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Equivalence of K-Functionals and Modulus of Smoothness for Fourier Transform

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Abstract

In Hilbert space $L^2(\mathbb{R}^n)$, we prove the equivalence between the modulus of smoothness and the K-functionals constructed by the Sobolev space corresponding to the Fourier transform. For this purpose, Using a spherical mean operator.

Keywords: Fourier Transform, Spherical Mean Operator, K-Functionals, Modulus of Smoothness.
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1. Introduction and Preliminaries

It is well known that integral Fourier transform are widely in mathematical physics. In this paper, the main result of the paper is the proof of the equivalence theorem for a K-functionals and a modulus of smoothness analog of the statement proved in [2]. For this paper, we use a spherical mean operator. Assume that $L^2(\mathbb{R}^n)$ the space of integrable functions f with the norm

$$\|f\| = \|f\|_2 = \left(\int_{\mathbb{R}^n} |f(x)|^2 dx\right)^{1/2}.$$

The Fourier transform for the function $f \in L^2(\mathbb{R}^n)$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix.\xi} dx.$$

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The inverse Fourier transform is defined by formula

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{ix.\xi} d\xi.$$

The Plancherel equality in [4].

$$\int_{\mathbb{R}^n} |f(x)|^2 dx = \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 d\xi.$$

Let the operator differential D is defined by

$$\mathrm{D} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2},$$

where $x = (x_1, ..., x_n)$, and $\mathrm{D}^0 f = f$, $\mathrm{D}^i f = \mathrm{D}(\mathrm{D}^{i-1} f)$, $i = 1, 2, ...$

Let $f \in \mathrm{L}^2(\mathbb{R}^n)$, we have

$$(\widehat{\mathrm{D}^m f})(\xi) = (-1)^m |\xi|^m \widehat{f}(\xi), \tag{1.1}$$

where $m \in \{1, 2, ...\}$.

Consider in $\mathrm{L}^2(\mathbb{R}^n)$ the spherical mean operator (see [3])

$$\mathrm{M}_h f(x) = \frac{1}{w_{n-1}} \int_{\mathbb{S}^{n-1}} f(x + hw) dw,$$

where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$, w_{n-1} its total surface measure with respect to the usual induced measure dw.

We have

$$\|\mathrm{M}_h f\| \leq \|f\| \ \ ; f \in \mathrm{L}^2(\mathbb{R}^n) \tag{1.2}$$

Let $j_\alpha(x)$ be a normalized Bessel function of the first kind, i.e.,

$$j_\alpha(x) = \frac{2^\alpha \Gamma(\alpha + 1) J_\alpha(x)}{x^\alpha},$$

where $J_\alpha(x)$ is a Bessel function of the first kind ([1], chap. 7). For any function $f(x) \in \mathrm{L}^2(\mathbb{R}^n)$ we define the finite differences of the first and the finite differences of the order m with a step $h > 0$.

$$\Delta_h f(x) = f(x) - \mathrm{M}_h f(x) = (\mathrm{E} - \mathrm{M}_h) f(x)$$

and

$$\Delta_h^m f(x) = \Delta_h(\Delta_h^{m-1} f(x)) = (\mathrm{E} - \mathrm{M}_h)^m f(x).$$

We define the generalized modulus of smoothness of the mthe order by the formula

$$w_m(f,\delta)_{2,n} = \sup_{0<h\leq\delta} \|\Delta_h^m f\|,$$

where $\delta > 0$ and $f \in \mathrm{L}^2(\mathbb{R}^n)$.
Let $\mathrm{W}_{2,n}^m$ be the Sobolev space construcred by the operator D,

$$\mathrm{W}_{2,n}^m = \{f \in \mathrm{L}^2(\mathbb{R}^n) : \ \mathrm{D}^j f \in \mathrm{L}^2(\mathbb{R}^n); j = 1,2,..,m\}$$

Let us define the K-functionals constructed by the spaces $\mathrm{L}^2(\mathbb{R}^n)$ and $\mathrm{W}_{2,n}^m$,

$$K(f,t;\mathrm{L}^2(\mathbb{R}^n);\mathrm{W}_{2,n}^m) = \inf\{\|f-g\| + t\|\mathrm{D}^m g\|, \ g \in \mathrm{W}_{2,n}^m\},$$

where $f \in \mathrm{L}^2(\mathbb{R}^n)$; $t > 0$.

