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Some statistical inferences on the upper record of Lomax distribution

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Abstract

In this paper, some inferential properties of the upper record of the Lomax distribution are investigated. The upper record of the Lomax distribution parameters are estimated by using methods, Moment (MME), Maximum Likelihood (MLE), Kullback-Leibler Divergence of the Survival function (DLS) and Baysian. In addition, These methods are compared using Monte-Carlo simulation. Finally, this new model is fitted on the real data and some of the comparative criteria are calculated to confirm the superiority of the proposed model to other models.

Keywords: Lomax distribution, Upper record, Entropy, Parameter estimation, Simulation.
2010 MSC: Primary 37Mxx, Secondary 65Cxx.

1. Introduction

One of the most widely used income distributions, especially in statistics, is the two-parameter Lomax distribution. This distribution, which is a special case of income distribution known as Singh-Maddala, was first introduced by Lomax in 1954 to investigate the performance of any business failure data. Lomax distribution is used in business, economics, real science, queuing theory, internet traffic modeling, and etc [2]. This distribution is more flexible than the classic Pareto distribution in fitting to income data. Since this distribution has only two parameters, and its cumulative distribution function (cdf) and its quantile function are simple, it can be easily interpreted relative to the Singh-Maddala distribution. Consequently, fitting to income data is very suitable. Ahsanullah [4] reviewed some of the distributional properties of the record values and moments up to second order of Lomax distribution. Balakrishnan and Ahsanullah [10] studied the relations for single and product moments

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of record values from Lomax distribution. Interested readers can refer to Resources [7], [9], [27], [31], [29], [17], [24], [15], and [14] for more details.

Nowadays, the concept of records is used in many daily life issues such as weather, sports, geophysics, seismology and so on. In fact, when we deal with sequential observations, that are more or less than their previous, values records are used. Records are very important when observations are difficult to obtain or when observations are being destroyed when subjected to an experimental test. The term record value was first introduced by Chandler [13]. The interested readers can refer to [6], [5], [8], [25], [30], [33], [20], [32], [22], [34], [21], and [18].

The term entropy in the word, refers to a change from order to disorder. The birth of the information theory can be attributed to Shannon's "mathematical theory of information" article, 1948. Shannon stated in his article that the amount of uncertainty in a random experiment is measured by the entropy of possible outcomes. Nyquist [26] and Hartley [19] also conducted studies in this field. After the introduction of Shannon entropy, other criteria for measuring uncertainty were expressed by generalizing it. Among them, one can mention the Renyi entropy, Tsallis entropy, relative entropy and mutual information. Relative entropy (Kullback-Leibler Divergence (DLS)) was first introduced by Kullback-Leibler in 1951 [28], which measures the distance between two distributions.

In this paper, we use the various methods to estimate the parameters of upper record of Lomax distribution. For this purpose, we will use the relatively new DLS method. Finally, the results show that using the DLS method is more efficient in estimating the scale parameter.

Now, we carry out some necessary definition in the subsequent discusion [16], [1], [11], [28], [28], and [8].

Definition 1.1. *If X is a random variable having an absolutely continuous cdf F(x), pdf f(x) and support S_x, then the Shannon entropy of X is defined*

$$H(X) = -\int_{S_x} f_X(x) \log f_X(x) dx. \tag{1.1}$$

Definition 1.2. *If X is a random variable having an absolutely continuous cdf $F(x)$, pdf $f(x)$ and support S_x, then Renyi entropy of order ρ of the random variable X is defined by*

$$H_\rho(X) = -\frac{1}{\rho - 1} \log \int_{S_x} f^\rho(x) dx \ , \ \forall \rho > 0 (\rho \neq 1). \tag{1.2}$$

Definition 1.3. *If X is a random variable having an absolutely continuous cdf $F(x)$, pdf $f(x)$ and support S_x, then Tsallis entropy of order ρ of the random variable X is defined by*

$$S_\rho(X) = \frac{1}{\rho - 1} \left[1 - \int_{S_x} f^\rho(x) dx \right], \rho \neq 1, \rho > 0. \tag{1.3}$$

Definition 1.4. *Let $X_1, X_2, ...$ be a sequence of positive, independent and identically distributed (iid) random variable from a non-increasing survival function $\bar{F}(x, \Theta) = P_\Theta(X > x)$ with support S_x and vector of parameters Θ. Define the empirical survival function of a random sample of size n by*

$$\bar{G}_n(x) = \sum_{i=0}^{n-1} I_{[X_{(i)}, X_{(i+1)})}(x), \tag{1.4}$$

where I is the indicator function and $(0 = X_{(0)} \leq) X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ are the ordered sample.

