

(G, ψ) –Ciric-Reich-Rus contraction on metric space endowed with a graph

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Abstract

In this paper, we introduce the (G, ψ) –Ciric-Reich-Rus contraction on metric space endowed with a graph, such that (X, d) is a metric space, and $V(G)$ is the vertices of G coincides with X . We give an example to show that our results generalize some known results

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1 Introduction and preliminaries

One of the most attractive areas of the fixed point theory is the existence of fixed points in a metric space respect to a given graph. Recently Jachymski [9] has given some generalizations of the Banach Contraction Principle to mappings on a metric space respect to a graph. In order to study ψ –Ciric-Reich-Rus type contraction, we need the following definitions, see also [1, 2, 3, 8, 10, 11, 12, 13, 14, 15, 16, 18, 19, 21].

Let (X, d) be a metric space, and Δ be the diagonal of $X \times X$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X , and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Delta$. Let G has no parallel edges, so one can identify G with the pair $(V(G), E(G))$. By G^{-1} we denote the graph obtained from G by reversing the direction of edges, and call it the reverse of graph G . Thus,

$$E(G^{-1}) = \{(x, y) \in X \times X | (y, x) \in E(G)\}.$$

\tilde{G} is the undirected graph that obtained from G by remove the direction of edges. So we have,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

A path from x to y of length N ($N \in \mathbf{N}$) is a sequence $(x_i)_{i=0}^N$ of $N + 1$ vertices such that $x_0 = x, x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \dots, N$. G is weakly connected if \tilde{G} is connected. $[x]_G$ is the equivalence class of relations \mathfrak{R} defined on $V(G)$ by the rule:

$$z \mathfrak{R} y \quad \text{if there is a path in } G \text{ from } z \text{ to } y.$$

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G_x is called the component of G which consists of all edges and vertices which are contained in some path beginning at x . If $f : X \rightarrow X$ is an operator, then

$$X^f := \{x \in X : (x, fx) \in E(G)\},$$

and the set of all fixed points of f is denoted by

$$F_f := \{x \in X : f(x) = x\}.$$

Definition 1.1. [4] The operator $f : X \rightarrow X$ is called a G -Ciric-Reich-Rus operator if:

1. for all $x, y \in X$, if $(x, y) \in E(G)$ then $(Tx, Ty) \in E(G)$;
2. there exist $\alpha, \beta, \gamma \in \mathbf{R}^+$ with $\alpha + \beta + \gamma \in (0, 1)$, such that for each $x, y \in X$, we have,

$$d(fx, fy) \leq \alpha d(x, y) + \beta d(x, fx) + \gamma d(y, fy).$$

Definition 1.2. [20] The operator $f : X \rightarrow X$ is called a Picard operator (PO) if:

- (i) f has a unique fixed point x^* ;
- (ii) for all $x \in X$, we have $\lim_{n \rightarrow \infty} T^n x = x^*$.

Definition 1.3. [20] The operator $f : X \rightarrow X$ is called a weakly Picard operator (WPO) if:

- (i) $F_f \neq \emptyset$;
- (ii) for all $x \in X$, we have $\lim_{n \rightarrow \infty} T^n x = x^*(x)$, where $x^*(x)$ is the fixed point of f which depended on x .

Definition 1.4. [9] A mapping $f : X \rightarrow X$ is called orbitally continuous if for all $x, y \in X$ and any sequence $(K_n)_{n \in \mathbf{N}}$ of positive integers, $f^{K_n} x \rightarrow y$, implies $f(f^{K_n} x) \rightarrow fy$ as $n \rightarrow \infty$.

Definition 1.5. [9] A mapping $f : X \rightarrow X$ is called orbitally G -continuous if for all $x, y \in X$ and any sequence $(K_n)_{n \in \mathbf{N}}$ of positive integers, $f^{K_n} x \rightarrow y$, $(f^{K_n} x, f^{K_{n+1}} x) \in E(G)$ imply $f(f^{K_n} x) \rightarrow fy$ as $n \rightarrow \infty$.