For brevity, we denote

$$K_m(f,t)_{2,n} = K(f,t;\mathrm{L}^2(\mathbb{R}^n);\mathrm{W}_{2,n}^m).$$

2. Main Result

$c, c_1, c_2, ...$ are positive constants

Lemma 2.1. *Let $f(x) \in \mathrm{L}^2(\mathbb{R}^n)$, then*

$$\|\Delta_h^m f\| \leq 2^m \|f\|$$

Proof . We use the proof of recurrence for m and formula (1.2). □

Lemma 2.2. *Let $f \in \mathrm{L}^2(\mathbb{R}^n)$, then*

$$(\widehat{\mathrm{M}_h f})(\xi) = j_{\frac{n-2}{2}}(h|\xi|)\widehat{f}(\xi) \tag{2.1}$$

Proof . (see Proposition 4 in [3]) □

Lemma 2.3. *For $x \in \mathbb{R}$ the following inequalities are fulfilled*

1. $|j_\alpha(x)| \leq 1$
2. $|1 - j_\alpha(x)| \leq 2|x|$
3. $|1 - j_\alpha(x)| \geq c$ *with* $|x| \geq 1$, *where* $c > 0$ *is a certain constant which depend only on* α.

Proof . (Analog of lemma 2.9 in [2]) □

Lemma 2.4. *Let $f \in \mathrm{W}_{2,n}^m$, $t > 0$. Then*

$$w_m(f,t)_{2,n} \leq c_1 t^m \|\mathrm{D}^m f\|.$$

Proof . Let $h \in (0,t]$, $\Delta_h^m f = (\mathrm{E} - \mathrm{M}_h)^m f$ is the finite difference with the step h. From formulas (1.1), (2.1) and the Parseval equality, we obtain

$$\|\Delta_h^m f\| = \|(1 - j_{\frac{n-2}{2}}(h|\xi|))^m \widehat{f}(\xi)\|; \quad \|\mathrm{D}^m f\| = \||\xi|^m \widehat{f}(\xi)\|$$

we have

$$\|\Delta_h^m f\| = h^m \left\| \frac{(1 - j_{\frac{n-2}{2}}(h|\xi|))^m}{(h|\xi|)^m} |\xi|^m \widehat{f}(\xi) \right\|.$$

From inequality 2 of lemma 2.3, we obtain

$$\|\Delta_h^m f\| \leq c_1 h^m \||\xi|^m \widehat{f}(\xi)\| = c_1 h^m \|\mathrm{D}^m f\| \leq c_1 t^m \|\mathrm{D}^m f\|,$$

where $c_1 = 2^m$. Calculating the supremum with respect to all $h \in (0,t]$, we obtain

$$w_m(f,t)_{2,n} \leq c_1 t^m \|\mathrm{D}^m f\|.$$

□

For any function $f \in \mathrm{L}^2(\mathbb{R}^n)$ and any number $\nu > 0$ let us define the function

$$P_\nu(f)(x) = F^{-1}(\widehat{f}(\xi)\chi_\nu(\xi)),$$

where $\chi_\nu(\xi) = 1$ for $|\xi| \leq \nu$ and $\chi_\nu(\xi) = 0$ for $|\xi| > \nu$, F^{-1} is the inverse Fourier transform.

One can easily prove that the function $P_\nu(f)$ is infinitely differentiable and belongs to all classes $\mathrm{W}_{2,n}^m$, $m \in \{1, 2, ...\}$

Lemma 2.5. *Let $f \in \mathrm{L}^2(\mathbb{R}^n)$. The following inequality is true*

$$\|f - P_\nu(f)\| \leq c_3 \|\Delta_{1/\nu}^m f\|, \ \nu > 0$$

Proof . Let $|1 - j_{\frac{n-2}{2}}(h|\xi|)| \geq c$ with $h|\xi| \geq 1$. Using the Parseval equality, we obtain

$$\|f - P_\nu(f)\| = \|(1 - \chi_\nu(\xi))\widehat{f}(\xi)\| = \left\| \frac{1 - \chi_\nu(\xi)}{(1 - j_{\frac{n-2}{2}}(\frac{|\xi|}{\nu}))^m} (1 - j_{\frac{n-2}{2}}(\frac{|\xi|}{\nu}))^m \widehat{f}(\xi) \right\|$$

Note that

$$\sup_{|\xi| \in \mathbb{R}} \frac{1 - \chi_\nu(\xi)}{|1 - j_{\frac{n-2}{2}}(\frac{|\xi|}{\nu})|^m} \leq \frac{1}{c^m}.$$

We have $\|f - P_\nu(f)\| \leq c^{-m} \|(1 - j_{\frac{n-2}{2}}(\frac{|\xi|}{\nu}))^m \widehat{f}(\xi)\| = c_3 \|\Delta_{1/\nu}^m f\|$, where $c_3 = c^{-m}$. □

Corollary 2.6. $\|f - P_\nu(f)\| \leq c_3 w_m(f, 1/\nu)_{2,n}$.

