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Hermitian solutions to the system of operator equations $T_iX = U_i$

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Abstract

In this article we consider the system of operator equations $T_iX = U_i$ for $i = 1, 2, 3, ..., n$, between Hilbert spaces and give necessary and sufficient conditions for the existence of common Hermitian solutions to this system of operator equations for arbitrary operators without the closedness condition. Also we study the Moore-Penrose inverse of a $n \times 1$ block operator matrix and then give the general form of common Hermitian solutions to this system of equations. Cosequently, we give the necessary and sufficient conditions for the existence of common Hermitian solutions to the system of operator equations $T_iXV_i = U_i$, for $i = 1, 2, 3, ..., n$ and also present the necessary conditions for solvability of the equation $\sum_{i=1}^{n} T_iX_i = U$.

Keywords: Operator equation; Hermitian solution; Common solution; Existence of solution; Moore Penrose inverse.
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1. Introduction

The main goal of this article is to study the system of operator equations

$$T_iX = U_i \quad for \quad i = 1, 2, ..., n \quad \forall n \in \mathbb{N}, \tag{1.1}$$

and present the necessary and sufficient conditions for the existence of common Hermitian solution to this system of equations for arbitrary operators. In fact, Hermitian and positive solutions to matrix

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equations or operator equations has long been a topic of interest because of its multiple applications in different areas as, for example in theories and applications of stability and control for discrete-time systems. Also these equations play important roles in system theory, such as, eigenstructure assignment [11], observer design [5], control of system with inpute constraint [10], and fault detection [12].
For instance, much progress has been made on the study of matrix and operator equation

$$TX = U, \tag{1.2}$$

and the system of matrix and operator equations

$$\begin{cases} \mathrm{T}_1 X = U_1, \\ \mathrm{T}_2 X = U_2, \end{cases} \tag{1.3}$$

(see, for example [13, 4, 8, 14, 7, 27]).
Also, Hermitian positive semidefinite solution to the matrix equation

$$TXV = U, \tag{1.4}$$

were studied by Khatri and Mitra in 1976 ([13]) and Zhang in 2004 ([28]), respectively. In particular, in the last few years the system of operator Eq. (1.3) has recieved considerable attention (see, for example [26, 1, 6, 2]).
Indeed, the necessary and sufficient conditions for the existence of a common solution, and the general common solution of the equation pair

$$\begin{cases} \mathrm{T}_1 X V_1 = U_1, \\ \mathrm{T}_2 X V_2 = U_2, \end{cases} \tag{1.5}$$

and the solvability of the equation

$$T_1 X V_1 + T_2 X V_2 = U, \tag{1.6}$$

were studied by many authours for matrices and for bounded linear operators between Banach or Hilbert spaces (see, [15, 16, 18, 25, 20, 21, 22, 19, 6]).
The necessary and sufficient conditions for the existence of the general common Hermitian and positive solution to some system operator equations such as

$$T_1 X = U_1,\ XT_2 = U_2,\ T_3 X T_3{}^* = U_3,\ T_4 X T_4{}^* = U_4,$$

and

$$T_1 X_1 = U_1,\ X_1 T_1^{'} = U_2,\ T_2 X_2 = U_3,\ X_2 T^{'}{}_2 = U_4,\ T_3 X_1 T_3{}^* + T_4 X_2 T_4{}^* = U_5,$$

for adjointable operators over Hilbert C^*-moduls has been studied by Wang and others in [23, 24], respectively.

In all above works, it is only considered the case in which T_i, U_i and V_i are matrices or closed range operators, but in 2010, Arias and Gonzalez ([1]) presented different results regarding the existence of solution and also the existence of positive solution to operator Eq. (1.4) for arbitrary operators.

In this article, at first we study the system of operator equations

$$T_i X = U_i \quad for \quad i = 1, 2, ..., n \quad \forall n \in \mathbb{N},$$

and present the necessary and sufficient conditions for the existence of common Hermitian solutions to this equations for arbitrary operators. In fact, we extend the Dajic and Koliha theorem ([7]) for

arbitrary operators with not necessarily closed range. Also, we present the general form of common Hermitian solution to this system of equations, by using the Moore-Penrose inverse of a $n \times 1$ block operator matrix.
Consequently, we present the necessary and sufficient conditions for the existence of common Hermitian solutions to the system of operator equations

$$T_i X V_i = U_i \quad for \quad i = 1, 2, ..., n \quad \forall n \in \mathbb{N}, \tag{1.7}$$

and also present the necessary conditions for the existence of solutions to the equation

$$\sum_{i=1}^{n} T_i X_i = U. \tag{1.8}$$

2. Preliminary

Along this work $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{G}$ denote complex Hilbert spaces with inner product $\langle ., . \rangle$, $\mathcal{L}(\mathcal{H}, \mathcal{K})$ is the set of linear operators and $\mathcal{B}(\mathcal{H}, \mathcal{K})$ is the set of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$.
By $\mathcal{B}(\mathcal{H})^+$ we denote the cone of positive operators of $\mathcal{B}(\mathcal{H})$, i.e,

$$\mathcal{B}(\mathcal{H})^+ = \{T \in \mathcal{B}(\mathcal{H}) | \langle T(\xi), \xi \rangle \geq 0, \forall \xi \in \mathcal{H}\}.$$

T^* denote the adjoint operator of T, $R(T)$ stands for the range of T and $N(T)$ for its null space. Given a closed subspace S of $\mathcal{H}$, P_S denotes the orthogonal projection onto S.
Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, the inner inverse of T is a linear operator as T^- such that $T^- : D(T^-) \subseteq \mathcal{K} \longrightarrow \mathcal{H}$ with $R(T) \subseteq D(T^-)$ and $TT^-T = T$. In [3], Ben-Israel show that for every $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, there exists at least an inner inverse T^- for T but it is not necessarily bounded. ($T^- \notin \mathcal{B}(\mathcal{K}, \mathcal{H})$, in general). We say that the operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is regular if there is an inner inverse $T^- \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. He proved that for given $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, there exists an inner inverse of T, T^-, such that $T^- \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ if and only if T has closed range.
If ,in addition, T^- satisfies $T^-TT^- = T^-$, then T^- is called a **generalized inverse** of T. Note that T^- is not unique, however there exists a unique generalized inverse of T which also satisfies

$$(TT^-)^* = TT^- \text{ and } (T^-T)^* = T^-T,$$

which is called the **Moore-Penrose generalized inverse** of T and it will be denoted by $T^\dagger$. Therefore, $T^\dagger$ is the unique generalized inverse of T which satisfying the four following Penrose equations:

i. $TT^\dagger T = T$,
ii. $T^\dagger TT^\dagger = T^\dagger$,
iii. $(TT^\dagger)^* = TT^\dagger$,
iv. $(T^\dagger T)^* = T^\dagger T$.

An operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ has the unique Moore-Penrose inverse $T^\dagger \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ if and only if T has closed range, or equivalently if and only if it is regular, ([17]). The assumption that $R(T)$ is closed can be avoided in general, however in that case the Moore-Penrose inverse is not bounded.
Also, if $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then we have the following properties, ([27]):

1. $(T^\dagger)^* = (T^*)^\dagger$,
2. If $T \geq 0$ then $T^\dagger = T^\dagger TT^\dagger = (T^{1/2}T^\dagger)^*(T^{1/2}T^\dagger) \geq 0$,

3. $T^{\dagger}T$ and $TT^{\dagger}$ both are projection and $T^{\dagger}T = P_{\overline{R(T^*)}}$ and $TT^{\dagger} = P_{\overline{R(T)}}|_{R(T)\oplus R(T)^{\perp}}$,
4. $(TT^*)^{\dagger} = (T^*)^{\dagger}T^{\dagger}$ and $(T^*T)^{\dagger} = T^{\dagger}(T^{\dagger})^*$,
5. $R(T^{\dagger}T) = R(T^{\dagger}) = R(T^*)$ so that $T^{\dagger}TT^* = T^*$,
6. $R(TT^{\dagger}) = R(T)$ so that $T^*TT^{\dagger} = (TT^{\dagger}T)^* = T^*$.

Throughout this work the next well-known theorem due to Douglas ([9]) about range inclusions of operators will be crucial.

Theorem 2.1. *(Douglas) Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $U \in \mathcal{B}(\mathcal{G}, \mathcal{K})$. The following conditions are equivalent:*

i. There exists $V \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ such that $TV = U$. (This means that the equation $TX = U$ has a solution)
ii. $R(U) \subseteq R(T)$.
iii. There exists a positive number λ such that $UU^* \leq \lambda TT^*$.

As a consequence of Dauglas Theorem, Arias and Gonzalez proved the next lemma. This fact will be used frequently along this work.

Lemma 2.2. *([1]; lemma 2.1) If $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $U \in \mathcal{B}(\mathcal{G}, \mathcal{K})$ such that $R(U) \subseteq R(T)$, Then $T^{\dagger}U \in \mathcal{B}(\mathcal{G}, \mathcal{H})$, even though $T^{\dagger} \notin \mathcal{B}(\mathcal{K}, \mathcal{H})$.*

The following theorems proved by Dajic and Koliha ([7]) in 2007. They presented conditions for the existence of Hermitian solutions of Eq. (1.2) and common Hermitian solution of Eq. (1.3) for closed range operators. Also they obtained the formula for the general form of these equations. In the next section, we will extend these theorems to a system of operator Eqs. (1.1) without the closedness condition.

Theorem 2.3. *([7]; Theorem 3.1) Let $T, U \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and let T be a closed range operator. Then the equation $TX = U$ has a Hermitian solution $X \in \mathcal{B}(\mathcal{H})$ if and only if $TT^-U = U$ and UT^* is Hermitian. The general form of Hermitian solution to Eq.* (1.2) *is*

$$X = T^-U + (I - T^-T)(T^-U)^* + (I - T^-T)S(I - T^-T)^*, \tag{2.1}$$

where $S \in \mathcal{B}(\mathcal{H})$ is Hermitian.

Theorem 2.4. *([7]; Theorem 4.2) Let $T_1, U_1 \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $T_2, U_2 \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ and let the operators T_1 and T_2 have closed range. Let $M = {T_2}^*(I - {T_1}^-T_1)$ has closed range, and let ${T_1}^-$, ${T_2}^-$ and M^- be inner inverses of T_1, T_2 and M, respectively. Then the equations*

$$\begin{cases} T_1X = U_1, \\ XT_2 = U_2, \end{cases} \tag{2.2}$$

have a common hermitian solution $X \in \mathcal{B}(\mathcal{H})$ if and only if $T_1{T_1}^-U_1 = U_1$, $U_2{T_2}^-T_2 = U_2$, $T_1U_2 = U_1T_2$ and $T_1{U_1}^*, T_2^*U_2$ are Hermitian.

The next theorem is proved by Arias and Gonzalez in [1]. They gave the necessary and sufficient conditions about the existence of solutions of Eq. (1.4) that will be crucial to prove our main results.

Theorem 2.5. *([1]; Proposition 3.3) Let $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $V \in \mathcal{B}(\mathcal{G},\mathcal{H})$ and $U \in \mathcal{B}(\mathcal{G},\mathcal{K})$. Then the following conditions are equivalent:*

i. The equation $TXV = U$ is solvable.
ii. $R(U) \subseteq R(T)$ and $R((T^\dagger U)^*) \subseteq R(V^*)$.
iii. $R(U) \subseteq R(T)$ and there exists $\tilde{Y} \in \mathcal{B}(\mathcal{H})$ such that $\tilde{Y}V = T^\dagger U$.

Moreover, if one of the previous conditions holds then every solution of $XV = T^\dagger U$ is also a solution of $TXV = U$. Also, for $\tilde{X} \in \mathcal{B}(\mathcal{H})$ such that $T\tilde{X}V = U$, we have that $P_{\overline{R(T^*)}}\tilde{X}$ is a solution of $XV = T^\dagger U$.

3. The main results.

In this section, at first we prove the following lemmas about Hermitian solution of Eq. (1.2) and the Moore-Penrose inverse of a $n \times 1$ block operator matrix $\begin{bmatrix} T_1 & T_2 & \cdots & T_n \end{bmatrix}^t$, (where A^t denote the transpose of A). Then we extend the Dajic and Koliha theorem (theorem 2.3) to give the necessary and sufficient conditions for the existence of common Hermitian solutions to the system of operator Eqs. (1.1) for arbitrary operators which has not necessarily closed range.