Definition 1.6. [18] Let us define the class $\Psi = \{\psi : \mathbf{R}^+ \rightarrow \mathbf{R}^+ \mid \psi \text{ is nondecreasing}\}$ which satisfies the following conditions:

- (i) $\psi(w) = 0$ if and only if $w = 0$;
- (ii) for every $(w_n) \in \mathbf{R}^+$, $\psi(w_n) \rightarrow 0$ if and only if $w_n \rightarrow 0$;
- (iii) for every $w_1, w_2 \in \mathbf{R}^+$, $\psi(w_1 + w_2) \leq \psi(w_1) + \psi(w_2)$.

In the next section, we state two fixed point theorems for (G, ψ) -Ciric-Reich-Rus type contraction.

2 Main results

In this section, we assume that (X, d) is a metric space, and G is a directed graph such that $V(G) = X$, $\Delta \subseteq E(G)$ and G has no parallel edges.

Definition 2.1. A mapping $f : X \rightarrow X$ is called (G, ψ) -Ciric-Reich-Rus contraction if:

- (i) for all $x, y \in X$ if $(x, y) \in E(G)$ then $(Tx, Ty) \in E(G)$;
- (ii) there exist $\alpha, \beta, \gamma \in \mathbf{R}^+$, with $\alpha + \beta + \gamma \in (0, 1)$, such that for each $(x, y) \in E(G)$ implies $\psi(d(fx, fy)) \leq \alpha \psi(d(x, y)) + \beta \psi(d(x, fx)) + \gamma \psi(d(y, fy))$.

The following Lemma is immediately.

Lemma 2.2. If $f : X \rightarrow X$ is a (G, ψ) -Ciric-Reich-Rus contraction then f is both a (G^{-1}, ψ) - Ciric - Reich - Rus contraction and a (\tilde{G}, ψ) - Ciric - Reich - Rus contraction.

Lemma 2.3. Let $f : X \rightarrow X$ be a (G, ψ) -Ciric-Reich-Rus with the constants α, β, γ . Then, for given $x \in X^f$, there exists $r(x) \geq 0$ such that

$$\psi(d(f^n x, f^{n+1} x)) \leq a^n r(x),$$

for all $n \in N$, where $a := \frac{\alpha + \beta}{1 - \gamma}$.

Proof . Assume that $x \in X^f$, then by induction, we have $(f^n x, f^{n+1} x) \in E(G)$, for each $n \in N$. So

$$\psi(d(f^n x, f^{n+1} x)) \leq \alpha\psi(d(f^{n-1} x, f^n x)) + \beta\psi(d(f^{n-1} x, f^n x)) + \gamma\psi(d(f^n x, f^{n+1} x)).$$

Hence

$$\psi(d(f^n x, f^{n+1} x)) \leq \frac{\alpha + \beta}{1 - \gamma} \psi(d(f^{n-1} x, f^n x)) \leq \dots \leq a^n \psi(d(x, f x)).$$

Set $r(x) := \psi(d(x, f x))$. \square

Lemma 2.4. Assume that (X, d) is a complete metric space and $f : X \rightarrow X$ is a (G, ψ) -Ciric-Reich-Rus contraction with the constants α, β, γ . Then, for each $x \in X^f$, there exists $x^*(x) \in X$ such that the sequence $(f^n x)_{n \in N}$ converges to $x^*(x)$ as $n \rightarrow \infty$.

