Int. J. Nonlinear Anal. Appl. 10 (2019) No. 2, 275-299
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2019.18455.2014

New inequalities for generalized m-convex functions via generalized fractional integral operators and their applications

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Abstract

In the present work, we prove a parametrized identity for a differentiable function via generalized integral operators. By applying the established identity and the new so–called generalized m-convex function, some generalized trapezium, Ostrowski and Simpson type integral inequalities have been discovered. Various special cases have been studied as well. Some applications of the present results to special means and new error estimates for the trapezium and midpoint quadrature formula have been investigated. It is hoped that the methods and techniques of this paper could further stimulate the research conducted in the field of integral inequalities.

Keywords: Trapezium inequality, Ostrowski inequality, Simpson inequality, convexity, general fractional integrals.
2010 MSC: Primary 26A51; Secondary 26A33, 26D07, 26D10, 26D15.

1. Introduction and preliminaries

The inequality below, known as Hermite–Hadamard inequality, is one of the most famous inequalities in the literature relevant to convex functions.

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Received: August 2019 *Revised:* November 2019

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function and $u_1, u_2 \in I$ with $u_1 < u_2$. Then the following inequality holds:*

$$f\left(\frac{u_1+u_2}{2}\right) \leq \frac{1}{u_2-u_1}\int_{u_1}^{u_2} f(x)dx \leq \frac{f(u_1)+f(u_2)}{2}. \tag{1.1}$$

The inequality (1.1) is also known as trapezium inequality.

The trapezium inequality has always been a source of great interest due to its wide ranging applications in the field of mathematical analysis. In recent decades, several authors have intensively investigates (1.1) motivated by the study of convex function. Interested readers are referred to [4]-[8],[17, 22, 23, 25, 29, 31],[36]-[40],[47, 49, 52, 53, 55, 56],

The following result is known as Ostrowski's inequality, (see [32] and the references cited therein) which provides an upper bound for the approximation of the integral average $\frac{1}{u_2-u_1}\int_{u_1}^{u_2} f(t)\,dt$ for $x \in [u_1, u_2]$.

Theorem 1.2. *Let $f : I \longrightarrow \mathbb{R}$ be a mapping differentiable on I° and let $u_1, u_2 \in I^\circ$ with $u_1 < u_2$. If $|f'(x)| \leq M$ for all $x \in [a, b]$, then*

$$\left|f(x) - \frac{1}{u_2-u_1}\int_{u_1}^{u_2} f(t)\,dt\right| \leq M(u_2-u_1)\left[\frac{1}{4} + \frac{\left(x-\frac{u_1+u_2}{2}\right)^2}{(u_2-u_1)^2}\right], \quad \forall x \in [u_1, u_2]. \tag{1.2}$$

For other recent results concerning Ostrowski type inequalities, see [1]-[3],[9]-[16],[20],[32]-[35],[41]-[43],[45, 46, 50, 54, 57]. It must be noted that Ostrowski's inequality is essential in all fields of mathematics, especially in approximation theory. Thus such types of inequalities have been studied extensively by several researches and there has been a plethora of generalizations, extensions and variants of these inequalities for various types of functions.
The following inequality is well known in the literature as Simpson's inequality.

Theorem 1.3. *Let $f : [u_1, u_2] \longrightarrow \mathbb{R}$ be four time differentiable on the interval (u_1, u_2) and having the fourth derivative bounded on (u_1, u_2), that is*

$$\|f^{(4)}\|_\infty = \sup_{x\in(u_1,u_2)} |f^{(4)}| < +\infty.$$

Then, we have

$$\left|\int_{u_1}^{u_2} f(x)dx - \frac{u_2-u_1}{3}\left[\frac{f(u_1)+f(u_2)}{2} + 2f\left(\frac{u_1+u_2}{2}\right)\right]\right| \leq \frac{1}{2880}\|f^{(4)}\|_\infty (u_2-u_1)^5. \tag{1.3}$$

The inequality (1.3) establishes an error bound for the classical Simpson quadrature formula, which is one of the most commonly used quadrature formulae in practical applications. In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Simpson type inequalities, see [30, 44, 51, 57].
In numerical analysis many quadrature rules have been established to approximate definite integrals. Ostrowski's inequality provides bounds for many numerical quadrature rules, see [15, 16].

The aim of this paper is to establish trapezium, Ostrowski and Simpson type generalized integral inequalities for generalized m-convex functions with respect to another function. Some applications to special means and new error bounds for midpoint and trapezium quadrature formula are obtained. Interestingly, special cases of some of our results constitute fractional integral inequalities. Hence, it is important to recall some essential facts relevant to fractional integrals.

Initially, let us present some preliminaries and definitions which will be helpful for further study.

Definition 1.4. *[37] Let $f \in L[u_1, u_2]$. Then k-fractional integrals of order $\alpha, k > 0$ with $u_1 \geq 0$ are defined by*

$$I_{u_1^+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{u_1}^{x} (x-t)^{\frac{\alpha}{k}-1} f(t)\, dt, \quad x > u_1$$

and

$$I_{u_2^-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{x}^{u_2} (t-x)^{\frac{\alpha}{k}-1} f(t)\, dt, \quad u_2 > x, \tag{1.4}$$

where $\Gamma_k(\cdot)$ stands for the k-gamma function.