Lemma 2.7. *The following inequality is true*

$$\|\mathrm{D}^m(P_\nu(f))\| \leq c_4 \nu^m \|\Delta_{1/\nu}^m f\|$$

Proof .

We use the Parseval equality

$$\|\mathrm{D}^m(P_\nu(f))\| = \|\widehat{\mathrm{D}^m(P_\nu(f))}\| = \||\xi|^m\chi_\nu(\xi)\widehat{f}(\xi)\| = \|\frac{|\xi|^m\chi_\nu(\xi)}{(1-j_{\frac{n-2}{2}}(\frac{|\xi|}{\nu}))^m}(1-j_{\frac{n-2}{2}}(\frac{|\xi|}{\nu}))^m\|.$$

Note that

$$\sup_{|\xi|\in\mathbb{R}}\frac{|\xi|^m\chi_\nu(\xi)}{|1-j_{\frac{n-2}{2}}(\frac{|\xi|}{\nu})|^m} = \nu^m\sup_{|\xi|\le\nu}\frac{(\frac{|\xi|}{\nu})^m}{|1-j_{\frac{n-2}{2}}(\frac{|\xi|}{\nu})|^m} = \nu^m\sup_{|t|\le 1}\frac{t^m}{|1-j_{\frac{n-2}{2}}(t)|^m}.$$

Let

$$c_4 = \sup_{|t|\le 1}\frac{t^m}{|1-j_{\frac{n-2}{2}}(t)|^m}.$$

then the formula is proved. □

Corollary 2.8. $\|\mathrm{D}^m(P_\nu(f))\| \le c_4\nu^m w_m(f,1/\nu)_{2,n}$.

The following theorem establishes the equivalence of the modulus of smoothness and the K-functional.

Theorem 2.9. *There exists positive constants c_5 and c_6 which satisfying the inequality*

$$c_5 w_m(f,\delta)_{2,n} \le K_m(f,\delta^m)_{2,n} \le c_6 w_m(f,\delta)_{2,n}$$

where $f \in \mathrm{L}^2(\mathbb{R}^n)$.

Proof . 1. We prove the inequality

$$c_5 w_m(f,\delta)_{2,n} \le K_m(f,\delta^m)_{2,n}$$

Let $h \in (0,\delta]$, $g \in \mathrm{W}^m_{2,n}$. We use Lemmas 2.1 and 2.4, we have

$$\begin{aligned}\|\Delta_h^m f\| &\le \|\Delta_h^m(f-g)\| + \|\Delta_h^m g\| \le 2^m\|f-g\| + c_1 h^m\|\mathrm{D}^m g\| \\ &\le c_7(\|f-g\| + \delta^m\|\mathrm{D}^m g\|),\end{aligned}$$

where $c_7 = \max(2^m, c_1)$. Calculating the supremum with respect to $h \in (0,\delta]$ and the infimum with respect to all possible functions $g \in \mathrm{W}^m_{2,n}$, we conclude $w_m(f,\delta)_{2,n} \le c_7 K_m(f,\delta^m)_{2,n}$, then the inequality is proved.

2. Now, we prove the inequality

$$K_m(f,\delta^m)_{2,n} \le c_6 w_m(f,\delta)_{2,n}.$$

Since $P_\nu(f) \in \mathrm{W}^m_{2,n}$, by the definition of K-functionals we obtain

$$K_m(f,\delta^m)_{2,n} \le \|f - P_\nu(f)\| + \delta^m\|\mathrm{D}^m(P_\nu(f))\|.$$

Using corollaries 2.6 and 2.8, we conclude

$$K_m(f,\delta^m)_{2,n} \le c_3 w_m(f,1/\nu)_{2,n} + c_4(\delta\nu)^m w_m(f,1/\nu)_{2,n}.$$

Since ν is an arbitrary positive value, choosing $\nu = \frac{1}{\delta}$, we obtain the inequality. □

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