Definition 1.5. *Let $\bar{F}(x,\Theta)$ be the true survival function with unknown parameters Θ and $\bar{G}_n(x)$ be the empirical survival function of a random sample of size n from $\bar{F}(x,\Theta)$. Define the Kullback-Leibler divergence of survival functions $\bar{G}_n(x)$ and F by*

$$DLS(\bar{G}_n\|\bar{F}) = \int_0^\infty \big(\bar{G}_n(x)\ln\frac{\bar{G}_n(x)}{\bar{F}(x)} - [\bar{G}_n(x) - \bar{F}(x)]\big)dx. \tag{1.5}$$

Definition 1.6. *Let $X_1, X_2, ...$ be a sequence of iid random variables having an absolutely continuous cdf F(x) and pdf f(x). An observation X_j is called an upper record value if its value exceeds that of all previous observations. Thus, X_j is an upper record if $X_j > X_i$ for every $i < j$. If we use the notation $X_{U(n)}$ for the r^{th} upper record statistic then, the pdf of the $X_{U(n)}$ is*

$$f_{X_{U(n)}}(x) = \frac{[-\ln(1-F(x))]^{n-1}}{\Gamma(n)} f(x), -\infty < x < \infty, \tag{1.6}$$

where Γ is the gamma function.

The rest of this paper is organized as follows. In the next section, some preliminary results and some studies on the entropy of the upper record values are presented, as well as the calculation of these results for the upper record of Lomax distribution (URLD). In section 3, we will estimate parameters of the URLD by using different methods. In section 4, we will compare the methods using the Monte Carlo simulation. Finally, in Section 5, by using a real data set, the uURLD will be compared to the upper record of several other income distributions.

2. The upper record value of Lomax distribution and its entropy

In this section, first, we introduce some preliminary results for the upper record values. In addition, some studies on the Shannon, Renyi and Tsallis entropies of the upper record values are expressed.

Corollary 2.1. *(See [3]). If use of the $U = -2\log[1-F(x)]$ transform in relation 1.6, we have*

$$X_{U(n)} \overset{d}{=} F_X^{-1}\Big[1 - \exp(-\frac{1}{2}\chi^2_{(2n)})\Big]. \tag{2.1}$$

Corollary 2.2. *(See [3]). The $100(1-\epsilon)\%$ confidence interval for $X_{U(n)}$ is*

$$P\Big[F_X^{-1}\big(1-\exp(-\frac{1}{2}\chi^2_{(2n,\frac{\epsilon}{2})})\big) < X_{U(n)} < F_X^{-1}\big(1-\exp(-\frac{1}{2}\chi^2_{(2n,1-\frac{\epsilon}{2})})\big)\Big] = 1-\epsilon. \tag{2.2}$$

Corollary 2.3. *(See [3]). The p^{th} quantile for $0 < p < 1$ of $X_{U(r)}$ is*

$$F^{-1}_{X_{U(n)}}(p) = F^{-1}\big[1-\exp(-\frac{1}{2}F^{-1}_{\chi^2_{(2n)}}(p))\big]. \tag{2.3}$$

Theorem 2.4. *(See [11]). Let $\{X_n, n \geq 1\}$ be a sequence of iid continuous random variables from the distribution F(x) with density function f(x) and entropy $H(X) < \infty$. Then for all $n \geq 1$*

$$H(X_{U_n}) = \sum_{i=1}^{n-1}(\log i - \frac{n-1}{i}) + (n-1)\gamma - \phi_f(n-1), \tag{2.4}$$

where

$$\phi_f(n) = \int_0^{+\infty} \frac{z^n}{n!}e^{-z}\log f(F^{-1}(1-e^{-z}))dz. \tag{2.5}$$

