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On single valued contractions of Banach and Kannan type in G-metric spaces endowed with a graph

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Abstract

In this paper, we discuss the existence of fixed points for Banach and Kannan contractions defined on G-metric spaces, which were introduced by Mustafa and Sims, endowed with a graph. Our results generalize and unify some recent results by Jachymski, Bojor and Mustafa and those contained therein. Moreover, we provide some examples to show that our results are substantial improvement of some known results in literature.

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1. Introduction and Preliminaries

Investigation of the existence and uniqueness of fixed points of certain mappings in the framework of metric spaces is one of the centers of interests in nonlinear functional analysis [1]. Fixed point theory has a wide application in almost all fields of quantitative sciences such as economics, biology, physics, chemistry, computer science and many branches of engineering. It is quite natural to consider various generalizations of metric spaces in order to address the needs of these quantitative sciences. Different mathematicians tried to generalize the usual notion of metric space (X, d). In the 1960s, Gähler ([8, 9]) tried to generalize the notion of metric and introduced the concept of 2-metric spaces inspired by the mapping that associated the area of a triangle to its three vertices. But different authors proved that there is no relation between these two functions [10]. Then, in 1992 Dhage

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([4, 5, 7]) in his Ph. D. thesis introduce a new class of generalized metric space called D-metric spaces. Unfortunately, both kinds of metrics appear not to have as good properties as their authors announced ([10, 13]). To overcome these drawbacks, in 2003 Mustafa and Sims [12] showed that most of the results claimed concerning of such spaces are invalid. Then they introduced a generalization of metric spaces (X, d), which are called G-metric spaces ([13, 14]). The G-metric space is defined as follows:

Definition 1.1 ([14]). *Let X be a nonempty set, and $G : X \times X \times X \to [0, +\infty)$ be a function satisfying:*

(G1) $G(x, y, z) = 0$ *if* $x = y = z$,

(G2) $0 < G(x, x, y)$; *for all* $x, y \in X$, *with* $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$; *for all* $x, y, z \in X$ *with* $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, *(symmetry in all three variables),*

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, *for all* $x, y, z, a \in X$, *(rectangle inequality).*

Then the function G is called a generalized metric, or, more specifically a G-metric on X, and the pair (X, G) is a G-metric spaces.

A G-metric space (X, G) is called symmetric G-metric space if $G(x, x, y) = G(x, y, y)$; for all $x, y \in X$.

Example 1.2 ([14]). *Let (X, d) be a metric space. The function $G : X \times X \times X \to [0, +\infty)$, defined as*

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}, \tag{1.1}$$

or

$$G(x, y, z) = d(x, y) + d(y, z) + d(z, x), \tag{1.2}$$

for all $x, y, z \in X$, is a G-metric on X.

Definition 1.3 ([14]). *Let (X, G) be a G-metric space, then a sequence $\{x_n\}$ is said to be G-Cauchy sequence if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$.*

Proposition 1.4 ([14]). *Let (X, G) be a G-metric space. the following are equivalent:*

1) *the sequence $\{x_n\}$ is G-Cauchy,*

2) *for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq N$.*

Definition 1.5 ([14]). *Let (X, G) be a G-metric space, and $\{x_n\}$ be a sequence of points of X. we say that $\{x_n\}$ is G-convergent to $x \in X$ if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$.*

Proposition 1.6 ([14]). *Let (X, G) be a G-metric space. the following statements are equivalent:*

1) *$\{x_n\}$ is G-convergent to x,*

2) $G(x_n, x, x) \to 0$ *as* $n \to +\infty$,

3) $G(x_n, x_n, x) \to 0$ *as* $n \to +\infty$,

4) $G(x_n, x_m, x) \to 0$ *as* $n, m \to +\infty$.

