 d(x_{m+1}, x_{j_0}) &\leq d(x_{m+1}, x_{j_0+1}) + d(x_{j_0+1}, x_{j_0}) \\
 &< \zeta + \frac{\delta'}{8} < \zeta + \frac{\delta'}{2},
 \end{aligned}$$

which means that $m + 1 \in \mathcal{U}$.

Case 2. Assume that $d(x_m, x_{j_0}) < \zeta$. We then have

$$\begin{aligned}
 d(x_{m+1}, x_{j_0}) &\leq d(x_{m+1}, x_m) + d(x_m, x_{j_0}) \\
 &< \zeta + \frac{\delta'}{8} < \zeta + \frac{\delta'}{2},
 \end{aligned}$$

which shows that $m + 1 \in \mathcal{U}$.

Therefore, from Cases 1 and 2, we get

$$d(x_m, x_{j_0}) < \zeta + \frac{\delta'}{2} \text{ for all } m \geq j_0. \tag{3.5}$$

For $m, n \in \mathbb{N}$ with $m \geq n \geq j_0$, by (3.5), we obtain

$$d(x_m, x_n) \leq d(x_m, x_{j_0}) + d(x_{j_0}, x_n) < 2\zeta + \delta' \leq 3\zeta < \epsilon,$$

which shows that $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X . By the completeness of X , there exists $v \in X$ such that $x_n \rightarrow v$ as $n \rightarrow \infty$. Now, we verify that $v \in \mathcal{F}(T)$. For any $n \in \mathbb{N}$, we have

- $K(x_n, v) = \frac{d(x_n, x_{n+1}) + d(v, Tv)}{2},$

- $C(x_n, v) = \frac{d(x_n, Tv) + d(v, x_{n+1})}{2}$,
- $I(x_n, v) = \frac{d(x_n, x_{n+1}) + d(v, x_{n+1})}{2}$,
- $J(x_n, v) = \frac{d(v, x_{n+1}) + d(v, Tv)}{2}$,
- $M(x_n, v) = \frac{d(x_n, x_{n+1}) + d(v, Tv) + d(v, x_{n+1})}{3}$,
- $P(x_n, v) = \frac{d(x_n, x_{n+1}) + d(x_n, Tv) + d(v, x_{n+1})}{3}$,
- $Q(x_n, v) = \frac{d(v, Tv) + d(x_n, Tv) + d(v, x_{n+1})}{3}$,
- $U(x_n, v) = \frac{d(x_n, x_{n+1}) + d(v, Tv) + d(x_n, Tv) + d(v, x_{n+1})}{4}$,
- $V(x_n, v) = \frac{d(x_n, v) + d(x_n, x_{n+1}) + d(v, Tv) + d(x_n, Tv) + d(v, x_{n+1})}{5}$.

Since $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$, we have $d(x_n, x_{n+1}) > 0$ and hence

$$S(x_n, v) = \max\{d(x_n, v), K(x_n, v), C(x_n, v), I(x_n, v), J(x_n, v), \\ M(x_n, v), P(x_n, v), Q(x_n, v), U(x_n, v), V(x_n, v)\} > 0$$

for all $n \in \mathbb{N}$. By (DR), we have

$$d(x_{n+1}, Tv) < S(x_n, v) \quad \text{for all } n \in \mathbb{N}.$$

Since $x_n \rightarrow v$ as $n \rightarrow \infty$, by taking the limit as $n \rightarrow \infty$ on the last inequality, we get

$$d(v, Tv) \leq \frac{2}{3}d(v, Tv)$$

which implies $d(v, Tv) = 0$. Therefore we obtain $v \in \mathcal{F}(T)$. We claim that $\mathcal{F}(T)$ is a singleton set. Suppose there exist $u, v \in \mathcal{F}(T)$ with $u \neq v$. So $d(u, v) > 0$. Since

$$K(u, v) = 0,$$

$$C(u, v) = d(u, v),$$

$$I(u, v) = J(u, v) = U(u, v) = \frac{1}{2}d(u, v),$$

$$M(u, v) = \frac{1}{3}d(u, v)$$

$$P(u, v) = Q(u, v) = \frac{2}{3}d(u, v),$$

and

$$V(u, v) = \frac{3}{5}d(u, v),$$

by (DR), we obtain

$$d(u, v) = d(Tu, Tv) < S(u, v) = d(u, v),$$

a contradiction. Therefore $\mathcal{F}(T)$ is a singleton set and T has a unique fixed point in X . The proof is completed.

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