Proof . Let $x \in X^f$. By Lemma 2.3, $\psi(d(f^n x, f^{n+1} x)) \leq a^n r(x)$. Hence

$$\sum_{n=0}^{\infty} \psi(d(f^n x, f^{n+1} x)) < \infty.$$

Thus,

$$\psi(d(f^n x, f^{n+1} x)) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Then, we have $d(f^n x, f^{n+1} x) \rightarrow 0$. So the sequence $(f^n x)_{n \in N}$ is a Cauchy sequence. Since the space X is complete, there exists $x^*(x) \in X$ such that the sequence $(f^n x)_{n \in N}$ converges to $x^*(x)$ as $n \rightarrow \infty$. \square

Theorem 2.5. Let (X, d) be a complete metric space endowed with a graph G , and let the triple (X, d, G) has the following condition: for any $(x_n)_{n \in N}$ in X , if $x_n \rightarrow x$ and $(x_n, x_{n+1}) \in E(G)$, for all $n \in N$, then there is a subsequence $(x_{k_n})_{n \in N}$ with $(x_{k_n}, x) \in E(G)$, for all $n \in N$. Let $f : X \rightarrow X$ be a (G, ψ) - Ciric - Reich - Rus contraction and f be orbitally G -continuous. Then the following statements hold.

- (i) $F_f \neq \emptyset$ if and only if $X^f \neq \emptyset$.
- (ii) If $X^f \neq \emptyset$ and G is weakly connected, then f is a weakly Picard operator.
- (iii) For any $X^f \neq \emptyset$, $f|_{[x]_{\tilde{G}}}$ is a weakly Picard operator.

Proof . First, we prove (iii). Let $x \in X^f$, by Lemma 2.4, there exists $x^* \in X$ such that $\lim_{n \rightarrow \infty} f^n x = x^*$. Since $x \in X^f$, $f^n x \in X^f$, for every $n \in N$. Now, assume that $(x, f x) \in E(G)$. By condition (P), there is a subsequence $(f^{k_n} x)_{n \in N}$ of $(f^n x)_{n \in N}$ such that $(f^{k_n} x, x^*) \in E(G)$, for each $n \in N$. Now we have a path in G by using the points $x, f x, \dots, f^{k_l} x, x^*$ and hence $x^* \in [x]_{\tilde{G}}$. On the other hand, since f is orbitally G -continuous, we have x^* is a fixed point for $f|_{[x]_{\tilde{G}}}$.

(i) is obtained using (iii), because $F_f \neq \emptyset$ if $X^f \neq \emptyset$. Now suppose that $F_f \neq \emptyset$. By using the assumption that $\Delta \subseteq E(G)$, we obtain $X^f \neq \emptyset$.

For proving (ii), let $x \in X^f$. Because G is weakly connected, we have $X = [x]_{\tilde{G}}$ and applying (iii) completes the proof. \square

Remark 2.6. Set $\psi(w) = w$ in Theorem 2.5, then Theorem 2.2 in [19] will be obtained immediately.

In the next we study the case that $f : X \rightarrow X$ as a (G, ψ) -Ciric-Reich-Rus contraction can be a Picard operator. So we need the following definition.

Definition 2.7. Let (X, d) be a metric space endowed with a graph G and $f : X \rightarrow X$ be a mapping. We say that the graph G has a f -path property, if for any path in G , $(x_i)_{i=0}^N$ from x to y such that $x_0 = x, x_N = y$ we have $f x_{i-1} = x_i$ for all $i = 1, \dots, N$.

Lemma 2.8. Let (X, d) be a metric space endowed with a graph G and $f : X \rightarrow X$ be a (G, ψ) -Ciric-Reich-Rus contraction such that the graph G has the f -path property. Then for any $x \in X$ and $y \in [x]_{\tilde{G}}$ two sequences $(f^n x)_{n \in \mathbb{N}}$ and $(f^n y)_{n \in \mathbb{N}}$ are equivalent.