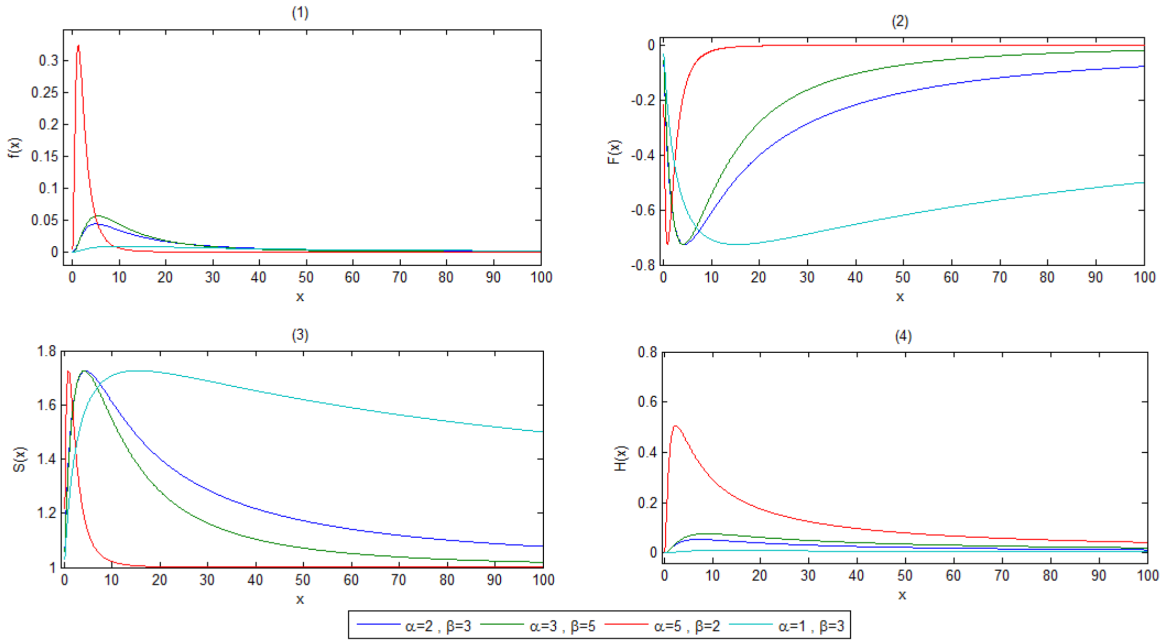

Figure 1: (1) Graph of pdf of 4^{th} URLD for different values (α, β), (2) Graph of cdf of 4^{th} URLD for different values (α, β), (3) Graph of survival function of 4^{th} URLD for different values (α, β), (4) Graph of hazard rate function of 4^{th} URLD for different values (α, β).

Theorem 2.5. *(See [11]). Under the assumptions of theorem 2.4 we have*

$$(i) \qquad H(X_{U_n}) \leq H(X_{U_n^*}) - n - B_n - E[\frac{\log f_Y(y)}{1 - F_Y(y)}] - \log M, \tag{2.6}$$

$$(ii) \qquad H(X_{U_n}) \geq H(X_{U_n^*}) - n - \log M, \tag{2.7}$$

where $M = f(m) < \infty$, m is the mode of the distribution, $Y = MX$ and

$$X_{U_n^*} \sim \Gamma(n, 1) \quad , \quad B_n = \frac{(n-1)^{(n-1)}}{(n-1)!} e^{-(n-1)}. \tag{2.8}$$

Lemma 2.6. *(See [1]). Let $\{X_n, n \geq 1\}$ be a sequence of iid continuous random variables from the distribution F(x) with density function f(x) and the quantile function $F^{-1}(.)$. Then the Renyi entropy of $X_{U(n)}$ can be expressed as*

$$H_\rho(X_{U_n}) = H_\rho(X_{U_n^*}) - \frac{\rho(n-1)+1}{\rho - 1} \log \rho - \frac{1}{\rho - 1} \log E_{g_n}\left[f^{\rho-1}(F^{-1}(1 - e^{-v_n}))\right], \tag{2.9}$$

where

$$X_{U_n^*} \sim \Gamma(n, 1) \ , \ v_n \sim g_n = \Gamma(\rho(n-1)+1, 1). \tag{2.10}$$

Lemma 2.7. *(See [12]). Let $X_1, X_2, ..., X_n$ be a random sample with size n from continuous cdf F(x) and pdf f(x). Let $X_{U(n)}$ denotes the n^{th} upper record. Then, the Tsallis entropy of $X_{U(n)}$ can be expressed as*

$$S_\rho(X_{U_n}) = \frac{1}{\rho - 1}\Big[1 - \frac{\Gamma(\rho(n-1)+1)}{\Gamma^\rho(n)} E(f^{\rho-1}\big(F^{-1}(1 - e^{-V_n})\big))\Big]. \tag{2.11}$$

Now, we will study the upper record of Lomax distribution. A continuous random variable X is said to be a be a two-parameter Lomax distribution with shape parameter α and scale parameter β and denoted by $X \sim Lomax(\alpha, \beta)$, if its cdf is

$$F(x) = 1 - (1 + \frac{x}{\beta})^{-\alpha} \quad ; \quad x, \alpha, \beta > 0. \tag{2.12}$$

In the following, based on the second section, we present the following results for the URLD:

1) Distribution of n^{th} upper record is

$$X_{U(n)} \overset{d}{=} \beta\big[(\exp(-\frac{\chi^2_{(2n)}}{2}))^{-\frac{1}{\alpha}} - 1\big]. \tag{2.13}$$