For $k = 1$, the k-fractional integrals yield Riemann–Liouville integrals. For $\alpha = k = 1$, the k-fractional integrals yield classical integrals.

Let K be a nonempty closed set in $\mathbb{R}^n$, and K° be the interior of K. We denote by $\langle ., . \rangle$ and $\|.\|$ the inner product and norm on $\mathbb{R}^n$, respectively. Let f, $\phi : K \to \mathbb{R}$ be continuous mappings.

Definition 1.5. *[38] The function f on the ϕ-convex set K is said to be ϕ-convex, if*

$$f(u_1 + te^{i\phi}(u_2 - u_1)) \leq (1-t)f(u_1) + tf(u_2), \qquad \forall\ u_1, u_2 \in K, \quad t \in [0,1].$$

The function f is said to be ϕ-concave iff $-f$ is ϕ-convex. Note that, every convex function is ϕ-convex but the converse does not hold in general.

Definition 1.6. *[18] The special function*

$$\mathbf{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \ \alpha \in C, \ R(\alpha) > 0, \ z \in C,$$

where Γ stands for the Gamma function, is called Mittag–Leffler function.

This function plays an essential role in fractional calculus.

We are now in the position to introduce the notions of generalized m-convex set and generalized m-convex function as follows:

Definition 1.7. *A non empty set K is said to be a generalized m-convex set for some fixed $m \in (0,1]$, if*

$$mu_1 + t\mathbf{E}_\alpha(u_2 - mu_1) \in K, \qquad \forall\ u_1, u_2 \in K, \quad t \in [0,1], \tag{1.5}$$

where $\alpha \in C$, $R(\alpha) > 0$.

Remark 1.8. *If we replace $m = 1$ and the function E_α with E_1, then the generalized m-convex set reduces to the ϕ-convex set. The generalized m-convex sets are nonconvex.*

Definition 1.9. *The function f is said to be generalized m-convex for some fixed $m \in (0,1]$, if*

$$f(mu_1 + t\mathbf{E}_\alpha(u_2 - mu_1)) \le (1-t)f(mu_1) + tf(u_2), \qquad \forall\ u_1, u_2 \in K, \quad t \in [0,1]. \tag{1.6}$$

Remark 1.10. *Clearly, a generalized m-convex function reduces to a ϕ-convex function if we set $m = 1$ and replace E_α by E_1, which is a special case of the Mittag–Leffler function. The generalized m-convex functions are nonconvex.*

Definition 1.11. *[26, 27] Let $g : [u_1, u_2] \longrightarrow \mathbb{R}$ be an increasing and positive monotone function on $[u_1, u_2]$, having a continuous derivative on (u_1, u_2). The left–sided fractional integral of f with respect to g on $[u_1, u_2]$ of order $\alpha > 0$ is defined by:*

$$I_{u_1+}^{\alpha,g} f(x) = \frac{1}{\Gamma(\alpha)} \int_{u_1}^{x} \frac{g'(u) f(u)}{[g(x) - g(u)]^{1-\alpha}}\, du,\ \ x > u_1, \tag{1.7}$$

provided that the integral exists. The right–sided fractional integral of f with respect to g on $[u_1, u_2]$ of order $\alpha > 0$ is defined by:

$$I_{u_2-}^{\alpha,g} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{u_2} \frac{g'(u) f(u)}{[g(u) - g(x)]^{1-\alpha}}\, du,\ \ x < u_2, \tag{1.8}$$

provided that the integral exists.

Jleli and Samet in [22], proved the Hadamard type inequality for Riemann-Liouville fractional integral of a convex function f with respect to another function g.
Also in [47], Sarikaya and Ertuğral defined a function $\phi : [0, +\infty) \longrightarrow [0, +\infty)$ satisfying the following conditions:

$$\int_0^1 \frac{\phi(t)}{t}\, dt < +\infty, \tag{1.9}$$

$$\frac{1}{A} \le \frac{\phi(s)}{\phi(r)} \le A \text{ for } \frac{1}{2} \le \frac{s}{r} \le 2, \tag{1.10}$$

$$\frac{\phi(r)}{r^2} \le B \frac{\phi(s)}{s^2} \text{ for } s \le r, \tag{1.11}$$

$$\left| \frac{\phi(r)}{r^2} - \frac{\phi(s)}{s^2} \right| \le C|r - s| \frac{\phi(r)}{r^2} \text{ for } \frac{1}{2} \le \frac{s}{r} \le 2, \tag{1.12}$$

where $A, B, C > 0$ are independent of $r, s > 0$. If $\phi(r) r^\alpha$ is increasing for some $\alpha \ge 0$ and $\frac{\phi(r)}{r^\beta}$ is decreasing for some $\beta \ge 0$, then ϕ satisfies (1.9)-(1.12), see [48]. Therefore, the left–sided and right–sided generalized integral operators are defined as follows:

$$_{u_1^+}I_\phi f(x) = \int_{u_1}^{x} \frac{\phi(x-t)}{x-t} f(t)\, dt, \quad x > u_1, \tag{1.13}$$

$$_{u_2^-}I_\phi f(x) = \int_{x}^{u_2} \frac{\phi(t-x)}{t-x} f(t)\, dt, \quad x < u_2. \tag{1.14}$$

The most important feature of generalized integrals is that they produce Riemann–Liouville fractional integrals, k-Riemann–Liouville fractional integrals, Katugampola fractional integrals, conformable fractional integrals, Hadamard fractional integrals, etc., see [21, 24, 47].
Recently, Farid in [19] generalised the above integral by introducing an increasing and positive monotone function g on $[u_1, u_2]$, having continuous derivative on (u_1, u_2). The generalized fractional integral operator defined by Farid may be given as follows.

