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$(G.\psi)-$Ciric-Reich-Rus contraction on metric space endowed with a graph

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Abstract

In this paper, we introduce the $(G,\psi)-$Ciric-Reich-Rus contraction on metric space endowed with a graph, such that (X,d) is a metric space, and $V(G)$ is the vertices of G coincides with X. We give an example to show that our results generalize some known results

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1. Introduction and preliminaries

One of the most attractive areas of the fixed point theory is the existence of fixed points in a metric space respect to a given graph. Recently Jachymski [9] has given some generalizations of the Banach Contraction Principle to mappings on a metric space respect to a graph. In order to study $\psi-$Ciric-Reich-Rus type contraction, we need the following definitions. (see also [1, 2, 3, 8, 10, 11, 12, 13, 14, 15, 16, 18, 19, 21])
Let (X,d) be a metric space, and Δ be the diagonal of $X \times X$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Delta$. Let G has no parallel edges, so one can identify G with the pair $(V(G),E(G))$.
By G^{-1} we denote the graph obtained from G by reversing the direction of edges, and call it the reverse of graph G. Thus,

$E(G^{-1}) = \{(x,y) \in X \times X | (y,x) \in E(G)\}$.
$\tilde{G}$ is the undirected graph that obtained from G by remove the direction of edges. So we have,

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$E(\tilde{G}) = E(G) \bigcup E(G^{-1})$.

A path from x to y of length $N (N \in \mathbf{N})$ is a sequence $(x_i)_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x, x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, ..., N$.
G is weakly connected if $\tilde{G}$ is connected. $[x]_G$ is the equivalence class of relations $\Re$ defined on $V(G)$ by the rule:

$z \Re y$ if there is a path in G from z to y.
G_x is called the component of G which consists of all edges and vertices which are contained in some path beginning at x.
If $f : X \to X$ is an operator, then

$X^f := \{x \in X : (x, fx)\} \in E(G)\}$,
and the set of all fixed points of f is denoted by

$F_f := \{x \in X : f(x) = x\}$.

Definition 1.1. *[4] The operator $f : X \to X$ is called a $G-$Ciric-Reich-Rus operator if:*

1. *for all $x, y \in X$ if $(x, y) \in E(G)$ then $(Tx, Ty) \in E(G)$;*
2. *There exists $\alpha, \beta, \gamma \in \boldsymbol{R}^+$ with $\alpha + \beta + \gamma \in (0, 1)$, such that for each $x, y \in X$ we have, $d(fx, fy) \leq \alpha d(x, y) + \beta d(x, fx) + \gamma d(y, fy)$.*

Definition 1.2. *[20] The operator $f : X \to X$ is called a Picard operator (PO) if:*

(i) *f has a unique fixed point x^*;*

(ii) *For all $x \in X$, we have $\lim_{n\to\infty} T^n x = x^*$.*

Definition 1.3. *[20] The operator $f : X \to X$ is called a weakly Picard operator (WPO) if:*

(i) $F_f \neq \varnothing$;

(ii) *for all $x \in X$, we have $\lim_{n\to\infty} T^n x = x^*(x)$.*
($x^*(x)$ is the fixed point of f which depened on x)

Definition 1.4. *[9] A mapping $f : X \to X$ is called orbitally continuous if for all $x, y \in X$ and any sequence $(K_n)_{n\in N}$ of positive integers,*
$f^{k_n} x \to y$, implise $f(f^{k_n} x) \to fy$ as $n \to \infty$.

Definition 1.5. *[9] A mapping $f : X \to X$ is called orbitally $G-$ continuous if for all $x, y \in X$ and any sequence $(K_n)_{n\in N}$ of positive integers,*
$f^{k_n} x \to y$, $(f^{k_n} x, f^{k_{n+1}} x) \in E(G)$ imply $f(f^{k_n} x) \to fy$ as $n \to \infty$.

Definition 1.6. *[18] Let us define the class $\Psi = \{\psi : \boldsymbol{R}^+ \to \boldsymbol{R}^+ \mid \psi$ is nondecreasing $\}$ which satisfies the following conditions:*

(i) *$\psi(w) = 0$ if and only if $w = 0$;*

(ii) *for every $(w_n) \in \boldsymbol{R}^+$, $\psi(w_n) \to 0$ if and only if $w_n \to 0$;*

(iii) *for every $w_1, w_2 \in \boldsymbol{R}^+, \psi(w_1 + w_2) \leq \psi(w_1) + \psi(w_2)$.*

In the next section, we state two fixed point theorems for $(G, \psi)-$Ciric-Reich-Rus type contraction.

2. Main results

In this section, we assume that (X,d) is a metric space, and G is a directed graph such that $V(G)=X$, $\triangle\subseteq E(G)$ and G has no parallel edges.

