Jensen’s inequality for GG-convex functions

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Abstract

In this paper, we obtain Jensen’s inequality for GG-convex functions. Also, we get inequalities alike to Hermite-Hadamard inequality for GG-convex functions. Some examples are given.

Keywords: Jensen’s inequality, GG-convex, Integral inequality.


1. Introduction

Let $\mu$ be a positive measure on $X$ such that $\mu(X) = 1$. If $f$ is a real-valued function in $L^1(\mu)$, $a < f(x) < b$ for all $x \in X$ and $\varphi$ is convex on $(a, b)$, then

$$\varphi \left( \int_X f \, d\mu \right) \leq \int_X (\varphi . f) \, d\mu$$

(1.1)

The inequality (1.1) is known as Jensen’s inequality [4], [7].

Definition 1.1. A function $\varphi : (a, b) \rightarrow (0, \infty)$, where $0 < a < b \leq \infty$, is called GG-convex or multiplicatively-convex (according to the geometric mean) if the inequality

$$\varphi(x^\lambda y^{1-\lambda}) \leq \varphi(x)^\lambda \varphi(y)^{1-\lambda}$$

(1.2)

holds, where $a < x < b$, $a < y < b$, and $0 \leq \lambda \leq 1$.

In this paper, first we prove Jensen’s inequality for GG-convex functions. Then as a result of Jensen’s inequality, we prove the geometric mean of positive numbers is not greater than the mean power of the same numbers of order $\alpha > 0$, that is

$$\sqrt[n]{a_1a_2\cdots a_n} \leq \left( \frac{a_1^\alpha a_2^\alpha \cdots a_n^\alpha}{n} \right)^{\frac{1}{\alpha}}$$

($\alpha > 0$, $a_1, a_2, \cdots a_n > 0$).

By GG-convexity of Gamma function on $[1, \infty]$, we obtain several interesting inequalities. Finally, we prove alike to Hermit-Hadamard inequality for GG-convex functions.

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2. Main results

First we need the following theorem.

**Theorem 2.1.** A function \( \varphi \) is GG-convex on \((a, b)\) if for \(0 < a < s < t < u < b\) the following inequality holds

\[
\frac{\ln \varphi(t) - \ln \varphi(s)}{\ln t - \ln s} \leq \frac{\ln \varphi(u) - \ln \varphi(t)}{\ln u - \ln t} \tag{2.1}
\]

**Proof.** Let \( \varphi \) be GG-convex and \( \lambda = \frac{\ln u - \ln t}{\ln u - \ln s} \), then \( t = s^{\lambda} u^{1-\lambda} \). Hence

\[
\varphi(t) \leq [\varphi(s)]^\lambda \ln u - \ln s [\varphi(u)]^{1-\lambda} \ln u - \ln s
\]

It follows that

\[
\frac{\ln \varphi(t)}{\ln u - \ln s} \leq \frac{\ln \varphi(s)}{\ln u - \ln s} + \frac{\ln t - \ln s}{\ln u - \ln s} \ln \varphi(t) \leq \frac{\ln u - \ln t}{\ln u - \ln s} \ln \varphi(s) + \frac{\ln t - \ln s}{\ln u - \ln s} \ln \varphi(u)
\]

\[
\frac{\ln u - \ln t}{\ln u - \ln s} (\ln \varphi(t) - \ln \varphi(s)) \leq \frac{\ln t - \ln s}{\ln u - \ln s} (\ln \varphi(u) - \ln \varphi(t))
\]

since \( s < t < u \), we obtain

\[
\frac{\ln \varphi(t) - \ln \varphi(s)}{\ln t - \ln s} \leq \frac{\ln \varphi(u) - \ln \varphi(t)}{\ln u - \ln t}
\]