Lemma 3.1. *Let $T, U \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and suppose the equation $TX = U$ has a solution $X \in \mathcal{B}(\mathcal{H})$, then the general form solution of Eq. (1.2) is*

$$X = T^\dagger U + (I - T^\dagger T)S, \quad \forall S \in \mathcal{B}(\mathcal{H}). \tag{3.1}$$

Proof . Suppose, Eq. (1.2) has a solution, so by Douglas theorem we have $R(U) \subseteq R(T)$, hence $T^\dagger U \in \mathcal{B}(\mathcal{H})$ and $TT^\dagger U = U$. So $X_0 = T^\dagger U$ is a particular solution of equation $TX = U$ and therefore the general form of solution of Eq. (1.2) is, $X = T^\dagger U + (I - T^\dagger T)S$, $\forall S \in \mathcal{B}(\mathcal{H})$. □

Lemma 3.2. *Let $T, U \in \mathcal{B}(\mathcal{H},\mathcal{K})$, then the equation $TX = U$ has a Hermitian solution $X \in \mathcal{B}(\mathcal{H})$ if and only if $R(U) \subseteq R(T)$ and UT^* is Hermitian. Then the general form of Hermitian solution to Eq. (1.2) is*

$$X = T^\dagger U + (I - T^\dagger T)(T^\dagger U)^* + (I - T^\dagger T)S(I - T^\dagger T), \tag{3.2}$$

where $S \in \mathcal{B}(\mathcal{H})$ is Hermitian.

Proof . If $R(U) \subseteq R(T)$, then by Douglas theorem $T^\dagger U \in \mathcal{B}(\mathcal{H})$ and the equation $TX = U$ has a solution. Besides, since UT^* is Hermitian, $X_0 = T^\dagger U + (I - T^\dagger T)(T^\dagger U)^*$ is a particular Hermitian solution of Eq. (1.2).
Converesly, suppose $X \in \mathcal{B}(\mathcal{H})$ be a Hermitian solution of Eq. (1.2), then $R(U) \subseteq R(T)$. Indeed, since $TU^* = T(TX)^* = TX^*T^* = TXT^*$, so TU^* and similary UT^* is Hermitian.
To find the general form of Hermitian solution of Eq. (1.2), suppose Eq. (1.2) has a Hermitian solution, then $X_0 = T^\dagger U + (I - T^\dagger T)(T^\dagger U)^*$ is a particular Hermitian solution of this equation. If $X \in \mathcal{B}(\mathcal{H})$ be an arbitrary Hermitian solution of Eq. (1.2), then $X - X_0$ is a Hermitian solution of equation $TZ = 0$. But by ([29], lemma 2.4), Z has the form $(I - T^\dagger T)S(I - T^\dagger T)$, where $S \in \mathcal{B}(\mathcal{H})$ and Hermitian, so X has the form of Eq. (3.2).
Converesly, if $X = T^\dagger U + (I - T^\dagger T)(T^\dagger U)^* + (I - T^\dagger T)S(I - T^\dagger T)$, where $S \in \mathcal{B}(\mathcal{H})$ be Hermitian, then it is obvious that X is a Hermitian solution of equation $TX = U$. □

Lemma 3.3. *Suppose $\mathcal{H}$, $\mathcal{K}_i$ be Hilbert spaces and $T_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i)$ such that $R(T_i^*) \cap R(T_j^*) = \{0\}$ for all $1 \leq i \neq j \leq n$. Then* $\begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}^\dagger = \begin{bmatrix} T_1{}^\dagger & T_2{}^\dagger & \cdots & T_n{}^\dagger \end{bmatrix}$.

Note that $\begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}^\dagger$ *exits uniquely but is not necessarily bounded.*

Proof . At first, since $R(T_i^*) \cap R(T_j^*) = \{0\}$, then

$$N(T_i) = R(T_i^*)^\perp \supseteq R(T_j^*) = R(T_j{}^\dagger).$$

So we have $T_iT_j{}^\dagger = 0 \quad \forall i \neq j \quad and \quad 1 \leq i, j \leq n$ and hence

$$TT^\dagger = \begin{bmatrix} T_1T_1{}^\dagger & 0 & \cdots & 0 \\ 0 & T_2T_2{}^\dagger & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & T_nT_n{}^\dagger \end{bmatrix}. \tag{3.3}$$

Now, we prove that $\begin{bmatrix} T_1^\dagger & T_2^\dagger & \cdots & T_n^\dagger \end{bmatrix}$ satisfies the Moore penrose conditions.

i. $\begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}^\dagger \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix} = \begin{bmatrix} T_1T_1{}^\dagger & 0 & \cdots & 0 \\ 0 & T_2T_2{}^\dagger & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & T_nT_n{}^\dagger \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}$,

ii. $\begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}^\dagger \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}^\dagger = \begin{bmatrix} T_1{}^\dagger & \cdots & T_n{}^\dagger \end{bmatrix} \begin{bmatrix} T_1T_1{}^\dagger & 0 & \cdots & 0 \\ 0 & T_2T_2{}^\dagger & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & T_nT_n{}^\dagger \end{bmatrix} = \begin{bmatrix} T_1{}^\dagger & \cdots & T_n{}^\dagger \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}^\dagger$,

iii. $\left(\begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}^\dagger\right)^* = \begin{bmatrix} (T_1T_1{}^\dagger) & 0 & \cdots & 0 \\ 0 & (T_2T_2{}^\dagger) & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & (T_nT_n{}^\dagger) \end{bmatrix}^* = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}^\dagger$,

iv. $\left(\begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}^\dagger \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}\right)^* = (T_1{}^\dagger T_1 + T_2{}^\dagger T_2 + \cdots + T_n{}^\dagger T_n)^* = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}^\dagger \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}$.

□

Example 3.4. *Suppose* $M = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$ *be a block matrix such that* $T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $T_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$. *Obviously* T_1 *and* T_2 *are not invertible, but there are uniquely Moore-Penrose inverse for them,* ${T_1}^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *and* ${T_2}^\dagger = \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}$.
Moreover, M *is not invertible but by simple calculation,* $M^\dagger = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$. *It means* $M^\dagger = \begin{bmatrix} {T_1}^\dagger & {T_2}^\dagger \end{bmatrix}$.
(It is obvious that $R({T_1}^*) \cap R({T_2}^*) = \{0\}$*).*

Example 3.5. *Suppose* $M = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$ *be a block matrix such that* $T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $T_2 = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}$ *and* $T_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. *Obviously* T_1, T_2 *and* T_3 *are not invertible, but there are uniquely Moore-Penrose inverse for them,* ${T_1}^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, ${T_2}^\dagger = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$ *and* ${T_3}^\dagger = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$. *By simple calculation we have,* $M^\dagger = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$. *It means* $M^\dagger = \begin{bmatrix} {T_1}^\dagger & {T_2}^\dagger & {T_3}^\dagger \end{bmatrix}$.
(It is obvious that $R({T_i}^*) \cap R({T_j}^*) = \{0\}$, *for* $i, j = 1, 2, 3$*).*