Definition 2.1. *A mapping $f:X\to X$ is called $(G,\psi)-Ciric-Reich-Rus$ contraction if:*

(i) for all $x,y\in X$ if $(x,y)\in E(G)$ then $(Tx,Ty)\in E(G)$;

(ii) there exists $\alpha,\beta,\gamma\in \boldsymbol{R}^+$, with $\alpha+\beta+\gamma\in(0,1)$, such that for each $(x,y)\in E(G)$ implies $\psi(d(fx,fy))\leq \alpha\psi(d(x,y))+\beta\psi(d(x,fx))+\gamma\psi(d(y,fy))$.

The following Lemma is immediately.

Lemma 2.2. *If $f:X\to X$ is a $(G,\psi)-Ciric-Reich-Rus$ contraction then f is both a $(G^{-1},\psi)-Ciric-Reich-Rus$ contraction and a $(\tilde{G},\psi)-Ciric-Reich-Rus$ contraction.*

Lemma 2.3. *Let $f:X\to X$ be a $(G,\psi)-Ciric-Reich-Rus$ with the constants α,β,γ. Then, for given $x\in X^f$, there exists $r(x)\geq 0$ such that*

$$\psi(d(f^nx,f^{n+1}x))\leq a^n r(x),$$

for all $n\in N$, where $a:=\dfrac{\alpha+\beta}{1-\gamma}$.

Proof . *Assume that $x\in X^f$, then by induction, we have $(f^nx,f^{n+1}x)\in E(G)$ for each $n\in N$. So $\psi(d(f^nx,f^{n+1}x))\leq\alpha\psi(d(f^{n-1}x,f^nx))+\beta\psi(d(f^{n-1}x,f^nx))+\gamma\psi(d(f^nx,f^{n+1}x))$.*
Hence $\psi(d(f^nx,f^{n+1}x))\leq\dfrac{\alpha+\beta}{1-\gamma}\psi(d(f^{n-1}x,f^nx))\leq\cdots\leq a^n\psi(d(x,fx))$. Set $r(x):=\psi(d(x,fx))$.
□

Lemma 2.4. *Assume that (X,d) is a complete metric space and $f:X\to X$ is a $(G,\psi)-Ciric-Reich-Rus$ contraction with the constants α,β,γ. Then, for each $x\in X^f$, there exists $x^*(x)\in X$ such that the sequence $(f^nx)_{n\in N}$ converges to $x^*(x)$ as $n\to\infty$.*

Proof . *Let $x\in X^f$. By Lemma 2.3, $\psi(d(f^nx,f^{n+1}x))\leq a^nr(x)$. Hence $\sum_{n=0}^{\infty}\psi(d(f^nx,f^{n+1}x))<\infty$. Thus $\psi(d(f^nx,f^{n+1}x))\to 0$ as $n\to\infty$.*
Then we have $d(f^nx,f^{n+1}x)\to 0$. So the sequence $(f^nx)_{n\in N}$ is a Cauchy sequence. Since the space X is complete, there exists $x^*(x)\in X$ such that the sequence $(f^nx)_{n\in N}$ converges to $x^*(x)$ as $n\to\infty$.
□

Theorem 2.5. *Let (X,d) be a complete metric space endowed with a graph G, and let the triple (X,d,G) has the following condition:*
For any $(x_n)_{n\in N}$ in X, if $x_n\to x$ and $(x_n,x_{n+1})\in E(G)$ for all $n\in N$, then there is a subsequence $(x_{k_n})_{n\in N}$ with $(x_{k_n},x)\in E(G)$ for all $n\in N$.
Let $f:X\to X$ be a $(G,\psi)-Ciric-Reich-Rus$ contraction and f be orbitally $G-$continuous. Then the following statements hold.

(i) $F_f\neq\varnothing$ if and only if $X^f\neq\varnothing$.

(ii) If $X^f\neq\varnothing$ and G is weakly connected, then f is a weakly Picard operator.

(iii) *For any $X^f \neq \varnothing$, $f\mid_{[x]_{\tilde{G}}}$ is a weakly Picard operator.*

Proof . *First we prove (iii). Let $x \in X^f$; by Lemma 2.4, there exists $x^* \in X$ such that $\lim_{n\to\infty} f^n x = x^*$. Since $x \in X^f$, then $f^n x \in X^f$ for every $n \in N$. Now assume that $(x, fx) \in E(G)$. By condition (P), there is a subsequence $(f^{k_n}x)_{n\in N}$ of $(f^n x)_{n\in N}$ such that $(f^{k_n}x, x^*) \in E(G)$ for each $n \in N$. Now we have a path in G by using the points $x, fx, \cdots, f^{k_l}x, x^*$ and hence $x^* \in [x]_{\tilde{G}}$. On the other hand since f is orbitally $G-$continuous, we have x^* is a fixed point forf $\mid_{[x]_{\tilde{G}}}$.*
(i) is obtained using (iii), because $F_f \neq \varnothing$ if $X^f \neq \varnothing$. Now suppose that $F_f \neq \varnothing$. By using the assumption that $\triangle \subseteq E(G)$, we obtain $X^f \neq \varnothing$.
For proving (ii) let $x \in X^f$. Because G is weakly connected, we have $X = [x]_{\tilde{G}}$ and (iii) complete the proof. □