Conversely let the inequality (2.1) holds, and \( \lambda \in [0, 1] \), \( a < x < y < b \), then \( x^{\lambda} y^{1-\lambda} \leq y \). By inequality (2.1) we have

\[
\frac{\ln \varphi(x^{\lambda} y^{1-\lambda}) - \ln \varphi(x)}{\ln x^{\lambda} y^{1-\lambda} - \ln x} \leq \frac{\ln \varphi(y) - \ln \varphi(x^{\lambda} y^{1-\lambda})}{\ln y - \ln x^{\lambda} y^{1-\lambda}}
\]

\[
\frac{\ln \varphi(x^{\lambda} y^{1-\lambda}) - \ln \varphi(x)}{(1-\lambda)(\ln y - \ln x)} \leq \frac{\ln \varphi(y) - \ln \varphi(x^{\lambda} y^{1-\lambda})}{\lambda(\ln y - \ln x)}
\]

\[
\ln \varphi(x^{\lambda} y^{1-\lambda}) \leq (1-\lambda) \ln \varphi(y) + \lambda \ln \varphi(x)
\]

\[
\varphi(x^{\lambda} y^{1-\lambda}) \leq \varphi(x)^{\lambda} \varphi(1-\lambda)(y)
\]

Thus \( \varphi \) is GG-convex. \( \square \)

By similar way to the convex functions we can prove that if \( \varphi \) is GG-convex on \((a, b)\), then \( \varphi \) is continuous on \((a, b)\).

**Theorem 2.2.** Let \( \mu \) be a positive measure on a \( \sigma \)-algebra \( \mathfrak{m} \) in a set \( X \), so that \( \mu(X) = 1 \). If \( f \) is a real function in \( L^1(\mu) \), \( 0 < a < f(x) < b \) for all \( x \in X \), and if \( \varphi \) is GG-convex on \((a, b)\), then

\[
\varphi \left( e \int_X \ln f d\mu \right) \leq e \int_X \ln(\varphi \circ f) d\mu \tag{2.2}
\]
Proof. Put $t = e^\int_X \ln f \, d\mu$. Then $a < t < b$. If $M$ is the supremum of quotients on the left side of (2.1), where $a < s < t$, then for any $u \in (t, b)$ we have

$$M \leq \frac{\ln \varphi(u) - \ln \varphi(t)}{\ln u - \ln t}.$$ 

It follows that

$$\frac{\ln \varphi(t) - \ln \varphi(s)}{\ln t - \ln s} \leq M \quad (a < s < b)$$

so

$$\ln \varphi(s) \geq \ln \varphi(t) + M (\ln s - \ln t).$$

Hence, for any $x \in X$, we have

$$\ln \varphi(f(x)) \geq \ln \varphi(t) + M (\ln f(x) - \ln t)$$

since $\varphi$ is continuous, $\varphi \circ f$ is measureable, and since $f \in L^1(\mu)$, by cancavity of $\psi(x) = \ln x$ and Jensen inequality (1.1) $\ln f \in L^1(\mu)$. By integrating both sides with respect to measure $\mu$ we obtain

$$\int_X \ln(\varphi \circ f) \, d\mu \geq \ln \varphi(t) + M \left( \int_X \ln f \, d\mu - \ln t \right) \quad (\mu(X) = 1)$$

Now set $t = e^{\int_X \ln f \, d\mu}$, it follows that

$$\int_X \ln(\varphi \circ f) \, d\mu \geq \ln \varphi \left( e^{\int_X \ln f \, d\mu} \right) + M \left( \int_X \ln f \, d\mu - \ln e^{\int_X \ln f \, d\mu} \right)$$

so

$$\ln \varphi \left( e^{\int_X \ln f \, d\mu} \right) \leq \int_X \ln(\varphi \circ f) \, d\mu$$

or

$$\varphi \left( e^{\int_X \ln f \, d\mu} \right) \leq e^{\int_X \ln(\varphi \circ f) \, d\mu}.$$ 

□

In [6], the author proved the following assertion.
Here we prove it in another way and a result of theorem 2.2.