Proof . Let $x \in X$ and $y \in [x]_{\tilde{G}}$. Then there exists a path $(x_i)_{i=0}^l$ in \tilde{G} from x to y such that $x_0 = x, x_l = y$ with $(x_{i-1}, x_i) \in E(G)$ and $f x_{i-1} = x_i$, for all $i = 1, \dots, l$. From Lemma 2.2, f is a (\tilde{G}, ψ) -Ciric-Reich-Rus. Then, for all $n \in \mathbb{N}$, $(f^n x_{i-1}, f^n x_i) \in E(\tilde{G})$, so

$$\begin{aligned} \psi(d(f^n x_{i-1}, f^n x_i)) &\leq \alpha\psi(d(f^{n-1} x_{i-1}, f^{n-1} x_i)) + \beta\psi(d(f^{n-1} x_{i-1}, f^n x_{i-1})) + \gamma\psi(d(f^{n-1} x_i, f^n x_i)) \\ &= \alpha\psi(d(f^{n-1} x_{i-1}, f^{n-1} x_i)) + \beta\psi(d(f^{n-1} x_{i-1}, f^{n-1} x_i)) + \gamma\psi(d(f^n x_{i-1}, f^n x_i)) \end{aligned}$$

then,

$$\psi(d(f^n x_{i-1}, f^n x_i)) \leq \frac{\alpha + \beta}{1 - \gamma} \psi(d(f^{n-1} x_{i-1}, f^{n-1} x_i)).$$

Hence, for all $n \in \mathbb{N}$,

$$\psi(d(f^n x_{i-1}, f^n x_i)) \leq a^n \psi(d(x_{i-1}, x_i)), \tag{2.1}$$

where $a = \frac{\alpha + \beta}{1 - \gamma}$. We know that $(f^n x_i)_{i=0}^l$ is a path in \tilde{G} from $f^n x$ to $f^n y$. Using the triangle inequality and (2.1),

$$\psi(d(f^n x, f^n y)) \leq \sum_{i=1}^l \psi(d(f^n x_{i-1}, f^n x_i)) \leq a^n \sum_{i=1}^l \psi(d(x_{i-1}, x_i)).$$

Letting $n \rightarrow \infty$, we get $d(f^n x, f^n y) \rightarrow 0$. \square

Theorem 2.9. Let (X, d) be a complete metric space endowed with a graph G , and $f : X \rightarrow X$ be a (G, ψ) -Ciric-Reich-Rus contraction such that the graph G has the f -path property and f be orbitally G -continuous. Let the triple (X, d, G) has the following condition: for any $(x_n)_{n \in \mathbb{N}}$ in X , if $x_n \rightarrow x$ and $(x_n, x_{n+1}) \in E(G)$, for all $n \in \mathbb{N}$, then there is a subsequence $(x_{k_n})_{n \in \mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$, for all $n \in \mathbb{N}$. Let there exists $z \in X$ such that $z \in X^f$, then the following statements hold:

- (1) $f|_{[z]_{\tilde{G}}}$ is a Picard operator;
- (2) if G is weakly connected, then f is a Picard operator.

Proof . (1) Using (iii) Theorem 2.5, there exists $x^*(z) \in [z]_{\tilde{G}}$ such that $\lim_{n \rightarrow \infty} f^n(z) = x^*(z)$, and $x^*(z)$ is a fixed point of f . Now if $y \in [z]_{\tilde{G}}$ and $\lim_{n \rightarrow \infty} f^n(y) = x^*(y)$. Then by Lemma 2.8 two sequences $(f^n z)_{n \in \mathbb{N}}$ and $(f^n y)_{n \in \mathbb{N}}$ are equivalent. Since both are convergent sequence, they are Cauchy sequences. Hence, they are Cauchy equivalent. This means $x^*(y) = x^*(z)$.

(2) Since $z \in X^f$ and G is weakly connected, we have $X = [z]_{\tilde{G}}$. Then we only need to apply (1). \square

Definition 2.10. [18] We say that mapping $f : X \rightarrow X$ is a (G, ψ) -contraction if the following hold:

- (i) f preserves edges of G , i.e, for all $x, y \in X$ if $(x, y) \in E(G)$ then $(fx, fy) \in E(G)$;
- (ii) f decreases the weight of G , that is, there exists $c \in (0, 1)$ such that for all $x, y \in X$ if

$$(x, y) \in E(G) \quad \text{then} \quad \psi(d(fx, fy)) \leq c\psi(d(x, y)).$$

In the following example we show that (G, ψ) -Ciric-Reich-Rus contraction is a generalization of (G, ψ) -contraction.