Example 3.6. *Suppose* $T = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$ *be a block operator matrix such that*

$$\begin{aligned} &T_1 : \mathbb{R}^4 \longrightarrow \mathbb{R}^2 \quad s.t \quad T_1(x, y, z, w) = (x, 0), \\ &T_2 : \mathbb{R}^4 \longrightarrow \mathbb{R} \quad s.t \quad T_2(x, y, z, w) = 2y, \\ &T_3 : \mathbb{R}^4 \longrightarrow \mathbb{R}^3 \quad s.t \quad T_3(x, y, z, w) = (0, 0, z + w). \end{aligned}$$

Obviously T_1, T_2 *and* T_3 *are not invertible, but there are uniquely Moore-Penrose inverse for them as follow:*

$$\begin{aligned} &{T_1}^\dagger : \mathbb{R}^2 \longrightarrow \mathbb{R}^4 \quad s.t \quad {T_1}^\dagger(x, y) = (x, 0, 0, 0), \\ &{T_2}^\dagger : \mathbb{R} \longrightarrow \mathbb{R}^4 \quad s.t \quad {T_2}^\dagger(x) = (0, 0.5x, 0, 0), \\ &{T_3}^\dagger : \mathbb{R}^3 \longrightarrow \mathbb{R}^4 \quad s.t \quad {T_3}^\dagger(x, y, z) = (0, 0, 0.5z, 0.5z). \end{aligned}$$

Moreover, we know that $T^\dagger$ *is a block operator matrix such that:*

$$T^\dagger : \mathbb{R}^2 \oplus \mathbb{R} \oplus \mathbb{R}^3 \longrightarrow \mathbb{R}^4 \quad s.t. \quad T^\dagger(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1, 0.5x_3, 0.5x_6, 0.5x_6). \tag{3.4}$$

Indeed,

$$\begin{bmatrix} T_1^\dagger & T_2^\dagger & T_3^\dagger \end{bmatrix} (\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}) = T_1^\dagger(X) + T_2^\dagger(Y) + T_3^\dagger(Z), \tag{3.5}$$

where $X \in \mathbb{R}^2$*,* $Y \in \mathbb{R}$ *and* $Z \in \mathbb{R}^3$*.*
By comparing Eqs. (3.4) *and* (3.5)*, we have* $T^\dagger = \begin{bmatrix} T_1^\dagger & T_2^\dagger & T_3^\dagger \end{bmatrix}$*.*
(Easily we can see that $R(T_i{}^*) \cap R(T_j{}^*) = \{0\}$*, for* $i, j = 1, 2, 3$*).*

Theorem 3.7. *Suppose* $\mathcal{H}$ *and* $\mathcal{K}_i$ *be Hilbert spaces and* $T_i, U_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i)$*, for* $i = 1, 2, ..., n \quad \forall n \in \mathbb{N}$*, such that* $R(T_i{}^*) \cap R(T_j{}^*) = \{0\}, \quad \forall 1 \le i \ne j \le n$*. The system operator equations* $T_i X = U_i$ *have a common Hermitian solution if and only if*

i. $R(U_i) \subseteq R(T_i)$, $\forall 1 \le i \le n$.
ii. $U_i T_i{}^*$ *is Hermitian,* $\forall 1 \le i \le n$.
iii. $T_i U_j{}^* = U_i T_j{}^*$, $\forall 1 \le i, j \le n$.

Then the general form of Hermitian solution to Eq. (1.1) *is*

$$X = \sum_{i=1}^{n} T_i{}^\dagger U_i + (I - \sum_{i=1}^{n} T_i{}^\dagger T_i)(\sum_{i=1}^{n} (T_i{}^\dagger U_i)^*) + (I - \sum_{i=1}^{n} T_i{}^\dagger T_i) S (I - \sum_{i=1}^{n} T_i{}^\dagger T_i), \tag{3.6}$$

where $S \in \mathcal{B}(\mathcal{H})$ *is Hermitian.*

Proof . Let

$$\mathbf{T} = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix} : \mathcal{H} \to \oplus_{i=1}^{n} \mathcal{K}_i \qquad s.t \qquad \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix} (h) = \begin{bmatrix} T_1(h) \\ T_2(h) \\ \vdots \\ T_n(h) \end{bmatrix},$$

$$\mathbf{U} = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix} : \mathcal{H} \to \oplus_{i=1}^{n} \mathcal{K}_i \qquad s.t \qquad \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix} (h) = \begin{bmatrix} U_1(h) \\ U_2(h) \\ \vdots \\ U_n(h) \end{bmatrix},$$

It is obvious that the system Eqs. (1.1) have a common Hermitian solution if and only if the equation $\mathbf{T}X = \mathbf{U}$ has a Hermitian solution. Clearly by lemma 3.2, the equation $\mathbf{T}X = \mathbf{U}$ has a Hermitian solution if and only if $R(\mathbf{U}) \subset R(\mathbf{T})$ and $\mathbf{T}\mathbf{U}^*$ is Hermitian.
Also, $R(\mathbf{U}) \subset R(\mathbf{T})$ if and only if $\mathbf{T}\mathbf{T}^\dagger\mathbf{U} = \mathbf{U}$. Now by assumption $R(T_i{}^*) \cap R(T_j{}^*) = \{0\}$, lemma 3.3 and condition (i), we have:

$$\begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix} \begin{bmatrix} T_1{}^\dagger & T_2{}^\dagger & \cdots & T_n{}^\dagger \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix} = \begin{bmatrix} T_1 T_1{}^\dagger & 0 & \cdots & 0 \\ 0 & T_2 T_2{}^\dagger & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & T_n T_n{}^\dagger \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix}$$

$$= \begin{bmatrix} T_1 T_1{}^\dagger U_1 \\ T_2 T_2{}^\dagger U_2 \\ \vdots \\ T_n T_n{}^\dagger U_n \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix}$$

hence $\mathbf{T}\mathbf{T}^{\dagger}\mathbf{U} = \mathbf{U}$.
Besides, from

$$\mathbf{T}\mathbf{U}^* = \begin{bmatrix} T_1{U_1}^* & T_1{U_2}^* & \cdots & T_1{U_n}^* \\ T_2{U_1}^* & T_2{U_2}^* & \cdots & T_2{U_n}^* \\ & & & \vdots \\ T_n{U_1}^* & T_n{U_2}^* & \cdots & T_n{U_n}^* \end{bmatrix}$$

and conditions (ii) and (iii), we have

$$(\mathbf{T}\mathbf{U}^*)^* = \begin{bmatrix} T_1{U_1}^* & T_1{U_2}^* & \cdots & T_1{U_n}^* \\ T_2{U_1}^* & T_2{U_2}^* & \cdots & T_2{U_n}^* \\ & & & \vdots \\ T_n{U_1}^* & T_n{U_2}^* & \cdots & T_n{U_n}^* \end{bmatrix}^*$$

$$= \begin{bmatrix} (T_1{U_1}^*)^* & (T_2{U_1}^*)^* & \cdots & (T_n{U_1}^*)^* \\ (T_1{U_2}^*)^* & (T_2{U_2}^*)^* & \cdots & (T_n{U_2}^*)^* \\ & & & \vdots \\ (T_1{U_n}^*)^* & (T_2{U_n}^*)^* & \cdots & (T_n{U_n}^*)^* \end{bmatrix} = \mathbf{T}\mathbf{U}^*.$$