Corollary 2.3. Let $f : [a, b] \rightarrow (0, \infty) \ (b > a > 0)$ be a continuous function and $\varphi : J \rightarrow (0, \infty)$ be a GG-convex function defined on an interval $J$ which includes the image of $f$. Then

$$\varphi \left( \frac{1}{e \ln b - \ln a} \left( \int_a^b \frac{\ln f(x)}{x} \, dx \right) \right) \leq \frac{1}{e \ln b - \ln a} \int_a^b \frac{\ln \varphi(f(x))}{x} \, dx \quad (2.3)$$

Proof. In theorem 2.2, put $X = [a, b]$ and $d\mu = \frac{dx}{x}$. □

In the following theorem we prove a version for the inverse of corollary 2.3.
Theorem 2.4. Let \( \varphi : (0, \infty) \to (0, \infty) \) be a function such that the inequality (2.3) holds, for every positive real bounded measurable function \( f \). Then \( \varphi \) is GG-convex.

Proof. Let \( \lambda \in [0,1] \) and \( c,d \in (0,\infty) \). Define

\[
 f(x) = \begin{cases} 
 c & a \leq x < b^{\lambda}a^{1-\lambda} \\
 d & b^{\lambda}a^{1-\lambda} \leq x \leq b 
\end{cases}
\]

we have

\[
 \frac{1}{\ln b - \ln a} \int_a^b \ln f(x) \frac{dx}{x} = \frac{1}{\ln b - \ln a} \left[ \int_a^{b^{\lambda}a^{1-\lambda}} (\ln c) \frac{dx}{x} + \int_{b^{\lambda}a^{1-\lambda}}^b (\ln d) \frac{dx}{x} \right] 
\]

\[
 = \lambda \ln c + (1-\lambda) \ln d 
\]

so

\[
 \varphi \left( \frac{1}{e \ln b - \ln a} \int_a^b \ln f(x) \frac{dx}{x} \right) = \varphi(e^{\lambda \ln c + (1-\lambda) \ln d}) = \varphi(c^{\lambda}d^{1-\lambda}) \quad (*)
\]

on the other hand we have

\[
 \frac{1}{\ln b - \ln a} \int_a^b \ln \varphi(f(x)) \frac{dx}{x} = \frac{1}{\ln b - \ln a} \left[ \int_a^{b^{\lambda}a^{1-\lambda}} \ln \varphi(c) \frac{dx}{x} + \int_{b^{\lambda}a^{1-\lambda}}^b \ln \varphi(d) \frac{dx}{x} \right] 
\]

\[
 = \lambda \ln \varphi(c) + (1-\lambda) \ln \varphi(d) 
\]

Hence

\[
 e^{\frac{1}{e \ln b - \ln a} \int_a^b \ln \varphi(f(x)) \frac{dx}{x}} = e^{\lambda \ln \varphi(c) + (1-\lambda) \ln \varphi(d)} = \varphi(c^{\lambda}d^{1-\lambda}) \quad (**)
\]

Now the (*) and (**) and (2.3) show that \( \varphi \) is GG-convex. □

Example 2.5. (1) Let \( X = \{x_1, x_2, \ldots, x_n\} \), \( \mu(\{x_i\}) = \frac{1}{n} \) and \( f(x_i) = a_i > 0 \). Then (2.2) becomes

\[
 \varphi \left( \frac{1}{e^n} (\ln a_1 + \ln a_2 + \cdots + \ln a_n) \right) \leq \frac{1}{e^n} (\ln \varphi(a_1) + \ln \varphi(a_2) + \cdots + \ln \varphi(a_n)) 
\]

Hence

\[
 \varphi \left( \sqrt[n]{a_1a_2\cdots a_n} \right) \leq \sqrt[n]{\varphi(a_1)\varphi(a_2)\cdots \varphi(a_n)} \quad (2.4)
\]