Definition 1.12. *The left and right–sided generalized fractional integral of a function f with respect to another function g may be given as follows, respectively:*

$$G_{u_1+}^{\phi,g}f(x)=\int_{u_1}^{x}\frac{\phi\left(g(x)-g(u)\right)}{g(x)-g(u)}g'(u)f(u)\,du,\ \ x>u_1, \tag{1.15}$$

$$G_{u_2-}^{\phi,g}f(x)=\int_{x}^{u_2}\frac{\phi\left(g(u)-g(x)\right)}{g(u)-g(x)}g'(u)f(u)\,du,\ \ x<u_2. \tag{1.16}$$

This operator generalizes the various fractional integrals of a function f with respect to another function g.

The following special cases are focussed in our study.

(i) If we take $\phi(u)=u$ then the operator (1.15) and (1.16) reduces to Riemann–Liouville integral of f with respect to function g.

$$I_{u_1+}^{g}f(x)=\int_{u_1}^{x}g'(u)f(u)\,du,\quad x>u_1, \tag{1.17}$$

$$I_{u_2-}^{g}f(x)=\int_{x}^{u_2}g'(u)f(u)\,du,\quad x<u_2. \tag{1.18}$$

If $g(u)=u$, then (1.17) and (1.18) will reduce to Riemann integral of f.

(ii) If we take $\phi(u)=\frac{u^{\alpha}}{\Gamma(\alpha)}$ then the operator (1.15) and (1.16) reduces to Riemann–Liouville fractional integral of f with respect to function g.

$$I_{u_1+}^{\phi,g}f(x)=\frac{1}{\Gamma(\alpha)}\int_{u_1}^{x}\left[g(x)-g(u)\right]^{\alpha-1}g'(u)f(u)\,du,\ \ x>u_1, \tag{1.19}$$

$$I_{u_2-}^{\phi,g}f(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{u_2}\left[g(u)-g(x)\right]^{\alpha-1}g'(u)f(u)\,du,\ \ x<u_2. \tag{1.20}$$

If $g(u)=u$, then (1.19) and (1.20) will reduce to left and right–sided Riemann–Liouville fractional integrals of f respectively.

(iii) If we take $\phi(u)=\frac{u^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ then the operator (1.15) and (1.16) reduces to k-Riemann–Liouville fractional integral of f with respect to function g.

$$I_{u_1+,k}^{\phi,g}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_{u_1}^{x}\left[g(x)-g(u)\right]^{\frac{\alpha}{k}-1}g'(u)f(u)\,du,\ \ x>u_1, \tag{1.21}$$

$$I_{u_2-,k}^{\phi,g}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_{x}^{u_2}\left[g(u)-g(x)\right]^{\alpha-1}g'(u)f(u)\,du,\ \ x<u_2. \tag{1.22}$$

If $g(u)=u$, then these operators in (1.21) and (1.22) reduces to k-fractional integral operators given in [37].

(iv) If we take $\phi_g(u)=u(g(u_2)-u)^{\alpha-1}$ for $\alpha\in(0,1)$, then the operator given in (1.15) and (1.16) reduces to conformable fractional integral operator of f with respect to a function g.

$$K_{u_1}^{\alpha,g}f(x)=\int_{u_1}^{x}\left[g(u)\right]^{\alpha-1}g'(u)f(u)\,du,\ \ x>u_1. \tag{1.23}$$

This operator (1.23) generalizes conformable fractional integral operator which was given by Khalil et al. in [28].

(v) If we take $\phi(u) = \frac{u}{\alpha}\exp(-Au)$, where $A = \frac{1-\alpha}{\alpha}$ and $\alpha \in (0,1)$, then the operator given in (1.15) and (1.16) reduces to fractional integral operator of f with respect to function g with exponential kernel.

$$J^{\alpha,g}_{u_1+} f(x) = \frac{1}{\alpha}\int_{u_1}^{x} \exp\left(-A(g(x)-g(u))\right) g'(u) f(u)\, d\, u, \ x > u_1, \tag{1.24}$$

$$J^{\alpha,g}_{u_2-} f(x) = \frac{1}{\alpha}\int_{x}^{u_2} \exp\left(-A(g(x)-g(u))\right) g'(u) f(u)\, d\, u, \ x < u_2. \tag{1.25}$$

Operators in (1.24) and (1.25) generalizes fractional integral operator with exponential kernel which was introduced by Kirane and Torebek in [29].

Motivated by the above literatures, the main objective of this paper is to discover in Section 2, an interesting identity with parameter λ in order to study some new bounds regarding trapezium, Ostrowski and Simpson type integral inequalities. By using the established identity as an auxiliary result, some new estimates for trapezium, Ostrowski and Simpson type integral inequalities for generalized m-convex functions via generalized integrals are obtained. It is pointed out that some new fractional integral inequalities have been deduced from main results. In Section 3, some applications to special means and new error estimates for the midpoint and trapezium quadrature formula are given. In Section 4, a briefly conlusion is given as well.