Definition 1.7 ([14]). *Let (X, G) be a G-metric space. A mapping $f : X \to X$ is said to be G-continuous if $\{f(x_n)\}$ is G-convergent to $f(x)$ where $\{x_n\}$ is any G-convergent sequence converging to x.*

Definition 1.8 ([14]). *A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).*

Proposition 1.9 ([14]). *Every G-metric space (X, G) induces a metric space (X, d_G) defined by*

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \qquad \text{for all } x, y \in X.$$

If (X, G) is symmetric, then

$$d_G(x, y) = 2\,G(x, y, y), \qquad \text{for all } x, y \in X.$$

However, if (X, G) is not symmetric, then it holds that:

$$\frac{3}{2}G(x, y, y) \le d_G(x, y) \le 3G(x, y, y), \qquad \text{for all } x, y \in X.$$

Theorem 1.10 ([13]). *Let (X, G) be a complete G-metric space and $T : X \to X$ be a mapping satisfying the following condition for all $x, y, z \in X$:*

$$G(Tx, Ty, Tz) \le kG(x, y, z),$$

where $k \in [0, 1)$. Then T has a unique fixed point.

Theorem 1.11 ([15]). *Let (X, G) be a complete G-metric space and $T : X \to X$ be a mapping satisfying the following condition for all $x, y, z \in X$:*

$$G(Tx, Ty, Tz) \le k\{G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)\},$$

where $k \in [0, 1/3)$. Then T has a unique fixed point.

We next review some notions in graph theory. All of them can be found in, e.g., [2].

Let X be a G-metric space. Consider a directed graph H with $V(H) = X$ and $E(H) \supseteq \{(x, x) : x \in X\}$, i.e., $E(H)$ contains all loops. Suppose further that H has no parallel edges. With these assumptions, we may denote H by the pair $(V(H), E(H))$. In this way, the G-metric space X is endowed with the graph H. The notation $\widetilde{H}$ is used to denote the undirected graph obtained from H by deleting the directions of the edges of H. Thus,

$$V(\widetilde{H}) = X \qquad E(\widetilde{H}) = \big\{(x, y) \in X \times X : (x, y) \in E(H) \ \vee \ (y, x) \in E(H)\big\}.$$

By a path in H from a vertex x to a vertex y, it is meant a finite sequence $(x_s)_{s=0}^{N}$ of vertices of H such that $x_0 = x$, $x_N = y$, and $(x_{s-1}, x_s) \in E(H)$ for $s = 1, \ldots, N$. A graph H is called connected if there is a path between any two vertices and is called weakly connected if $\widetilde{H}$ is connected, i.e., there exists an undirected path in H between its each two vertices.

If H is such that $E(H)$ is symmetric, then for $x \in V(H)$, the symbol $[x]_H$ denotes the equivalence class of the relation $\mathcal{R}$ defined on $V(H)$ by the rule:

$$y\mathcal{R}z \text{ if there is a path in } H \text{ from } y \text{ to } z.$$

Recall that if $T : X \to X$ is an operator, then by $\mathrm{Fix}T = \{x \in X \ : \ T(x) = x\}$ we denote the set of all fixed points of T. Denote also

$$X_T = \{x \in X \ : \ (x, Tx) \in E(H)\}.$$

Moreover, we may treat H as a G-weighted graph by assigning to each three vertices x, y and z in X the G-distance $G(x, y, z)$.

2. Main results

The aim of this paper is to study the existence of fixed points for Banach and Kannan H-contractions in G-metric spaces endowed with a graph H by introducing the concept of Banach and Kannan H-contractions according to the articles of Jakhymski and Bojor [11, 3].

Definition 2.1. *Let (X, G) be a G-metric space with a graph H and $T : X \to X$ be a mapping. We call T a Banach H-contraction if T preserves edges of H, i.e.,*

$$\forall x, y \in X, \ (x, y) \in E(H) \Rightarrow (Tx, Ty) \in E(H),$$

and there exists $k \in [0, 1)$ such that $G(Tx, Ty, Tz) \leq kG(x, y, z)$ for all x, y and z that are on a path to length at most 2 in H.

It might be valuable if we discuss these contractions a little. Our first proposition follows immediately from Definition 2.1.