2) Probability density function $X_{U(n)}$ is

$$f_{X_{U(n)}}(x) = \frac{\alpha^n}{\beta\Gamma(n)}[\log(1 + \frac{x}{\beta})]^{n-1}(1 + \frac{x}{\beta})^{-(\alpha+1)}. \tag{2.14}$$

3) Survival function $X_{U(n)}$ is

$$\bar{F}_{X_{U(n)}}(x) = \big(1 + \frac{x}{\beta}\big)^{-\alpha} \sum_{j=0}^{n-1} \frac{\Big(\alpha \log\big(1 + \frac{x_{(i)}}{\beta}\big)\Big)^j}{j!}. \tag{2.15}$$

4) Hazard rate function $X_{U(n)}$ is

$$H(x) = \frac{\frac{\alpha^n}{\beta\Gamma(n)}[\log(1 + \frac{x}{\beta})]^{n-1}(1 + \frac{x}{\beta})^{-(\alpha+1)}}{\big(1 + \frac{x}{\beta}\big)^{-\alpha} \sum_{j=0}^{n-1} \frac{1}{j!}\Big(\alpha \log\big(1 + \frac{x_{(i)}}{\beta}\big)\Big)^j}. \tag{2.16}$$

5) Moment generating function $X_{U(n)}$ is

$$M_{X_{U(n)}}(t) = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{t^m}{m!} \alpha^n \beta^m \frac{(-1)^k \binom{m}{k}}{(\alpha - m + k)^n}. \tag{2.17}$$

6) The $100(1-\epsilon)\%$ confidence interval for $X_{U(n)}$ is

$$\Big(\beta\big[(\exp(-\frac{\chi^2_{(2n,\frac{\epsilon}{2})}}{2}))^{-\frac{1}{\alpha}} - 1\big], \beta\big[(\exp(-\frac{\chi^2_{(2n,1-\frac{\epsilon}{2})}}{2}))^{-\frac{1}{\alpha}} - 1\big]\Big). \tag{2.18}$$

7) The p^{th} quantile for $0 < p < 1$ of $X_{U(n)}$ is

$$q_{U(n)}(p) = \beta\Big[(\exp(-\frac{1}{2}F_{\chi^2(2n)}(p)))^{-\frac{1}{\alpha}} - 1\Big]. \tag{2.19}$$

8) Entropy of the upper record $X_{U(n)}$ is

$$H(X_{U_n}) = \sum_{i=1}^{n-1}(\log i - \frac{n-1}{i}) + (n-1)\gamma - \log(\frac{\alpha}{\beta}) + \frac{n(\alpha+1)}{\alpha}. \tag{2.20}$$

9) Bounds for entropy of the upper record values $X_{U(n)}$ is

$$H(X_{U_n^*}) - n - \log(\frac{\alpha}{\beta}) \leq H(U_n) < \infty. \tag{2.21}$$

10) Renyi entropy of the upper record $X_{U(n)}$ is

$$H_\rho(X_{U_n}) = H_\rho(X_{U_n^*}) - \frac{\rho(n-1)+1}{\rho-1}\log\rho - \log\frac{\alpha}{\beta} + \frac{\rho(n-1)}{\rho-1}\log\big(\rho + \frac{1}{\alpha}(\rho-1)\big). \tag{2.22}$$

11) Tsallis entropy of the upper record $X_{U(n)}$ is

$$S_\rho(X_{U_n}) = \frac{1}{\rho-1}\left[1 - \frac{\Gamma(\rho(n-1)+1)}{\Gamma^\rho(n)}(\frac{\alpha}{\beta})^{\rho-1}[\frac{\alpha+1}{\alpha}(\rho-1)+1]^{-\rho(n-1)-1}\right]. \tag{2.23}$$

3. Estimation of parameters of the URLD

In this section, we use four methods to estimate the shape parameter α and the scale parameter β of the URLD. Suppose that $X_1, X_2, ..., X_n$ is a random sample from r^{th} upper record of Lomax distribution.

Moment method (MME):

In this method, after equating the moments of population and the sample, we have the following equations:

$$(I) \qquad \bar{x} = \beta\left[(\frac{\alpha}{\alpha-1})^r - 1\right],$$

$$(II) \qquad \bar{x^2} = \beta^2\left[(\frac{\alpha}{\alpha-2})^r - 2(\frac{\alpha}{\alpha-1})^r + 1\right],$$

$$(III) \qquad s^2 = \beta^2\left[(\frac{\alpha}{\alpha-2})^{2r} - (\frac{\alpha}{\alpha-1})^r\right].$$

Here, we consider three cases:

- If α is known and β is unknown, according to equation (I), the estimator β is

$$\tilde{\beta}_{\alpha_{known}} = \frac{\bar{x}}{[(\frac{\alpha}{\alpha-1})^r - 1]}. \tag{3.1}$$