2. Main results

Throughout this study, let $P = [mu_1, u_2]$, where $u_1 < u_2$ for some fixed $m \in (0,1]$ and for all $t \in [0,1]$. For brevity, we define

$$\mathbf{\Upsilon}^{\phi,g}_{m}(x,t) = \int_0^t \frac{\phi\left(g\left(mu_1 + u\mathbf{E}_\alpha(x-mu_1)\right) - g(mu_1)\right)}{g\left(mu_1 + u\mathbf{E}_\alpha(x-mu_1)\right) - g(mu_1)} \times g'\left(mu_1 + u\mathbf{E}_\alpha(x-mu_1)\right)\, d\, u < +\infty \tag{2.1}$$

and

$$\mathbf{\Theta}^{\phi,g}_{m}(x,t) = \int_t^1 \frac{\phi\left(g\left(mx + \mathbf{E}_\alpha(u_2-mx)\right) - g\left(mx + u\mathbf{E}_\alpha(u_2-mx)\right)\right)}{g\left(mx + \mathbf{E}_\alpha(u_2-mx)\right) - g\left(mx + u\mathbf{E}_\alpha(u_2-mx)\right)} \times g'\left(mx + u\mathbf{E}_\alpha(u_2-mx)\right)\, d\, u < +\infty, \tag{2.2}$$

where g is an increasing and positive monotone function on P, having continuous derivative on $P^\circ = (mu_1, u_2)$.

For establishing some new results regarding general fractional integrals we need to prove the following lemma.

Lemma 2.1. *Let $f : P \longrightarrow \mathbb{R}$ be a differentiable mapping on P° and $\lambda \in \mathbb{R}$. If $f' \in L(P)$ and $\mathbf{E}_\alpha(u_2 - mu_1) > 0$, then the following identity for generalized fractional integrals hold:*

$$\frac{\mathbf{E}_\alpha(x-mu_1) f\left(mu_1 + t\mathbf{E}_\alpha(x-mu_1)\right) + \mathbf{E}_\alpha(u_2-mx) f(mx)}{\mathbf{E}_\alpha(u_2-mu_1)}$$

$$-\frac{\lambda}{\mathbf{E}_\alpha(u_2-mu_1)}\times\left[\frac{\mathbf{E}_\alpha(x-mu_1)f\left(mu_1+t\mathbf{E}_\alpha(x-mu_1)\right)}{\mathbf{\Upsilon}_m^{\phi,g}(x,1)}+\frac{\mathbf{E}_\alpha(u_2-mx)f(mx)}{\mathbf{\Theta}_m^{\phi,g}(x,0)}\right]$$

$$+\frac{\lambda}{\mathbf{E}_\alpha(u_2-mu_1)}\times\left[\frac{\mathbf{E}_\alpha(x-mu_1)f(mu_1)}{\mathbf{\Upsilon}_m^{\phi,g}(x,1)}+\frac{\mathbf{E}_\alpha(u_2-mx)f(mx+\mathbf{E}_\alpha(u_2-mx))}{\mathbf{\Theta}_m^{\phi,g}(x,0)}\right]$$

$$-\frac{1}{\mathbf{E}_\alpha(u_2-mu_1)}\times\left[\frac{G^{\phi,g}_{(mu_1+\mathbf{E}_\alpha(x-mu_1))^-}f(mu_1)}{\mathbf{\Upsilon}_m^{\phi,g}(x,1)}+\frac{G^{\phi,g}_{(mx)^+}f(mx+\mathbf{E}_\alpha(u_2-mx))}{\mathbf{\Theta}_m^{\phi,g}(x,0)}\right]$$

$$=\frac{\mathbf{E}^2_\alpha(x-mu_1)}{\mathbf{\Upsilon}_m^{\phi,g}(x,1)\mathbf{E}_\alpha(u_2-mu_1)}\times\int_0^1\left[\mathbf{\Upsilon}_m^{\phi,g}(x,t)-\lambda\right]f'\left(mu_1+t\mathbf{E}_\alpha(x-mu_1)\right)dt \tag{2.3}$$

$$-\frac{\mathbf{E}^2_\alpha(u_2-mx)}{\mathbf{\Theta}_m^{\phi,g}(x,0)\mathbf{E}_\alpha(u_2-mu_1)}\times\int_0^1\left[\mathbf{\Theta}_m^{\phi,g}(x,t)-\lambda\right]f'\left(mx+t\mathbf{E}_\alpha(u_2-mx)\right)dt.$$

We denote

$$T_{f,\mathbf{\Upsilon}_m^{\phi,g},\mathbf{\Theta}_m^{\phi,g}}(\lambda;x,u_1,u_2)=\frac{\mathbf{E}^2_\alpha(x-mu_1)}{\mathbf{\Upsilon}_m^{\phi,g}(x,1)\mathbf{E}_\alpha(u_2-mu_1)} \tag{2.4}$$

$$\times\int_0^1\left[\mathbf{\Upsilon}_m^{\phi,g}(x,t)-\lambda\right]f'\left(mu_1+t\mathbf{E}_\alpha(x-mu_1)\right)dt$$

$$-\frac{\mathbf{E}^2_\alpha(u_2-mx)}{\mathbf{\Theta}_m^{\phi,g}(x,0)\mathbf{E}_\alpha(u_2-mu_1)}\times\int_0^1\left[\mathbf{\Theta}_m^{\phi,g}(x,t)-\lambda\right]f'\left(mx+t\mathbf{E}_\alpha(u_2-mx)\right)dt.$$