Proposition 2.2. *Let X be a G-metric space with a graph H. If a mapping T from X into itself is a Banach H-contraction, then T is also a Banach $\widetilde{H}$-contraction.*

We now give some examples.

Example 2.3. *Let X be a G-metric space with any arbitrary graph H. Since $E(H)$ contains all loops, each constant mapping $T : X \to X$ is a Banach H-contraction.*

Example 2.4. *Let X be a G-metric space and H_0 be the complete graph $(X, X \times X)$. Then Banach H_0-contractions are precisely the Banach contractions in G-metric space.*

Example 2.5. *Let $\preceq$ be a partial order on a G-metric space X and consider a graph H_1 by $V(H_1) = X$ and $E(H_1) = \big\{(x, y) \in X \times X : x \preceq y\big\}$. Then Banach H_1-contractions are precisely the nondecreasing ordered H-contractions.*

Theorem 2.6. *Let X be a complete G-metric space endowed with a graph H and the triple (X, G, H) have the following property:*

- $(*)$ *if $\{x_n\} \to x$ is a sequence in X whose consecutive terms are adjacent, then there exists a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ of $\{x_n\}$such that whose consecutive terms are adjacent and every term's is adjacent to x.*

Then a Banach $\widetilde{H}$-contraction $T : X \to X$ has a fixed point if and only if $X_T \neq \emptyset$.

Proof . $(\Rightarrow)$ It is trivial, since $\mathrm{Fix}(T) \subseteq X_T$.

$(\Leftarrow)$ Let $x_0 \in X$ with $(x_0, Tx_0) \in E(H)$, and $x_n = T^n(x_0)$

$$G(x_n, x_{n+1}, x_{n+1}) \leq kG(x_{n-1}, x_n, x_n).$$

Continuing in the same argument, we will find

$$G(x_n, x_{n+1}, x_{n+1}) \leq k^n G(x_0, x_1, x_1).$$

Further, for all $n, m \in N; n < m$, we have

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G(x_{n+2}, x_{n+3}, x_{n+3}) + ... + G(x_{m-1}, x_m, x_m) \\ &\leq (k^n + k^{n+1} + k^{n+2} + ... + k^{m-1})G(x_0, x_1, x_1) \\ &\leq \frac{k^n}{1-k} G(x_0, x_1, x_1). \end{aligned}$$

Then, $\lim G(x_n, x_m, x_m) = 0$, as $n, m \to \infty$. So $\{x_n\}$ is G-Cauchy sequence. Since (X, G) is complete, there exists $u \in X$ such that $\{x_n\}$ is G-convergent to u. From the hypothesis, we have

$$G(x_{n_k}, T(u), T(u)) \leq kG(x_{n_{k-1}}, u, u).$$

Taking the limit as $k \to \infty$, and since function G is continuous on its variable, $G(u, T(u), T(u)) \leq kG(u, u, u)$. So, $G(u, T(u), T(u)) = 0$. Which implies that $T(u) = u$. Then T on (X, G) has a fixed point. □

As an immediate consequence of Theorem 2.6 , we obtain the following

Corollary 2.7. *Let (X, G) be complete, and let the triple (X, G, H) have the property (*). Suppose also that $T : X \to X$ be a Banach H-contraction. Then the following statements hold.*

1) $\mathrm{cardFix}(T) = \mathrm{card}\{[x]_{\widetilde{H}} : x \in X_T\}$

2) $\mathrm{Fix}(T) = \emptyset$ *if and only if* $X_T = \emptyset$

3) T has a unique fixed point if and only if there exists $x_0 \in X_T$ such that $X_T \subseteq [x]_{\widetilde{H}}$.