- If β is known and α is unknown, according to equation (I), the estimator α is

$$\tilde{\alpha}_{\beta_{known}} = [1 - (1 + \frac{\bar{x}}{\beta})^{-\frac{1}{r}}]^{-1}. \tag{3.2}$$

- If α and β are both unknown then, equating the sample coefficient of variation (CV) and the population CV (we observe that the population CV is independent of the β) we have

$$CV = \frac{\sqrt{Var(X)}}{E(X)} = \frac{s}{\bar{x}} = \frac{\sqrt{(\frac{\alpha}{\alpha-2})^r - (\frac{\alpha}{\alpha-1})^{2r}}}{(\frac{\alpha}{\alpha-1})^r - 1}, \tag{3.3}$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$ and $s^2 = \frac{1}{n-1}\sum_{i=1}^n (x_i - \bar{x})^2$. Now, using equation 3.3 and with the help of suitable numerical methods, α is estimated. In order to obtain the moment estimator β, it is sufficient to put the estimated value of α in equation (I).

Maximum Likelihood method (MLE):
In this method, the logarithm of the likelihood function for the URLD is

$$\log L = nr\log\alpha - n\log\beta + (r-1)\sum_{i=1}^{n}\log[\log(1+\frac{x_i}{\beta})] - (\alpha+1)\sum_{i=1}^{n}\log(1+\frac{x_i}{\beta}). \tag{3.4}$$

Now, we have to maximize the 3.4 to the parameters. In this method, we consider three cases:

- If β is known and α is unknown: differentiate with respect to α then, equating to zero and solving with respect α. Then, the estimate of the MLE of parameter α is obtained as follows

$$\hat{\alpha} = \frac{nr}{\sum_{i=1}^{n}\log(1+\frac{x_i}{\beta})}. \tag{3.5}$$

- If α is known and β is unknown: again differentiate with respect to β then, equating to zero and solving with respect β. Then, the estimate of the MLE of parameter β is obtained as follows

$$\frac{\partial\log L}{\partial\beta} = -n - (r-1)\sum_{i=1}^{n}\frac{\frac{x_i}{\beta+x_i}}{\log(1+\frac{x_i}{\beta})} + (\alpha+1)\sum_{i=1}^{n}\frac{x_i}{\beta+x_i} = 0. \tag{3.6}$$

  The solution of the above equation, which is a function based only on β, yields the estimate of the MLE of parameter β by using appropriate numerical methods.

- If α and β are both unknown then, in this case, by setting 3.5 in 3.4 and then taking a derivative of it with respect to β, we have

$$\frac{\partial\log L}{\partial\beta} = nr\frac{\sum_{i=1}^{n}\frac{x_i}{\beta+x_i}}{\sum_{i=1}^{n}\log(1+\frac{x_i}{\beta})} - n - (r-1)\sum_{i=1}^{n}\frac{\frac{x_i}{\beta+x_i}}{\log(1+\frac{x_i}{\beta})} + \sum_{i=1}^{n}\frac{x_i}{\beta+x_i} = 0. \tag{3.7}$$

  By solving this equation, which is a function based only on β, yields the estimate of the MLE of parameter β by using appropriate numerical methods. Then by putting it in 3.5, the MLE estimate of the parameter α can also be calculated.

Bayesian Method:
To estimate the URLD in the Bayesian method, we assume the previous information of α and β are independent of each other, so $\pi(\alpha,\beta) = \pi(\alpha)\pi(\beta)$. In this method, we use the three prior distributions uniform, normal and triangular (increasing and decreasing) using the Monte Carlo simulation (In the relevant section, the results are presented).

Kullback-Leibler Divergence of Survival function method (DLS):
In order to estimate the r^{th} URLD by using DLS method, by replacing with $\bar{F}_{X_{U(r)}}$ (the survival function of r^{th} upper record) in equation 1.5, it is as follows

$$\begin{aligned} DLS &= \sum_{i=0}^{n-1}(1-\frac{i}{n})\log(1-\frac{i}{n})\Delta x_{i+1} - \big[\bar{x}_n - \beta\big((\frac{\alpha}{\alpha-1})^r - 1\big)\big] \\ &+ \frac{\alpha\beta}{n}\sum_{i=1}^{n}\big((1+\frac{x_{(i)}}{\beta})\log(1+\frac{x_{(i)}}{\beta}) - \frac{x_{(i)}}{\beta}\big) \\ &- \sum_{i=0}^{n-1}(1-\frac{i}{n})\int_{x_{(i)}}^{x_{(i+1)}}\log\left(\sum_{j=0}^{r-1}\frac{\big(\alpha\log(1+\frac{x}{\beta})\big)^j}{j!}\right)dx, \end{aligned} \tag{3.8}$$