Proof . Integrating by parts equation (2.4) and changing the variables of integration, we have

$$T_{f,\mathbf{\Upsilon}_m^{\phi,g},\mathbf{\Theta}_m^{\phi,g}}(\lambda;x,u_1,u_2)=\frac{\mathbf{E}^2_\alpha(x-mu_1)}{\mathbf{\Upsilon}_m^{\phi,g}(x,1)\mathbf{E}_\alpha(u_2-mu_1)}$$

$$\times\left\{\int_0^1\mathbf{\Upsilon}_m^{\phi,g}(x,t)f'\left(mu_1+t\mathbf{E}_\alpha(x-mu_1)\right)dt-\lambda\int_0^1 f'\left(mu_1+t\mathbf{E}_\alpha(x-mu_1)\right)dt\right\}$$

$$-\frac{\mathbf{E}^2_\alpha(u_2-mx)}{\mathbf{\Theta}_m^{\phi,g}(x,0)\mathbf{E}_\alpha(u_2-mu_1)}$$

$$\times\left\{\int_0^1\mathbf{\Theta}_m^{\phi,g}(x,t)f'\left(mx+t\mathbf{E}_\alpha(u_2-mx)\right)dt-\lambda\int_0^1 f'\left(mx+t\mathbf{E}_\alpha(u_2-mx)\right)dt\right\}$$

$$=\frac{\mathbf{E}^2_\alpha(x-mu_1)}{\mathbf{\Upsilon}_m^{\phi,g}(x,1)\mathbf{E}_\alpha(u_2-mu_1)}\times\left\{\frac{\mathbf{\Upsilon}_m^{\phi,g}(x,t)f\left(mu_1+t\mathbf{E}_\alpha(x-mu_1)\right)}{\mathbf{E}_\alpha(x-mu_1)}\right|_0^1-\frac{1}{\mathbf{E}_\alpha(x-mu_1)}$$

$$\times\int_0^1\frac{\phi\left(g\left(mu_1+t\mathbf{E}_\alpha(x-mu_1)\right)-g(mu_1)\right)}{g\left(mu_1+t\mathbf{E}_\alpha(x-mu_1)\right)-g(mu_1)}$$

$$\times g'\left(mu_1+t\mathbf{E}_\alpha(x-mu_1)\right)f\left(mu_1+t\mathbf{E}_\alpha(x-mu_1)\right)dt$$

$$\left.-\frac{\lambda}{\mathbf{E}_\alpha(x-mu_1)}f\left(mu_1+t\mathbf{E}_\alpha(x-mu_1)\right)\right|_0^1\Bigg\}-\frac{\mathbf{E}^2_\alpha(u_2-mx)}{\mathbf{\Theta}_m^{\phi,g}(x,0)\mathbf{E}_\alpha(u_2-mu_1)}$$

$$\times\left\{\frac{\mathbf{\Theta}_m^{\phi,g}(x,t)f\left(mx+t\mathbf{E}_\alpha(u_2-mx)\right)}{\mathbf{E}_\alpha(u_2-mx)}\right|_0^1$$

$$-\frac{1}{\mathbf{E}_\alpha(u_2-mx)}\times\int_0^1\frac{\phi\left(g\left(mx+\mathbf{E}_\alpha(u_2-mx)\right)-g\left(mx+t\mathbf{E}_\alpha(u_2-mx)\right)\right)}{g\left(mx+\mathbf{E}_\alpha(u_2-mx)\right)-g\left(mx+t\mathbf{E}_\alpha(u_2-mx)\right)}$$

$$\times g'\left(mx+t\mathbf{E}_\alpha(u_2-mx)\right)f\left(mx+t\mathbf{E}_\alpha(u_2-mx)\right)\,dt$$

$$\left.-\frac{\lambda}{\mathbf{E}_\alpha(u_2-mx)}f\left(mx+t\mathbf{E}_\alpha(u_2-mx)\right)\right|_0^1\Bigg\}$$

$$=\frac{\mathbf{E}_\alpha(x-mu_1)f\left(mu_1+t\mathbf{E}_\alpha(x-mu_1)\right)+\mathbf{E}_\alpha(u_2-mx)f(mx)}{\mathbf{E}_\alpha(u_2-mu_1)}$$

$$-\frac{\lambda}{\mathbf{E}_\alpha(u_2-mu_1)}\times\left[\frac{\mathbf{E}_\alpha(x-mu_1)f\left(mu_1+t\mathbf{E}_\alpha(x-mu_1)\right)}{\mathbf{\Upsilon}_m^{\phi,g}(x,1)}+\frac{\mathbf{E}_\alpha(u_2-mx)f(mx)}{\mathbf{\Theta}_m^{\phi,g}(x,0)}\right]$$

$$+\frac{\lambda}{\mathbf{E}_\alpha(u_2-mu_1)}\times\left[\frac{\mathbf{E}_\alpha(x-mu_1)f(mu_1)}{\mathbf{\Upsilon}_m^{\phi,g}(x,1)}+\frac{\mathbf{E}_\alpha(u_2-mx)f(mx+\mathbf{E}_\alpha(u_2-mx))}{\mathbf{\Theta}_m^{\phi,g}(x,0)}\right]$$

$$-\frac{1}{\mathbf{E}_\alpha(u_2-mu_1)}\times\left[\frac{G^{\phi,g}_{(mu_1+\mathbf{E}_\alpha(x-mu_1))^-}f(mu_1)}{\mathbf{\Upsilon}_m^{\phi,g}(x,1)}+\frac{G^{\phi,g}_{(mx)^+}f(mx+\mathbf{E}_\alpha(u_2-mx))}{\mathbf{\Theta}_m^{\phi,g}(x,0)}\right].$$