Example 2.8. *Take the complete G-metric space $X = [0, +\infty)$ with the G-distance $G(x, y, z) = d(x, y)+d(x, z)+d(y, z)$, where d is Euclidean metric on X, and consider the graph H with $V(H) = X$ and*

$$(x, y) \in E(H) \Leftrightarrow \begin{cases} x, y \in [0, 1], x \leq y, \\ \textit{or} \\ x, y \in (n, n+1], \ \textit{for some } n = 1, 2, \ldots, x \leq y. \end{cases}$$

Let T be defined as

$$T(x) = \begin{cases} \frac{1}{2}x, & x \in [0, 1], \\ (n-1) + \frac{1}{2}(x - n), & x \in (n, n+1], n \textit{ is even}, \\ n - \frac{1}{2}(x - n), & x \in (n, n+1], n \textit{ is odd}, \end{cases}$$

Then T is not a Banach G-contraction, because T is not continuous, but T is a Banach H-contraction with a constant $k = \frac{1}{2}$.

Definition 2.9. *Let (X, G) be a G-metric space with a graph H and $T : X \to X$ be a mapping. We call T a Kannan H-contraction if*

K1) *T preserves edges of H;*

K2) *there exists $k \in [0, \frac{1}{3})$ such that*

$$G(Tx, Ty, Tz) \leq k\{G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)\}$$

for all x, y and z that are on a path to length at most 2 in H.

Example 2.10. *Any constant function $T : X \to X$ is a Kannan H-contraction, since $E(H)$ contains all loops.*

Example 2.11. *Any Kannan G-contraction is a H_0-contraction, where the graph H_0 is defined by $E(H_0) := X \times X$.*

Example 2.12. *Let $\preceq$ be a partial order on a G-metric space X and let H_1 be the graph introduced in Example 2.5. Then Kannan H_1-contractions are precisely the nondecreasing ordered Kannan H_1-contractions.*

Remark 2.13. *It is clear that every Kannan H-contraction is Kannan $\widetilde{H}$-contraction, but the converse is not true.*

Our next result is about the existence of fixed points for Kannan H-contractions.

Theorem 2.14. *Let X be a complete G-metric space endowed with a graph H and the triple (X, G, H) have property $(*)$. Then a Kannan $\widetilde{H}$-contraction $T : X \to X$ has a fixed point if and only if $X_T \neq \emptyset$.*

Proof . $(\Rightarrow)$ It is trivial, since $\mathrm{Fix}(T) \subseteq X_T$.

$(\Leftarrow)$ Suppose that $x_0 \in X_T$, and $x_n = T^n(x_0)$.

$$G(x_n, x_{n+1}, x_{n+1}) \leq kG(x_{n-1}, x_n, x_n) + 2kG(x_n, x_{n+1}, x_{n+1}).$$

Then,

$$G(x_n, x_{n+1}, x_{n+1}) \leq \tfrac{k}{1-2k} G(x_{n-1}, x_n, x_n).$$

Let $\lambda = \frac{k}{1-2k}$, and since $0 \leq k < \frac{1}{3}$, $0 \leq \lambda < 1$. So,

$$G(x_n, x_{n+1}, x_{n+1}) \leq \lambda G(x_{n-1}, x_n, x_n).$$

Continuing in the same argument, we will get

$$G(x_n, x_{n+1}, x_{n+1}) \leq \lambda^n G(x_0, x_1, x_1).$$

Further, for all $n, m \in N; n < m$, we have

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G(x_{n+2}, x_{n+3}, x_{n+3}) + ... + G(x_{m-1}, x_m, x_m) \\ &\leq (\lambda^n + \lambda^{n+1} + \lambda^{n+2} + ... + \lambda^{m-1}) G(x_0, x_1, x_1) \\ &\leq \frac{\lambda^n}{1-\lambda} G(x_0, x_1, x_1). \end{aligned}$$

Then, $\lim G(x_n, x_m, x_m) = 0$, as $n, m \to \infty$. So $\{x_n\}$ is G-Cauchy sequence. Since (X, G) is complete, there exists $u \in X$ such that $\{x_n\}$ is G-convergent to u in (X, G). To prove that T has a fixed point, let $T(u) \neq u$. From the hypothesis, we have,

$$G(x_{n_{k+1}}, T(u), T(u)) \leq k\{G(x_{n_k}, x_{n_{k+1}}, x_{n_{k+1}}) + 2G(u, T(u), T(u))\}.$$