where $\Delta_{x_{i+1}} = x_{i+1} - x_i$ and $x_0 = 0$.
We must now minimize DLS to parameters α and β. As a result, we have the following equations:

$$\begin{aligned} \frac{\partial DLS}{\partial \alpha} &= -\frac{\beta r \alpha^{r-1}}{(\alpha-1)r+1} + \frac{\beta}{n}\sum_{i=1}^{n}\big((1+\frac{x_{(i)}}{\beta})\log(1+\frac{x_{(i)}}{\beta}) - \frac{x_{(i)}}{\beta}\big) \\ &- \sum_{i=0}^{n-1}(1-\frac{i}{n})\int_{x_{(i)}}^{x_{(i+1)}} \log\left(\sum_{j=0}^{r-1}\frac{\alpha^{j-1}\big(\log(1+\frac{x}{\beta})\big)^j}{(j-1)!}\right)dx = 0, \end{aligned} \tag{3.9}$$

$$\begin{aligned} \frac{\partial DLS}{\partial \beta} &= -1 + \frac{\alpha}{n}\sum_{i=1}^{n}\big((1+\frac{x_{(i)}}{\beta})\log(1+\frac{x_{(i)}}{\beta}) - \frac{x_{(i)}}{\beta}\big) - \frac{\alpha}{n\beta}\sum_{i=1}^{n} x_{(i)}\log(1+\frac{x_{(i)}}{\beta}) \\ &+ (\frac{\alpha}{\alpha-1})^r - \sum_{i=0}^{n-1}(1-\frac{i}{n})\int_{x_{(i)}}^{x_{(i+1)}} \frac{\sum_{j=0}^{r-1}\frac{x\alpha}{\beta^2}\frac{\left(\alpha\log(1+\frac{x}{\beta})\right)^{j-1}}{(j-1)!}}{\sum_{j=0}^{r-1}\frac{\left(\alpha\log(1+\frac{x}{\beta})\right)^{j}}{j!}}dx = 0. \end{aligned} \tag{3.10}$$

Solving the above equations, using appropriate numerical methods, leads to the DLS estimation of the parameters α and β.

4. Simulation studies

According to the results, estimating the parameters using these methods does not have a closed form. As a result, appropriate numerical methods can be applied to evaluate the performance of these methods. For this purpose, MATLAB software is used.

First, we generate the random sample $x_1, x_2, ..., x_n$ from the 4^{th} URLD with $\alpha_{true} = 5$ and $\beta_{true} = 7$. Then, we estimate the parameters with the help of the above methods. Now, we repeat this process $j = 10000$ times. Finally, the sample mean values $\bar{\alpha}$ and $\bar{\beta}$ and the sample variance s_α^2 and s_β^2 are calculated for the comparison of the mentioned methods. Where α_k and β_k are the estimated 4^{th} URLD shape and scale parameter from k^{th} sample. Certainly, the method in which $\frac{\bar{\alpha}}{\alpha_{true}}$ and $\frac{\bar{\beta}}{\beta_{true}}$ are closer to one, provides more unbiased estimators.

In the Baysian method, note that for all three prior distributions, in the denominator of posterior distributions we have a dual integral, which is not a simple calculation. To overcome this problem, the "importance sampling " method is used. To investigate the effect of sample size on estimates, the above process is repeated for samples of size $n = 10, 20, 30, ..., 100$ and the results are shown in Figures 2 and 3.

Comparison of α estimators:

- According to Figure 2 (a), the use of MME, MLE and Baysian methods, especially with the normal and increasing triangular prior, yields estimators very close to the real value. In the MLE method, for the small sample size, the estimators differ slightly from the actual value. But, with increasing sample size, they are very close to the real value.
- According to of Figure 2 (c), which shows $\frac{\bar{\alpha}}{\alpha_{true}}$, this ratio has a very close to one in methods of MLE and Baysian. So, we can say that these two methods are very suitable in terms of unbiased criterion.