The proof of Lemma 2.1 is completed. □

Remark 2.2. **a** *Taking* $m=1$, $\lambda=0$, $\mathbf{E}_\alpha(x-mu_1)=x-mu_1$, $\mathbf{E}_\alpha(u_2-mx)=u_2-mx$, $\mathbf{E}_\alpha(u_2-mu_1)=u_2-mu_1$ *and* $g(t)=\phi(t)=t$ *in Lemma 2.1, we get the following Ostrowski type identity:*

$$T_f(x,u_1,u_2)=f(x)-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}f(t)\,dt.$$

b *Taking* $m=1$, $\lambda=1$, $\mathbf{E}_\alpha(x-mu_1)=x-mu_1$, $\mathbf{E}_\alpha(u_2-mx)=u_2-mx$, $\mathbf{E}_\alpha(u_2-mu_1)=u_2-mu_1$ *and* $g(t)=\phi(t)=t$ *in Lemma 2.1, we get the following Hermite–Hadamard type identity:*

$$\overline{T_f}(x,u_1,u_2)=\frac{(x-u_1)f(u_1)+(u_2-x)f(u_2)}{u_2-u_1}-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}f(t)\,dt.$$

c *Taking* $m=1$, $x=\frac{u_1+u_2}{2}$, $\mathbf{E}_\alpha(x-mu_1)=x-mu_1$, $\mathbf{E}_\alpha(u_2-mx)=u_2-mx$, $\mathbf{E}_\alpha(u_2-mu_1)=u_2-mu_1$ *and* $g(t)=\phi(t)=t$ *in Lemma 2.1, we get the following Simpson type identity:*

$$T_f(\lambda;u_1,u_2)=\lambda\left[\frac{f(u_1)+f(u_2)}{2}\right]+(1-\lambda)f\left(\frac{u_1+u_2}{2}\right)-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}f(t)\,dt.$$

Theorem 2.3. *Let* $f:P\longrightarrow\mathbb{R}$ *be a differentiable mapping on* P° *and* $\lambda\in[0,1]$. *If* $|f'|^q$ *is generalized* m*-convex on* P *and* $\mathbf{E}_\alpha(u_2-mu_1)>0$, *then for* $q>1$ *and* $p^{-1}+q^{-1}=1$, *the following inequality for generalized fractional integrals hold:*

$$\left|T_{f,\mathbf{\Upsilon}_m^{\phi,g},\mathbf{\Theta}_m^{\phi,g}}(\lambda;x,u_1,u_2)\right|$$

$$\leq \frac{\mathbf{E}_\alpha^2(x-mu_1)}{\sqrt[q]{2}\boldsymbol{\Upsilon}_m^{\phi,g}(x,1)\mathbf{E}_\alpha(u_2-mu_1)}\sqrt[p]{B_{\boldsymbol{\Upsilon}_m}^{\phi,g}(x;\lambda,p)}\times\sqrt[q]{|f'(mu_1)|^q+|f'(x)|^q} \tag{2.5}$$
$$+\frac{\mathbf{E}_\alpha^2(u_2-mx)}{\sqrt[q]{2}\boldsymbol{\Theta}_m^{\phi,g}(x,0)\mathbf{E}_\alpha(u_2-mu_1)}\sqrt[p]{B_{\boldsymbol{\Theta}_m}^{\phi,g}(x;\lambda,p)}\times\sqrt[q]{|f'(mx)|^q+|f'(u_2)|^q},$$

where

$$B_{\boldsymbol{\Upsilon}_m}^{\phi,g}(x;\lambda,p)=\int_0^1\left|\boldsymbol{\Upsilon}_m^{\phi,g}(x,t)-\lambda\right|^p dt,\quad B_{\boldsymbol{\Theta}_m}^{\phi,g}(x;\lambda,p)=\int_0^1\left|\boldsymbol{\Theta}_m^{\phi,g}(x,t)-\lambda\right|^p dt. \tag{2.6}$$