Taking the limit as $k \to \infty$, and since function G is continuous on its variable, $G(u, T(u), T(u)) \leq 2kG(u, T(u), T(u))$. This contradiction implies that $T(u) = u$. Then T on X has a fixed point. $\square$

Finally, we note that the conditions of Theorem 2.1 and condition (K2) of Theorem 2.9 are independent to each other, and thus Theorem 2.1 and Theorem 2.9 could not contain each other as a special case.

Example 2.15. *Let G be the G-metric defined by (1.2) on $\mathbb{R}$, where d is the usual Euclidean metric on $\mathbb{R}$. Define a mapping $T : \mathbb{R} \to \mathbb{R}$ by $Tx = \frac{x}{\alpha}$, for all $x \in \mathbb{R}$. Then for each $\alpha > 3$, T is a Banach H_0-contraction with the constants $k = \frac{1}{\alpha}$. Indeed, given for any $x, y, z \in \mathbb{R}$, we have*

$$G(Tx, Ty, Tz) = \frac{1}{\alpha} G(x, y, z).$$

On the other hand, if we let $y, z = 0$ and any $x \neq 0$ we see that

$$G(Tx, T0, T0) = \frac{2}{\alpha}|x| \text{ and } G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz) = \frac{4}{\alpha}|x|.$$

Therefore, since $k < \frac{1}{3}$, (K2) fails to hold and T is not a Kannan H_0-contraction.

Example 2.16. *Let G be the G-metric defined by (1.2) on $\mathbb{R}$, where d is usual Euclidean metric on $\mathbb{R}$. Now, consider a mapping $T : \mathbb{R} \to \mathbb{R}$ defined by*

$$T(x) = \begin{cases} \frac{1}{9}, & x \neq 1, \\ \frac{1}{10}, & x = 1, \end{cases}$$

Then T is a Kannan H_0-contraction with the constant $k = \frac{1}{42}$. Indeed, given any $x, y, z \in \mathbb{R}$, we have the following three possible cases:

Case 1: *If $x = y = z = 1$ or $x, y, z \neq 1$, then (K2) holds trivially, since $Tx = Ty = Tz$;*

Case 2: *If $x = 1$ and $y, z \neq 1$, then*

$$\begin{aligned} G(T1, Ty, Tz) &= G\left(\frac{1}{10}, \frac{1}{9}, \frac{1}{9}\right) \\ &= \frac{1}{45} \\ &< k\{G(1, T1, T1) + G(y, Ty, Ty) + G(z, Tz, Tz)\} \\ &= k\left\{\frac{9}{5} + 2|y - \frac{1}{9}| + 2|z - \frac{1}{9}|\right\} \end{aligned}$$

Case 3: *Finally, if $x, y = 1$ and $z \neq 1$, then*

$$\begin{aligned} G(T1, T1, Tz) &= G\left(\frac{1}{10}, \frac{1}{10}, \frac{1}{9}\right) \\ &= \frac{1}{45} \\ &< k\{G(1, T1, T1) + G(1, T1, T1) + G(z, Tz, Tz)\} \\ &= k\left\{\frac{9}{5} + \frac{9}{5} + 2|z - \frac{1}{9}|\right\} \end{aligned}$$

where $k = \frac{1}{42}$. But T is not a Banach H_0-contraction; for if we let $x = 1$ and $y, z = \frac{89}{90}$, then

$$G\left(T1, T\frac{89}{90}, T\frac{89}{90}\right) = \frac{1}{45} \quad \text{and} \quad G\left(1, \frac{89}{90}, \frac{89}{90}\right) = \frac{1}{45}.$$

Therefore, for each $k \in [0, 1)$ we have $G(T1, T\frac{89}{90}, T\frac{89}{90}) > kG(1, \frac{89}{90}, \frac{89}{90})$.

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