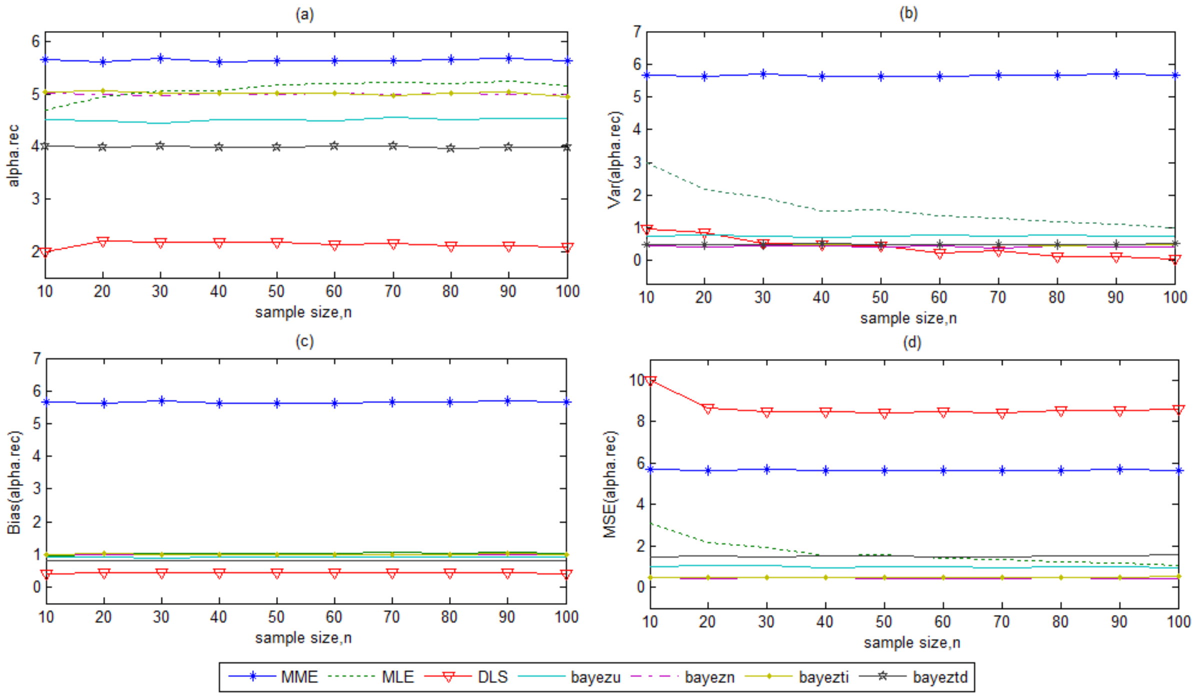

Figure 2: Comparison of α estimators.

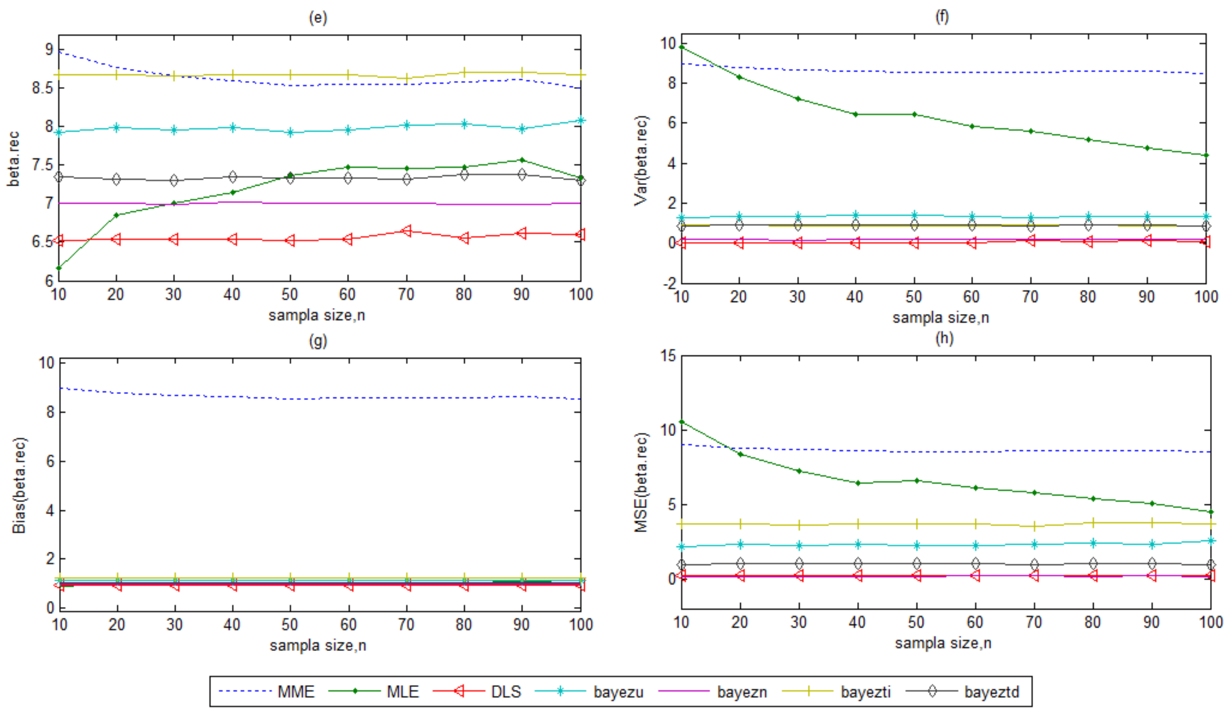

Figure 3: Comparison of β estimators.