Remark 2.6. *Set $\psi(w) = w$ in Theorem 2.5, then Theorem 2.2 in [19] obtain immediately.*

In the next we study the case that $f : X \to X$ as a $(G, \psi) - Ciric - Reich - Rus$ contraction can be a Picard operator. So we need the following definition.

Definition 2.7. *Let (X, d) be a metric space endowed with a graph G and $f : X \to X$ be a mapping. We say that the graph G has a $f-$path property, if for any path in G, $(x_i)_{i=0}^N$ from x to y such that $x_0 = x, x_N = y$ we have $fx_{i-1} = x_i$ for all $i = 1, \cdots, N$.*

Lemma 2.8. *Let (X, d) be a metric space endowed with a graph G and $f : X \to X$ be a $(G, \psi) - Ciric - Reich - Rus$ contraction such that the graph G has the $f-$path property. Then for any $x \in X$ and $y \in [x]_{\tilde{G}}$ two sequences $(f^n x)_{n\in N}$ and $(f^n y)_{n\in N}$ are equivalent.*
Proof . *Let $x \in X$, and let $y \in [x]_{\tilde{G}}$; then there exists a path $(x_i)_{i=0}^l$ in $\tilde{G}$ from x to y such that $x_0 = x, x_l = y$ with $(x_{i-1}, x_i) \in E(G)$ and $fx_{i-1} = x_i$ for all $i = 1, \cdots, l$. From Lemma 2.2, f is a $(\tilde{G}, \psi) - Ciric - Reich - Rus$. Then for all $n \in N$ $(f^n x_{i-1}, f^n x_i) \in E(\tilde{G})$, so*

$$\begin{aligned}\psi(d(f^n x_{i-1}, f^n x_i)) \leq\ & \alpha\psi(d(f^{n-1}x_{i-1}, f^{n-1}x_i)) + \beta\psi(d(f^{n-1}x_{i-1}, f^n x_{i-1})) + \gamma\psi(d(f^{n-1}x_i, f^n x_i))\\ =\ & \alpha\psi(d(f^{n-1}x_{i-1}, f^{n-1}x_i)) + \beta\psi(d(f^{n-1}x_{i-1}, f^{n-1}x_i)) + \gamma\psi(d(f^n x_{i-1}, f^n x_i))\end{aligned}$$

then,

$$\psi(d(f^n x_{i-1}, f^n x_i)) \leq \frac{\alpha+\beta}{1-\gamma}\psi(d(f^{n-1}x_{i-1}, f^{n-1}x_i)).$$

Hence, for all $n \in N$

$$\psi(d(f^n x_{i-1}, f^n x_i)) \leq a^n \psi(d(x_{i-1}, x_i)), \tag{2.1}$$

where $a = \dfrac{\alpha+\beta}{1-\gamma}$. We know that $(f^n x_i)_{i=0}^l$ is a path in $\tilde{G}$ from $f^n x$ to $f^n y$. Using the triangle inequality and (2.1),

$$\psi(d(f^n x, f^n y)) \leq \sum_{i=1}^{l} \psi(d(f^n x_{i-1}, f^n x_i)) \leq a^n \sum_{i=1}^{l} \psi(d(x_{i-1}, x_i)).$$

Letting $n \to \infty$, we get $d(f^n x, f^n y) \to 0$. □

Theorem 2.9. *Let (X, d) be a complete metric space endowed with a graph G, and $f : X \to X$ be a (G, ψ)-Ciric-Reich-Rus contraction such that the graph G has the $f-$path property and f be orbitally $G-$continuous. Let the triple (X, d, G) has the following condition:*
For any $(x_n)_{n\in N}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in N$, then there is a subsequence $(x_{k_n})_{n\in N}$ with $(x_{k_n}, x) \in E(G)$ for all $n \in N$. Let there exists $z \in X$ such that $z \in X^f$, then the following statements hold:

(1) $f\mid_{[z]_{\tilde{G}}}$ is a Picard operator;

(2) if G is weakly connected, then f is a Picard operator.