Now we investigate this inequality for \( \varphi(x) = e^{x^\alpha} \) and \( \varphi(x) = \Gamma(x) \)

(i) \( \varphi(x) = e^{x^\alpha} \) (\( \alpha > 0 \)) is GG-convex on \((0,\infty)\) (see [1]). The inequality (2.4) implies that

\[
 e^{(\sqrt[n]{a_1a_2\cdots a_n})^\alpha} \leq \sqrt[n]{e^{a_1^\alpha}e^{a_2^\alpha}\cdots e^{a_n^\alpha}} = \left( e^{a_1^\alpha + a_2^\alpha + \cdots + a_n^\alpha} \right)^{\frac{1}{n}} 
\]

\[
 \implies \sqrt[n]{a_1a_2\cdots a_n} \leq \left( \frac{a_1^\alpha + a_2^\alpha + \cdots + a_n^\alpha}{n} \right)^{\frac{1}{\alpha}} \quad (\alpha > 0)
\]
(ii) \( \varphi(x) = \Gamma(x) \) is GG-convex on \([1, \infty)\). The inequality (2.4) follows that

\[
\Gamma \left( \sqrt[n]{\prod_{i=1}^{n} a_i} \right) \leq \sqrt[n]{\prod_{i=1}^{n} \Gamma(a_i)}
\]

put \( a_k = x + \frac{k}{m}, \ k = 0, 1, 2, \ldots, m - 1 \) \( (x \geq 1) \). Then

\[
\Gamma \left( \sqrt[m]{\prod_{k=0}^{m-1} \left( x + \frac{k}{m} \right)} \right) \leq \sqrt[m]{\prod_{k=0}^{m-1} \Gamma(x + \frac{k}{m})}
\]

By Gauss multiplication formula \( \prod_{k=0}^{m-1} \Gamma(x + \frac{k}{m}) = (2\pi)^{\frac{m-1}{2}} \frac{1}{m^{1/2-m} \Gamma(mx)} [8] \) we obtain

\[
\Gamma \left( \sqrt[m]{x(x + \frac{1}{m}) \ldots \left( x + \frac{m-1}{m} \right)} \right) \leq (2\pi)^{\frac{m-1}{2m}} \frac{1}{m^{2m-1} \sqrt{m-1}!}
\]

Especially for \( x = 1 \) we have

\[
\Gamma \left( \sqrt[m]{\frac{(2m-1)!}{(m!)^m}} \right) \leq (2\pi)^{\frac{m-1}{2m}} \frac{1}{m^{2m-1} \sqrt{m-1}!}
\]

(2) \( \Gamma(x) \) is GG-convex on \([1, \infty)\). Hence (2.3) becomes

\[
\Gamma \left( e^{\frac{1}{\ln b - \ln a} \int_{a}^{b} \ln f(t) \frac{dt}{t}} \right) \leq e^{\frac{1}{\ln b - \ln a} \int_{a}^{b} \ln \Gamma(f(t)) \frac{dt}{t}}
\]

Especially for \( f(t) = \ln t \ (e \leq a < t < b) \) we have

\[
\Gamma \left( e^{\frac{1}{\ln b - \ln a} \int_{a}^{b} \ln(\ln t) \frac{dt}{t}} \right) \leq e^{\frac{1}{\ln b - \ln a} \int_{a}^{b} \ln \Gamma(\ln t) \frac{dt}{t}}
\]

By change of variable \( \ln t = x, \frac{dt}{t} = dx \),

\[
\Gamma \left( e^{\frac{1}{\ln b - \ln a} \int_{\ln a}^{\ln b} \ln x dx} \right) \leq e^{\frac{1}{\ln b - \ln a} \int_{\ln a}^{\ln b} \ln \Gamma(x) dx}
\]