- According to of Figure 2 (b) and (d), the MLE (with increasing sample size) and Baysian methods have a lower Mean Square Error (MSE) than other methods. These two criteria also confirm the appropriateness of these two methods.

Comparison of β estimators:

- According to Figure 3 (e), the use of the DLS and Baysian methods, especially with the normal prior, yields estimators very close to real value. Other methods provide some estimators that are slightly different from the real value. However, in β estimation, the DLS method has a relatively good performance. Also, the MLE method has a better performance with increasing sample size.
- According to Figure 3 (g), which shows $\frac{\bar{\beta}}{\beta_{true}}$, this ratio, in all methods, except for MME, has a value close to one.
- According to Figure 3 (f) and (h), MLE (with increasing sample size), DLS and Baysian methods have the lowest MSE.

5. Application with real data set

In this section, by using a real data set, the URLD with the Upper Record of Pareto Distribution(URPD), Upper Record of Weibull Distribution(URWD) and Upper Record of Sing-Maddala Distribution(URSMD) are compared with the probability density function of:

$$\begin{aligned}
f_{URLD}(x) &= \frac{\alpha^n}{\beta\Gamma(n)}[\log(1+\frac{x}{\beta})]^{n-1}(1+\frac{x}{\beta})^{-(\alpha+1)};\ x>0,\ \alpha,\beta>0,\\
f_{URPD}(x) &= \frac{\alpha^n\beta^\alpha}{\Gamma(n)}\frac{1}{x^{\alpha+1}}\big[\log(1+\frac{x}{\beta})\big]^{n-1}\ ;\ x\geq\beta\ ,\ \alpha,\beta>0,\\
f_{URWD}(x) &= \frac{\alpha}{\beta\Gamma(n)}(\frac{x}{\beta})^{n\alpha-1}\ ;\ x>0,\ \alpha,\beta>0,\\
f_{URSMD}(x) &= \frac{\alpha\lambda^n}{\beta}\big[\log(1+(\frac{x}{\beta})^\alpha)\big]^{n-1}(\frac{x}{\beta})^{\alpha-1}(1+(\frac{x}{\beta})^\alpha)^{-(\lambda+1)}\ ;\ x>0,\ \alpha,\beta,\lambda>0.
\end{aligned}$$

These data include 14 records of the income (1959 dollars) of families for whites for the period 1949-1966, and derived from Source [23]. These data are:

$$4.188, 4.332, 4.513, 4.720, 4.935, 5.083, 5.398, 5.633, 5.732, 5.828, 6.086, 6.294, 6.534, 6.750.$$

Table (1.1), contains the values of Akaike Information Criterion (AIC), Baysian information criterion(BIC) and contains goodness of fit test statistics Kolmogorov-Smirnov (K-S) with its corresponding p-value. The required numerical evaluations are implemented using the R software. As you can see in Table (1.1), the smallest values of information criterion values and goodness of fit test statistics and the largest p-value are related to the URLD. Therefore, it can be concluded that the URLD, in fitting this data, performs better than other distributions.

6. Conclusion

In this paper, we examined the URLD and we obtained some inferential properties such as, pdf, cdf, survival function, hazard rate function, moment generating function, quantile function, confidence interval, and Shanon, Renyi and Tsallis entropy. In addition, we estimated the URLD

Table 1: Goodness of fit test statistics for mean income level of families for whites (1949-1966).

Distribution	AIC	BIC	K-S	P-Value
URLD	9.361821	10.63994	0.3156798	0.08901
URPD	9.920529	11.19864	0.5048093	0.0007909
URWD	10.82335	12.10146	0.8876496	1.023e-13
URSMD	14.5175	15.79562	134.3688	$< 2.2e - 16$.

parameters in four methods, MME, MLE, DLS, and Bayesian, and compared these estimators with Monte Carlo simulation, and showed that, in the α estimate, the use of the Baysian method with the appropriate prior and the method of MLE (in large sample size), leads to suitable estimators and in β estimation, the use of DLS, Baysian with the appropriate prior and MLE (in large sample size) methods is recommended. Finally, to prove the superiority of the URLD performance, compared to some other income distributions, we fitted this model to the real data and showed that the URLD, in fitting these data, was more appropriate than the URPD, URWD, and URSMD.

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