So, $\mathbf{T}\mathbf{U}^*$ is Hermitian and then by Lemma (3.2), the equation $\mathbf{T}X = \mathbf{U}$ has a Hermitian solution. To prove the converse, suppose the system Eqs. (1.1) have a common Hermitian solution, then the equation $T_iX = U_i$ for every $i = 1, 2, \cdots, n$ have a Hermitian solution, so by lemma 3.2, the condition (i) and (ii) is verified. Indeed, for proving the condition (iii), if $X_0 \in \mathcal{B}(\mathcal{H})$ be a common Hermitian solution of the system Eqs. (1.1), then we have for every $i = 1, 2, \cdots, n$, $T_iX_0 = U_i$ and $X_0{T_i}^* = {U_i}^*$. So,

$$T_i{U_j}^* = T_iX_0{T_j}^* = U_i{T_j}^*, \quad \forall i, j = 1, 2, \cdots, n.$$

Now, we present the general form of common Hermitiam solution of Eq. (1.1). By the result of Douglas, since $R(U_i) \subset R(T_i)$, so ${T_i}^{\dagger}U_i \in \mathcal{B}(\mathcal{H})$ and consequently $\mathbf{T}^{\dagger}\mathbf{U} \in \mathcal{B}(\mathcal{H})$. Then by lemma 3.2, the general form of Hermitian solution to equation $\mathbf{T}\mathbf{X} = \mathbf{U}$ is:

$$X = \mathbf{T}^{\dagger}\mathbf{U} + (I - \mathbf{T}^{\dagger}\mathbf{T})(\mathbf{T}^{\dagger}\mathbf{U})^* + (I - \mathbf{T}^{\dagger}\mathbf{T})S(I - \mathbf{T}^{\dagger}\mathbf{T})^*, \tag{3.7}$$

where $S \in \mathcal{B}(\mathcal{H})$ is Hermitian. Simple calculation shows that X has the form of Eq. (3.6). □

Now, we give an example about the Theorem 3.7:
suppose,
$T_1 : l^2(\mathbb{N}) \longrightarrow l^2(\mathbb{N})$ *s.t.* $T_1(x_1, x_2, \cdots) \longrightarrow (x_1, \frac{1}{2}x_4, \frac{1}{3}x_7, \cdots)$,
$U_1 : l^2(\mathbb{N}) \longrightarrow l^2(\mathbb{N})$ *s.t.* $U_1(x_1, x_2, \cdots) \longrightarrow (\frac{1}{2}x_1, 0, 0, \cdots)$,
$T_2 : l^2(\mathbb{N}) \longrightarrow l^2(\mathbb{N})$ *s.t.* $T_2(x_1, x_2, \cdots) \longrightarrow (x_2, \frac{1}{2}x_5, \frac{1}{3}x_8, \cdots)$,
$U_2 : l^2(\mathbb{N}) \longrightarrow l^2(\mathbb{N})$ *s.t.* $U_2(x_1, x_2, \cdots) \longrightarrow (\frac{1}{3}x_2, 0, 0, \cdots)$,
$T_3 : l^2(\mathbb{N}) \longrightarrow l^2(\mathbb{N})$ *s.t.* $T_3(x_1, x_2, \cdots) \longrightarrow (x_3, \frac{1}{2}x_6, \frac{1}{3}x_9, \cdots)$,
$U_3 : l^2(\mathbb{N}) \longrightarrow l^2(\mathbb{N})$ *s.t.* $U_3(x_1, x_2, \cdots) \longrightarrow (\frac{1}{5}x_3, 0, 0, \cdots)$.
By some calculaton we can check the theorem.

As a corollary of above theorem, we prove the theorem 2.4, ([7]), about the common Hermitian solution of system operator Eqs. (2.2), under some easier condition and without the closedness conditions of range of operators.

Corollary 3.8. *Let $T_1, U_1 \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $T_2, U_2 \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ and let $R(T_2) \subseteq N(T_1)$. Then the system operator equations*

$$\begin{cases} T_1 X = U_1, \\ X T_2 = U_2, \end{cases} \tag{3.8}$$

have a common Hermitian solution $X \in \mathcal{B}(\mathcal{H})$ if and only if $T_1 T_1^{-} U_1 = U_1$, $U_2 T_2^{-} T_2 = U_2$, $T_1 U_2 = U_1 T_2$ and $T_1 U_1^*, T_2^* U_2$ are Hermitian.

Proof . Since, $N(T_1) = R(T_1^*)^{\perp}$, so $R(T_2) \subseteq N(T_1)$ if and only if $R(T_2) \subseteq R(T_1^*)^{\perp}$ if and only if $R(T_2) \cap R(T_1^*) = \{0\}$.
Besides, if $T_1 T_1^{-} U_1 = U_1$ then $R(U_1) \subseteq R(T_1)$ and $T_1 T_1^{\dagger} U_1 = U_1$. Conversly, if $R(U_1) \subseteq R(T_1)$ then $T_1 T_1^{\dagger} U_1 = U_1$ and so $T_1 T_1^{-} U_1 = T_1 T_1^{-} T_1 T_1^{\dagger} U_1 = T_1 T_1^{\dagger} U_1 = U_1$. Hence $R(U_1) \subseteq R(T_1)$ is equivalent to $T_1 T_1^{-} U_1 = U_1$ and by same way, $R(U_2^*) \subseteq R(T_2^*)$ is equivalent to $U_2 T_2^{-} T_2 = U_2$.
so by theorem 3.7, the proof is obvious. □

As a corollary of theorem 2.5, lemma 3.2 and theorem 3.7, we present the necessary and sufficient conditions for the existence of common Hermitian solution to system operator Eqs.(1.7). For this purpose, at first, we present the general form of solution of Eq. (1.4) without the closedness condition. Subsequently, we characterize the Hermitian solution of Eq. (1.4) and then extend our results to the system of operator Eqs (1.7).