Proof . From Lemma 2.1, generalized m-convexity of $|f'|^q$, Hölder's inequality and properties of the modulus, we have

$$\left|T_{f,\boldsymbol{\Upsilon}_m^{\phi,g},\boldsymbol{\Theta}_m^{\phi,g}}(\lambda;x,u_1,u_2)\right|$$
$$\leq \frac{\mathbf{E}_\alpha^2(x-mu_1)}{\boldsymbol{\Upsilon}_m^{\phi,g}(x,1)\mathbf{E}_\alpha(u_2-mu_1)}\times\int_0^1\left|\boldsymbol{\Upsilon}_m^{\phi,g}(x,t)-\lambda\right|\left|f'\left(mu_1+t\mathbf{E}_\alpha(x-mu_1)\right)\right|dt$$
$$+\frac{\mathbf{E}_\alpha^2(u_2-mx)}{\boldsymbol{\Theta}_m^{\phi,g}(x,0)\mathbf{E}_\alpha(u_2-mu_1)}\times\int_0^1\left|\boldsymbol{\Theta}_m^{\phi,g}(x,t)-\lambda\right|\left|f'\left(mx+t\mathbf{E}_\alpha(u_2-mx)\right)\right|dt$$
$$\leq \frac{\mathbf{E}_\alpha^2(x-mu_1)}{\boldsymbol{\Upsilon}_m^{\phi,g}(x,1)\mathbf{E}_\alpha(u_2-mu_1)}$$
$$\times\left(\int_0^1\left|\boldsymbol{\Upsilon}_m^{\phi,g}(x,t)-\lambda\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(mu_1+t\mathbf{E}_\alpha(x-mu_1)\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{\mathbf{E}_\alpha^2(u_2-mx)}{\boldsymbol{\Theta}_m^{\phi,g}(x,0)\mathbf{E}_\alpha(u_2-mu_1)}$$
$$\times\left(\int_0^1\left|\boldsymbol{\Theta}_m^{\phi,g}(x,t)-\lambda\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(mx+t\mathbf{E}_\alpha(u_2-mx)\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{\mathbf{E}_\alpha^2(x-mu_1)}{\boldsymbol{\Upsilon}_m^{\phi,g}(x,1)\mathbf{E}_\alpha(u_2-mu_1)}\sqrt[p]{B_{\boldsymbol{\Upsilon}_m}^{\phi,g}(x;\lambda,p)}\left(\int_0^1\left[(1-t)|f'(mu_1)|^q+t|f'(x)|^q\right]dt\right)^{\frac{1}{q}}$$
$$+\frac{\mathbf{E}_\alpha^2(u_2-mx)}{\boldsymbol{\Theta}_m^{\phi,g}(x,0)\mathbf{E}_\alpha(u_2-mu_1)}\sqrt[p]{B_{\boldsymbol{\Theta}_m}^{\phi,g}(x;\lambda,p)}\left(\int_0^1\left[(1-t)|f'(mx)|^q+t|f'(u_2)|^q\right]dt\right)^{\frac{1}{q}}$$
$$=\frac{\mathbf{E}_\alpha^2(x-mu_1)}{\sqrt[q]{2}\boldsymbol{\Upsilon}_m^{\phi,g}(x,1)\mathbf{E}_\alpha(u_2-mu_1)}\sqrt[p]{B_{\boldsymbol{\Upsilon}_m}^{\phi,g}(x;\lambda,p)}\times\sqrt[q]{|f'(mu_1)|^q+|f'(x)|^q}$$
$$+\frac{\mathbf{E}_\alpha^2(u_2-mx)}{\sqrt[q]{2}\boldsymbol{\Theta}_m^{\phi,g}(x,0)\mathbf{E}_\alpha(u_2-mu_1)}\sqrt[p]{B_{\boldsymbol{\Theta}_m}^{\phi,g}(x;\lambda,p)}\times\sqrt[q]{|f'(mx)|^q+|f'(u_2)|^q}.$$

The proof of Theorem 2.3 is completed. □
We point out some special cases of Theorem 2.3.

Corollary 2.4. *Taking $p=q=2$ in Theorem 2.3, we get*

$$\begin{aligned}&\left|T_{f,\mathbf{\Upsilon}_m^{\phi,g},\mathbf{\Theta}_m^{\phi,g}}(\lambda;x,u_1,u_2)\right|\\ &\leq \frac{\mathbf{E}_\alpha^2(x-mu_1)}{\sqrt{2}\mathbf{\Upsilon}_m^{\phi,g}(x,1)\mathbf{E}_\alpha(u_2-mu_1)}\sqrt{B_{\mathbf{\Upsilon}_m}^{\phi,g}(x;\lambda,2)}\times\sqrt{|f'(mu_1)|^2+|f'(x)|^2}\\ &+\frac{\mathbf{E}_\alpha^2(u_2-mx)}{\sqrt{2}\mathbf{\Theta}_m^{\phi,g}(x,0)\mathbf{E}_\alpha(u_2-mu_1)}\sqrt{B_{\mathbf{\Theta}_m}^{\phi,g}(x;\lambda,2)}\times\sqrt{|f'(mx)|^2+|f'(u_2)|^2}.\end{aligned} \tag{2.7}$$

Corollary 2.5. *Taking $|f'|\leq K$ in Theorem 2.3, we have*

$$\left|T_{f,\mathbf{\Upsilon}_m^{\phi,g},\mathbf{\Theta}_m^{\phi,g}}(\lambda;x,u_1,u_2)\right|\leq\frac{K}{\mathbf{E}_\alpha(u_2-mu_1)} \tag{2.8}$$

$$\times\left[\frac{\mathbf{E}_\alpha^2(x-mu_1)}{\mathbf{\Upsilon}_m^{\phi,g}(x,1)}\sqrt[p]{B_{\mathbf{\Upsilon}_m}^{\phi,g}(x;\lambda,p)}+\frac{\mathbf{E}_\alpha^2(u_2-mx)}{\mathbf{\Theta}_m^{\phi,g}(x,0)}\sqrt[p]{B_{\mathbf{\Theta}_m}^{\phi,g}(x;\lambda,p)}\right].$$