Example 2.11. Let $X = [0, 1]$ and $d(x, y) = |x - y|$. Define the graph G by

$$E(G) = \{(0, 0), (0, 1)\} \cup \{(x, y) \in (0, 1] \times [0, 1] \mid x \geq y\}.$$

$$f : X \rightarrow X \quad \text{by} \quad fx = \begin{cases} \frac{x}{2}, & x \in (0, 1]; \\ \frac{3}{4}, & x = 0. \end{cases}$$

Then G is weakly connected, and f is a (G, ψ) -Ciric-Reich-Rus contraction with constants, $\alpha = \frac{1}{8}, \beta = \frac{3}{4}, \gamma = \frac{1}{16}, \psi(w) = \frac{w}{2}$. But f is not (G, ψ) -contraction, because if we consider

$$\psi(d(f(0), f(\frac{1}{2}))) \leq c\psi(d(0, \frac{1}{2})).$$

Then, we have $\frac{1}{4} \leq c\frac{1}{4}$, which is a contradiction, since $c \in [0, 1)$.

Definition 2.12. The mapping $f : X \rightarrow X$ is called a (G, ψ) -Kannan mapping if:

- (i) for all $x, y \in X$, if $(x, y) \in E(G)$ then $(fx, fy) \in E(G)$;
- (ii) there exists a constant $a \in (0, 1)$ such that for all $x, y \in X, (x, y) \in E(G)$ then,

$$\psi(d(fx, fy)) \leq a[\psi(d(x, fx)) + \psi(d(y, fy))].$$

Corollary 2.13. Let (X, d) be a complete metric space endowed with a graph G , and $f : X \rightarrow X$ be a (G, ψ) -contraction such that the graph G has the f -path property and f be orbitally G -continuous. Let the triple (X, d, G) has the following condition: for any $(x_n)_{n \in \mathbb{N}}$ in X , if $x_n \rightarrow x$ and $(x_n, x_{n+1}) \in E(G)$, for all $n \in \mathbb{N}$, then there is a subsequence $(x_{k_n})_{n \in \mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$, for all $n \in \mathbb{N}$. Let there exists $z \in X$ such that $z \in X^f$, then the following statements hold:

- (1) $f|_{[z]_{\tilde{G}}}$ is a Picard operator;
- (2) if G is weakly connected, then f is a Picard operator.

Proof . If f is a (G, ψ) -contraction with constant $c \in [0, 1)$, then f is a (\tilde{G}, ψ) -Ciric-Reich-Rus contraction with constants $\alpha = c, \beta = \gamma = 0$. Hence according to Theorem 2.9, f is a Picard operator. \square

Corollary 2.14. Let (X, d) be a complete metric space endowed with a graph G , and $f : X \rightarrow X$ be a (G, ψ) -Kannan mapping such that the graph G has the f -path property and f be orbitally G -continuous. Let the triple (X, d, G) has the following condition: for any $(x_n)_{n \in \mathbb{N}}$ in X , if $x_n \rightarrow x$ and $(x_n, x_{n+1}) \in E(G)$, for all $n \in \mathbb{N}$, then there is a subsequence $(x_{k_n})_{n \in \mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$, for all $n \in \mathbb{N}$. Let there exists $z \in X$ such that $z \in X^f$, then the following statements hold:

- (1) $f|_{[z]_{\tilde{G}}}$ is a Picard operator;
- (2) if G is weakly connected, then f is a Picard operator.

Proof . If f is a (G, ψ) -Kannan with constant $a \in [0, 1)$, then f is a (\tilde{G}, ψ) -Ciric-Reich-Rus contraction with constants $\alpha = 0, \beta = \gamma = a$. Hence according to Theorem 2.9, f is a Picard operator. \square

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