Lemma 3.9. *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $V \in \mathcal{B}(\mathcal{G}, \mathcal{H})$, $U \in \mathcal{B}(\mathcal{G}, \mathcal{K})$. If the operator equation $TXV = U$ is solvable then the general form solution of this equation is:*

$$X = T^{\dagger} U V^{\dagger} + S - T^{\dagger} T S V V^{\dagger}, \qquad \forall S \in \mathcal{B}(\mathcal{H}). \tag{3.9}$$

Proof . Suppose that the equation $TXV = U$ is solvable.Then by theorem 2.5, $R((T^{\dagger}U)^*) \subseteq R(V^*)$ and the equation $XV = T^{\dagger}U$ is solvable and every solution of equation $XV = T^{\dagger}U$ is a solution of equation $TXV = U$.
Now, suppose $X_0 = T^{\dagger} U V^{\dagger}$. Since $R((T^{\dagger}U)^*) \subseteq R(V^*)$ then $V^* V^{*\dagger} (T^{\dagger}U)^* = (T^{\dagger}U)^*$ and also $T^{\dagger} U V^{\dagger} V = T^{\dagger} U$. Therefore $X_0 = T^{\dagger} U V^{\dagger}$ is a special solution of $TXV = U$.
Now, let X be a solution of equation $TXV = U$, then $X - X_0$ is a solution of equation $TZV = 0$. So, $Z = S - T^{\dagger} T S V V^{\dagger}$, for $S \in \mathcal{B}(\mathcal{H})$ and so X is the form of Eq. (3.9).
Converesly if

$$X = T^{\dagger} U V^{\dagger} + S - T^{\dagger} T S V V^{\dagger}, \qquad S \in \mathcal{B}(\mathcal{H}),$$

So,

$$\begin{aligned} TXV &= TX_0 V + (TSV - TT^{\dagger} T S V V^{\dagger} V) \\ &= TX_0 V = U, \end{aligned}$$

and hence $X = T^{\dagger} U V^{\dagger} + S - T^{\dagger} T S V V^{\dagger}$, $S \in \mathcal{B}(\mathcal{H})$ is general solution of $TXV = U$. □

Theorem 3.10. *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $V \in \mathcal{B}(\mathcal{G}, \mathcal{H})$, $U \in \mathcal{B}(\mathcal{G}, \mathcal{K})$ be such that $R(V) \subseteq \overline{R(T^*)}$, then the following are equivalent:*

1. *The equation $TXV = U$ has a Hermitian solution $X \in \mathcal{B}(\mathcal{H})$.*
2. *$R(U) \subseteq R(T)$ and the equation $XV = T^{\dagger}U$ has a Hermitian solution.*
3. *$R(U) \subseteq R(T)$, $R((T^{\dagger}U)^*) \subseteq R(V^*)$ and $V^* T^{\dagger} U$ is Hermitian.*

The general form of common Hermitian solution of this equation is:

$$X = T^{\dagger}UV^{\dagger} + (T^{\dagger}UV^{\dagger})^{*}(I - VV^{\dagger}) + (I - TT^{\dagger})S(I - TT^{\dagger}) + (I - VV^{\dagger})S^{'}(I - VV^{\dagger}), \tag{3.10}$$

where $S, S^{'} \in \mathcal{B}(\mathcal{H})$ *are Hermitian.*

Proof . $(1 \longrightarrow 2)$, Suppose $X_0 \in \mathcal{B}(\mathcal{H})$ be a Hermitian solution of the equation $TXV = U$, then $TX_0V = U$ and $R(U) \subseteq R(T)$. Furthermore, as $R(V) \subseteq \overline{R(T^*)}$, $T^{\dagger}TV = V$ and $T^{\dagger}TX_0T^{\dagger}TV = T^{\dagger}TX_0V = T^{\dagger}U$. Hence $Y_0 = T^{\dagger}TX_0T^{\dagger}T$ is a Hermitian solution of the equation $XV = T^{\dagger}U$.
$(2 \longrightarrow 1)$, Suppose $Y_0 \in \mathcal{B}(\mathcal{H})$ be the Hermitian solution of the equation $XV = T^{\dagger}U$, then $Y_0V = T^{\dagger}U$ and since $R(U) \subseteq R(T)$ we have $TY_0V = TT^{\dagger}U = U$.
$(2 \longleftrightarrow 3)$, By lemma 3.2, obviously 2 and 3 are equivalent.
Now, we find the general form of Hermitian solution of this equation.
Suppose the equation $TXV = U$ has a Hermitian solution, then the equation $XV = T^{\dagger}U$ and consequently the equation $V^*X = (T^{\dagger}U)^*$ has a Hermitian solution. So every particular Hermitian solution of the equation $V^*X = (T^{\dagger}U)^*$ is a particular Hermitian solution of the equation $TXV = U$. So by lemma 3.2, $X_0 = T^{\dagger}UV^{\dagger} + (T^{\dagger}UV^{\dagger})^*(I - VV^{\dagger})$ is a particular Hermitian solution of the equation $XV = (T^{\dagger}U)$ and $TXV = U$.
Now, suppose X be a Hermitian solution of the equation $TXV = U$, then $X - X_0$ is a Hermitian solution of the equation $TZV = 0$. So

$$X - X_0 = (I - TT^{\dagger})S(I - TT^{\dagger}) + (I - VV^{\dagger})S^{'}(I - VV^{\dagger}), where$$

S ,S$^{'} \in \mathcal{B}(\mathcal{H})$ are Hermitian.

Hence X is the form of Eq. (3.10).
□

Theorem 3.11. *Let* $T_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i)$, $V_i \in \mathcal{B}(\mathcal{G}_i, \mathcal{H})$, $U_i \in \mathcal{B}(\mathcal{G}_i, \mathcal{K}_i)$, $\forall i = 1, 2, \cdots, n$ *be such that* $R(T_i{}^*) \cap R(T_j{}^*) = \{0\}$ *and* $R(V_i) \subseteq R(T_i{}^*)$, $\forall 1 \leq i \neq j \leq n$. *If* $\forall i = 1, 2, \cdots, n$, *the equation* $T_iXV_i = U_i$ *be solvable, then the system operator equations* $T_iXV_i = U_i$ *have a common Hermitian solution if and only if*

1. $V_i{}^*T_i{}^{\dagger}U_i$ *is Hermitian,* $\forall 1 \leq i \leq n$,
2. $V_i{}^*V_i{}^{*\dagger}(T_i{}^{\dagger}U_i)^* = (T_i{}^{\dagger}U_i)^*$, $\forall 1 \leq i \leq n$,
3. $V_i{}^*T_j{}^{\dagger}U_j = (V_j{}^*T_i{}^{\dagger}U_i)^*$, $\forall 1 \leq i \neq j \leq n$,