Now put \( a = e^p \) and \( b = e^{p+1} \) \( (p \geq 1) \)

\[
\Gamma \left( e^{\int_{p}^{p+1} \ln x dx} \right) \leq e^{\int_{p}^{p+1} \ln \Gamma(x) dx}
\]
By easy calculations we see that
\[
\int_p^{p+1} \ln x \, dx = \ln \left( \frac{(p+1)^{p+1}}{p^p} \right) - 1 \quad \text{and} \quad \int_p^{p+1} \ln \Gamma(x) \, dx = -p + p \ln p + \ln \sqrt{2\pi}
\]
so
\[
\Gamma \left( \ln \left( \frac{(p+1)^{p+1}}{p^p} \right) - 1 \right) \leq -p + p \ln p + \ln \sqrt{2\pi}
\]
or
\[
\Gamma \left( \frac{(p+1)^{p+1}}{ep^p} \right) \leq \sqrt{2\pi}p^p e^{-p}
\]

In the following theorem we obtain inequalities alike to Hermite-Hadamard inequality for GG-convex functions.

\textbf{Theorem 2.6.} Let \( f : [a, b] \rightarrow (0, \infty) \) be a GG-convex function \((b > a > 0)\). Then the following inequalities hold:

\[
f(\sqrt{ab}) \leq e^{\frac{\ln b - \ln a}{2}} \int_a^b f(x) \left( \frac{\ln f(x)}{x} \right) dx \leq \frac{1}{\ln b - \ln a} \int_a^b \sqrt{f(x) f(abx)} \, dx \leq \sqrt{f(a) f(b)} \tag{2.5}
\]

\textbf{Proof.} Since \( f \) is GG-convex, the corollary 2.3 implies that

\[
\frac{1}{e^{\ln b - \ln a}} \int_a^b \frac{\ln f(x)}{x} \, dx \geq f \left( \frac{1}{e^{\ln b - \ln a}} \int_a^b \frac{\ln x}{x} \, dx \right)
\]

\[
= f \left( e^{2(\ln b - \ln a)(\ln^2 b - \ln^2 a)} \right) = f(\sqrt{ab})
\]

For the proof of middle part, since \( \varphi(f) = \ln t \) is concave, by Jensen’s inequality (1.1) we get

\[
\ln \left( \frac{1}{\ln b - \ln a} \int_a^b \sqrt{f(x) f(abx)} \, dx \right) \geq \frac{1}{\ln b - \ln a} \int_a^b \left[ \frac{1}{2} \ln f(x) + \frac{1}{2} \ln f(abx) \right] \, dx
\]

\[
= \frac{1}{\ln b - \ln a} \int_a^b \ln f(x) \, dx
\]

Because by change of variable, \( \frac{ab}{x} = t, \, dx = -\frac{ab}{t^2} \, dt \) we see that

\[
\int_a^b \ln f\left( \frac{ab}{x} \right) \, dx = \int_a^b \ln f(t) \, dt
\]

so

\[
\frac{1}{e^{\ln b - \ln a}} \int_a^b \frac{\ln f(x)}{x} \, dx \leq \frac{1}{\ln b - \ln a} \int_a^b \sqrt{f(x) f(abx)} \, dx
\]
For the proof of right side of (2.5), by change of variable $x = a^{1-t}b^t = a(b/a)^t$, $dx = a \ln a b/((b/a)^t)dt$ and GG-convexity of $f$ we obtain

$$\frac{1}{\ln b - \ln a} \int_a^b \sqrt{f(x)f\left(\frac{ab}{x}\right)} \frac{dx}{x} = \frac{1}{\ln b - \ln a} \int_0^1 \sqrt{f(a^{1-t}b^t)f(a^t b^{1-t})} \frac{a \ln b/((b/a)^t)}{a((b/a)^t)} dt$$

$$= \int_0^1 \sqrt{f(a^{1-t}b^t)f(a^t b^{1-t})} dt$$

$$\leq \int_0^1 \sqrt{f^{1-t}(a)f(t)(b)f(t)(a)f^{1-t}(b)} dt = \sqrt{f(a)f(b)}.$$

□

References