Proof . (1) *Using (iii) Theorem 2.5, there exists $x^*(z) \in [z]_{\tilde{G}}$ such that $\lim_{n\to\infty} f^n(z) = x^*(z)$, and $x^*(z)$ is a fixed point of f. Now if $y \in [z]_{\tilde{G}}$ and $\lim_{n\to\infty} f^n(y) = x^*(y)$. Then by Lemma 2.8 two sequences $(f^n z)_{n\in N}$ and $(f^n y)_{n\in N}$ are equivalent. Since both are convergent sequence, then they are Cauchy sequences. Hence they are Cauchy equivalent. This means $x^*(y) = x^*(z)$.*
(2) *Since $z \in X^f$ and G is weakly connected, we have $X = [z]_{\tilde{G}}$. Then we only need to apply* (1). □

Definition 2.10. *[18] We say that mapping $f : X \to X$ is a $(G, \psi)-$contraction if the following hold:*

(i) f preserves edges of G, i.e, for all $x, y \in X$ if $(x, y) \in E(G)$ then $(fx, fy) \in E(G)$);

(ii) f decreases the weight of G, that is, there exists $c \in (0, 1)$ such that for all $x, y \in X$ if

$$(x, y) \in E(G) \qquad then \quad \psi(d(fx, fy)) \leq c\psi(d(x, y)).$$

In the following example we show that $(G, \psi)-$Ciric-Reich-Rus contraction is a generalization of $(G, \psi)-$contraction.

Example 2.11. *Let $X = [0, 1]$ and $d(x, y) =\mid x - y \mid$. Define the graph G by*
$E(G) = \{(0, 0), (0, 1)\} \bigcup \{(x, y) \in (0, 1] \times [0, 1] \qquad x \geqslant y\}.$

$f : X \to X$ and

$$fx = \begin{cases} \frac{x}{2}, & x \in (0, 1]; \\ \frac{3}{4}, & x = 0. \end{cases}$$

G is weakly connected, and f is a $(G, \psi)-$Ciric-Reich-Rus contraction with constants, $\alpha = \frac{1}{8}, \beta = \frac{3}{4}, \gamma = \frac{1}{16}, \psi(w) = \dfrac{w}{2}$. But f is not $(G, \psi)-$contraction, because if we consider

$$\psi(d(f(0), f(\frac{1}{2})) \leq c\psi(d(0, \frac{1}{2})$$

Then we have $\frac{1}{4} \leq c\frac{1}{4}$ which is a contradiction since $c \in [0, 1)$.

Definition 2.12. *The mapping $f : X \to X$ is called a $(G, \psi)-$Kannan mapping if:*

(i) for all $x, y \in X$ if $(x, y) \in E(G)$ then $(fx, fy) \in E(G)$);

(ii) there exists a constant $a \in (0,1)$ such that for all $x, y \in X, (x,y) \in E(G)$ then,

$$\psi(d(fx,fy)) \leq a[\psi(d(x,fx)) + \psi(d(y,fy))].$$

Corollary 2.13. *Let (X,d) be a complete metric space endowed with a graph G, and $f : X \to X$ be a $(G,\psi)-$ contraction such that the graph G has the $f-$path property and f be orbitally $G-$continuous. Let the triple (X,d,G) has the following condition:*
For any $(x_n)_{n\in N}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in N$, then there is a subsequence $(x_{k_n})_{n\in N}$ with $(x_{k_n}, x) \in E(G)$ for all $n \in N$. Let there exists $z \in X$ such that $z \in X^f$, then the following statements hold:

(1) $f \mid_{[z]_{\tilde{G}}}$ is a Picard operator;

(2) if G is weakly connected, then f is a Picard operator.

Proof . *If f is a $(G,\psi)-$contraction with constant $c \in [0,1)$, then f is a $(\tilde{G},\psi)-$Ciric-Reich-Rus contraction with constants $\alpha = c, \beta = \gamma = 0$. Hence according to Theorem 2.9, f is a Picard operator.* □

Corollary 2.14. *Let (X,d) be a complete metric space endowed with a graph G, and $f : X \to X$ be a $(G,\psi)-$Kannan mapping such that the graph G has the $f-$path property and f be orbitally $G-$continuous. Let the triple (X,d,G) has the following condition:*
For any $(x_n)_{n\in N}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in N$, then there is a subsequence $(x_{k_n})_{n\in N}$ with $(x_{k_n}, x) \in E(G)$ for all $n \in N$. Let there exists $z \in X$ such that $z \in X^f$, then the following statements hold:

(1) $f \mid_{[z]_{\tilde{G}}}$ is a Picard operator;

(2) if G is weakly connected, then f is a Picard operator.

Proof . *If f is a $(G,\psi)-$Kannan with constant $a \in [0,1)$, then f is a $(\tilde{G},\psi)-Ciric-Reich-Rus$ contraction with constants $\alpha = 0, \beta = \gamma = a$. Hence according to Theorem 2.9, f is a Picard operator.* □

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