Proof . Let $\mathbf{T} = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}$, $\mathbf{V} = \begin{bmatrix} V_1 & V_2 & \cdots & V_n \end{bmatrix}$ and $\mathbf{U} = \begin{bmatrix} U_1 & W_{12} & \cdots & W_{1n} \\ W_{21} & U_2 & \cdots & W_{2n} \\ \vdots & & \ddots & \\ W_{n1} & W_{n2} & \cdots & U_n \end{bmatrix}$,
for some $W_{ij} \in \mathcal{B}(\mathcal{G}_j, \mathcal{K}_i)$, $\forall\ 1 \leq i \neq j \leq n$.
Obviously the system operator Eqs. (1.7) have a common Hermitian solution if and only if the equation $\mathbf{T}X\mathbf{V} = \mathbf{U}$ has a Hermitian solution. Moreover, suppose $h \in R(\mathbf{V})$, then

$$\exists \quad \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} \in \oplus_{i=1}^{n}\mathcal{G}_i \quad s.t \quad V(\begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}) = \oplus_{i=1}^{n}V_i(g_i) = h,$$

so,

$$h \in \oplus_{i=1}^{n} R(V_i) \quad and \quad R(\mathbf{V}) \subseteq \oplus_{i=1}^{n} R(V_i),$$

and

$$since \quad R(V_i) \subseteq R({T_i}^*), \quad then \quad R(\mathbf{V}) \subseteq \oplus_{i=1}^{n} R({T_i}^*). \tag{3.11}$$

Indeed, if $h \in \oplus_{i=1}^{n} R({T_i}^*)$, then exits $h_i \in R({T_i}^*)$ such that $h = \oplus_{i=1}^{n} h_i$. Furthermore, as $h_i \in R({T_i}^*)$, then exits $k_i \in \mathcal{K}_i$ such that ${T_i}^*(k_i) = h_i$ and $\mathbf{T}^*(\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}) = \oplus_{i=1}^{n} {T_i}^*(k_i) = h$. So, $h \in R(\mathbf{T}^*)$ and then $\oplus_{i=1}^{n} R({T_i}^*) \subseteq R(\mathbf{T}^*)$. Therefore, from Eq. (3.11), $R(\mathbf{V}) \subseteq R(\mathbf{T}^*)$.
So by theorem 3.10, the equation $\mathbf{TXV} = \mathbf{U}$ has a Hermitian solution if and only if $\mathbf{TT}^\dagger\mathbf{U} = \mathbf{U}$, $\mathbf{V}^*\mathbf{V}^{*\dagger}(\mathbf{T}^\dagger\mathbf{U})^* = (\mathbf{T}^\dagger\mathbf{U})^*$ and $\mathbf{V}^*\mathbf{T}^\dagger\mathbf{U}$ is Hermitian.
Suppose $W_{ij} = T_iV_j$ for every $1 \le i \ne j \le n$.
We have $\mathbf{TT}^\dagger\mathbf{U} = \mathbf{U}$ if and only if

$$\begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix} \begin{bmatrix} {T_1}^\dagger & \cdots & {T_n}^\dagger \end{bmatrix} \begin{bmatrix} U_1 & W_{12} & \cdots & W_{1n} \\ W_{21} & U_2 & \cdots & W_{2n} \\ \\ \vdots & & \ddots & \\ W_{n1} & W_{n2} & \cdots & U_n \end{bmatrix} = \begin{bmatrix} U_1 & W_{12} & \cdots & W_{1n} \\ W_{21} & U_2 & \cdots & W_{2n} \\ \\ \vdots & & \ddots & \\ W_{n1} & W_{n2} & \cdots & U_n \end{bmatrix}$$

$$\Longleftrightarrow \quad \begin{bmatrix} T_1{T_1}^\dagger U_1 & T_1{T_1}^\dagger W_{12} & \cdots & T_1{T_1}^\dagger W_{1n} \\ T_2{T_2}^\dagger W_{21} & T_2{T_2}^\dagger U_2 & \cdots & T_2{T_2}^\dagger W_{2n} \\ \\ \vdots & & \ddots & \\ T_n{T_n}^\dagger W_{n1} & T_n{T_n}^\dagger W_{n2} & \cdots & T_1{T_1}^\dagger U_n \end{bmatrix} = \begin{bmatrix} U_1 & W_{12} & \cdots & W_{1n} \\ W_{21} & U_2 & \cdots & W_{2n} \\ \\ \vdots & & \ddots & \\ W_{n1} & W_{n2} & \cdots & U_n \end{bmatrix}$$

and it is equivalent to $R(U_i) \subseteq R(T_i)$ and $R(W_{ij}) \subseteq R(T_i)$, for all $1 \le i \ne j \le n$. So $\mathbf{TT}^\dagger\mathbf{U} = \mathbf{U}$.
Indeed, by some calculation we have $\mathbf{V}^*\mathbf{V}^{*\dagger}(\mathbf{T}^\dagger\mathbf{U})^* = (\mathbf{T}^\dagger\mathbf{U})^*$ if and only if

$$\begin{bmatrix} V_1^*V_1^{*\dagger} & 0 & & \cdots & 0 \\ 0 & V_2^*V_2^{*\dagger} & & \cdots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ 0 & 0 & & \cdots & V_n^*V_n^{*\dagger} \end{bmatrix} \begin{bmatrix} (T_1^\dagger U_1 + \sum_{j=1, j\ne 1}^{n} {T_j}^\dagger W_{j1})^* \\ (T_2^\dagger U_2 + \sum_{j=1, j\ne 2}^{n} {T_j}^\dagger W_{j2})^* \\ \vdots \\ (T_n^\dagger U_n + \sum_{j=1, j\ne n}^{n} {T_j}^\dagger W_{jn})^* \end{bmatrix}$$

$$= \begin{bmatrix} (T_1^\dagger U_1 + \sum_{j=1, j\ne 1}^{n} {T_j}^\dagger W_{j1})^* \\ (T_2^\dagger U_2 + \sum_{j=1, j\ne 2}^{n} {T_j}^\dagger W_{j2})^* \\ \vdots \\ (T_n^\dagger U_n + \sum_{j=1, j\ne n}^{n} {T_j}^\dagger W_{jn})^* \end{bmatrix}$$

and it is equivalent to

$$V_i^*V_i^{*\dagger}((T_i^\dagger U_i)^* + \sum_{j=1, j\ne i}^{n} ({T_j}^\dagger W_{ji})^*) = (T_i^\dagger U_i)^* + \sum_{j=1, j\ne i}^{n} ({T_j}^\dagger W_{ji})^*. \tag{3.12}$$

Since every equation $T_iXV_i = U_i$ is solvable and has Hermitian solution, then for $i = 1, 2, \cdots, n$, $V_i^* V_i^{*\dagger}(T_i^\dagger U_i)^* = (T_i^\dagger U_i)^*$. So the Eq. (3.12) is eqivalent to condition (2) and $R(\sum_{j=1,j\neq i}^{n} (T_j{}^\dagger W_{ji})^*) \subseteq R(V_i{}^*)$ for every $1 \leq i, j \leq n$. So by $W_{ij} = T_iV_j$ and simple calculation we have $\mathbf{V}^*\mathbf{V}^{*\dagger}(\mathbf{T}^\dagger\mathbf{U})^* = (\mathbf{T}^\dagger\mathbf{U})^*$.
Moreover,

$$\mathbf{V}^*\mathbf{T}^\dagger\mathbf{U} = \begin{bmatrix} V_1^*T_1{}^\dagger & V_1^*T_2{}^\dagger & \cdots & V_1^*T_n{}^\dagger \\ V_2^*T_1{}^\dagger & V_2^*T_2{}^\dagger & \cdots & V_2^*T_n{}^\dagger \\ \vdots & \ddots & & \vdots \\ V_n^*T_1{}^\dagger & V_n^*T_2{}^\dagger & \cdots & V_n^*T_n{}^\dagger \end{bmatrix} \begin{bmatrix} U_1 & W_{12} & \cdots & W_{1n} \\ W_{21} & U_2 & \cdots & W_{2n} \\ & & & \\ \vdots & & \ddots & \\ W_{n1} & W_{n2} & \cdots & U_n \end{bmatrix}$$

$$= \begin{bmatrix} V_1^*T_1{}^\dagger U_1 + V_1^* \sum_{j=1,j\neq 1}^{n} T_j{}^\dagger W_{j1} & \cdots & V_1^*T_n{}^\dagger U_n + V_1^* \sum_{j=1,j\neq n}^{n} T_j{}^\dagger W_{jn} \\ V_2^*T_1{}^\dagger U_1 + V_2^* \sum_{j=1,j\neq 1}^{n} T_j{}^\dagger W_{j1} & \cdots & V_2^*T_n{}^\dagger U_n + V_2^* \sum_{j=1,j\neq n}^{n} T_j{}^\dagger W_{jn} \\ \vdots & & \vdots \\ V_n^*T_1{}^\dagger U_1 + V_n^* \sum_{j=1,j\neq 1}^{n} T_j{}^\dagger W_{j1} & \cdots & V_n^*T_n{}^\dagger U_n + V_n^* \sum_{j=1,j\neq n}^{n} T_j{}^\dagger W_{jn} \end{bmatrix}, \tag{3.13}$$

so $\mathbf{V}^*\mathbf{T}^\dagger\mathbf{U}$ is Hermitian if and only if the conditions (1) and (3) are verified and $V_i{}^*T_j{}^\dagger W_{ji}$ is Hermitian for all $1 \leq i \neq j \leq n$, and we have for all $1 \leq i \neq k, j \leq n$,

$$(V_i{}^* \sum_{j=1,j\neq k}^{n} (T_j{}^\dagger W_{jk}))^* = V_k{}^* \sum_{j=1,j\neq i}^{n} (T_j{}^\dagger W_{ji}).$$

So by $W_{ij} = T_iV_j$ and simple calculation, $\mathbf{V}^*\mathbf{T}^\dagger\mathbf{U}$ is Hermitian.
To prove converse, suppose $X_0 \in \mathcal{B}(\mathcal{H})$ be a common Hermitian solution of Eq. (1.1), then by theorem 3.10, the conditions (1) and (2) are verified. Moreover, if suppose $W_{ij} = T_iX_0V_j$, then X_0 is a Hermitian solution of equation $\mathbf{TXV} = \mathbf{U}$. So $\mathbf{V}^*\mathbf{T}^\dagger\mathbf{U}$ is Hermitian and by Eq. (3.13), the condition (3) is verified. □

Theorem 3.12. *Suppose $U, T_i \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $\forall i = 1, 2, \cdots, n$. If $R(U) \subseteq R(T_n)$ then the equation $\sum_{i=1}^{n} T_iX_i = U$ is solvable.*

Proof . Let $K_T = (I - TT^\dagger)$ and suppose $K_T^{(1)} := K_T$, $K_T^{(n)} := K_{K_T^{(n-1)}}$. By Dauglas theorem, the equation $TX = U$ has a solution if and only if $R(U) \subseteq R(T)$ or equivalently $K_TU = 0$.
Indeed, the equation $T_1X_1 + T_2X_2 = U$ is solvable if and only if the equation $K_{T_1}(U - T_2X_2) = 0$ is solvable and it is equivalent to $R(K_{T_1}U) \subseteq R(K_{T_1}T_2)$.
Moreover, the equation $T_1X_1 + T_2X_2 + T_3X_3 = U$ is solvable if and only if the equation $K_{T_1}(U - T_2X_2 - T_3X_3) = 0$ is solvable and it is equivalent to $R(K_{K_{T_1}T_2}K_{T_1}U) \subseteq R(K_{K_{T_1}T_2}K_{T_1}T_3)$.
So by induction, the equation $T_1X_1 + T_2X_2 + T_3X_3 + \cdots + T_nX_n = U$ is solvable if and only if the equation

$$K_{T_1}(U - T_2X_2 - T_3X_3 - \cdots - T_nX_n) = 0$$

is solvable and it is equivalent to

$$\begin{aligned} &R(K^{(n-2)}_{K_{T_1}T_2K_{T_1}T_3\cdots K_{T_1}T_{n-1}} K^{(n-3)}_{K_{T_1}T_2K_{T_1}T_3\cdots K_{T_1}T_{n-2}} \cdots K_{T_1}U) \\ &\subseteq R(K^{(n-2)}_{K_{T_1}T_2K_{T_1}T_3\cdots K_{T_1}T_{n-1}} K^{(n-3)}_{K_{T_1}T_2K_{T_1}T_3\cdots K_{T_1}T_{n-2}} \cdots K_{T_1}T_n). \end{aligned} \tag{3.14}$$

So, if $R(U) \subseteq R(T_n)$ then the Eq. (3.14) is verified. Hence the equation $T_1X_1 + T_2X_2 + T_3X_3 + \cdots + T_nX_n = U$ is solvable. □

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