Corollary 2.6. *Taking $m=1$, $\lambda=0$, $\mathbf{E}_\alpha(x-mu_1)=x-mu_1$, $\mathbf{E}_\alpha(u_2-mx)=u_2-mx$, $\mathbf{E}_\alpha(u_2-mu_1)=u_2-mu_1$ and $g(t)=\phi(t)=t$ in Theorem 2.3, we get the following Ostrowski type inequality:*

$$\left|T_f(x,u_1,u_2)\right|\leq\frac{1}{\sqrt[q]{2}\sqrt[p]{p+1}(u_2-u_1)} \tag{2.9}$$

$$\times\left\{(x-u_1)^2\sqrt[q]{|f'(u_1)|^q+|f'(x)|^q}+(u_2-x)^2\sqrt[q]{|f'(x)|^q+|f'(u_2)|^q}\right\}.$$

Corollary 2.7. *Taking $x=\frac{u_1+u_2}{2}$ in Corollary 2.6, we get the following midpoint type inequality:*

$$\left|T_f(u_1,u_2)\right|\leq\frac{(u_2-u_1)}{4\sqrt[q]{2}\sqrt[p]{p+1}} \tag{2.10}$$

$$\times\left\{\sqrt[q]{|f'(u_1)|^q+\left|f'\left(\frac{u_1+u_2}{2}\right)\right|^q}+\sqrt[q]{\left|f'\left(\frac{u_1+u_2}{2}\right)\right|^q+|f'(u_2)|^q}\right\}.$$

Corollary 2.8. *Taking $m=1$, $\lambda=1$, $\mathbf{E}_\alpha(x-mu_1)=x-mu_1$, $\mathbf{E}_\alpha(u_2-mx)=u_2-mx$, $\mathbf{E}_\alpha(u_2-mu_1)=u_2-mu_1$ and $g(t)=\phi(t)=t$ in Theorem 2.3, we get the following trapezium type inequality:*

$$\left|\overline{T_f}(x,u_1,u_2)\right|\leq\frac{1}{\sqrt[q]{2}\sqrt[p]{p+1}(u_2-u_1)} \tag{2.11}$$

$$\times\left\{(x-u_1)^2\sqrt[q]{|f'(u_1)|^q+|f'(x)|^q}+(u_2-x)^2\sqrt[q]{|f'(x)|^q+|f'(u_2)|^q}\right\}.$$

Corollary 2.9. *Taking $m=1$, $\lambda=\frac{1}{3}$, $\mathbf{E}_\alpha(x-mu_1)=x-mu_1$, $\mathbf{E}_\alpha(u_2-mx)=u_2-mx$, $\mathbf{E}_\alpha(u_2-mu_1)=u_2-mu_1$ and $g(t)=\phi(t)=t$ in Theorem 2.3, we get the following Simpson type inequality:*

$$\left|T_f\left(\frac{1}{3};u_1,u_2\right)\right|\leq\frac{1}{\sqrt[q]{2}(u_2-u_1)}\sqrt[p]{\frac{2^{p+1}+1}{3^{p+1}(p+1)}} \tag{2.12}$$

$$\times\left\{(x-u_1)^2\sqrt[q]{|f'(u_1)|^q+|f'(x)|^q}+(u_2-x)^2\sqrt[q]{|f'(x)|^q+|f'(u_2)|^q}\right\}.$$

Corollary 2.10. *Taking $\lambda = 0$ and $\phi(t) = t$ in Theorem 2.3, we get*

$$\left|T_{f,\boldsymbol{\Upsilon}^g_m,\boldsymbol{\Theta}^g_m}(0;x,u_1,u_2)\right| \leq \frac{1}{\sqrt[q]{2}\mathbf{E}_\alpha(u_2-mu_1)} \tag{2.13}$$

$$\times\Bigg\{\sqrt[q]{\mathbf{E}_\alpha(x-mu_1)}\sqrt[p]{B^g_1(x;p)}\times\sqrt[q]{|f'(mu_1)|^q+|f'(x)|^q}$$

$$+\sqrt[q]{\mathbf{E}_\alpha(u_2-mx)}\sqrt[p]{B^g_2(x;p)}\times\sqrt[q]{|f'(mx)|^q+|f'(u_2)|^q}\Bigg\},$$

where

$$B^g_1(x;p) = \int_{mu_1}^{mu_1+\mathbf{E}_\alpha(x-mu_1)}\big[g(t)-g(mu_1)\big]^p\,dt \tag{2.14}$$

and

$$B^g_2(x;p) = \int_{mx}^{mx+\mathbf{E}_\alpha(u_2-mx)}\big[g(mx+\mathbf{E}_\alpha(u_2-mx))-g(t)\big]^p\,dt. \tag{2.15}$$

Corollary 2.11. *Taking $\lambda = 0$ and $\phi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 2.3, we have*

$$\left|T_{f,\boldsymbol{\Upsilon}^g_m,\boldsymbol{\Theta}^g_m}(0;x,u_1,u_2)\right| \leq \frac{1}{\sqrt[q]{2}\mathbf{E}_\alpha(u_2-mu_1)} \tag{2.16}$$

$$\times\Bigg\{\sqrt[q]{\mathbf{E}_\alpha(x-mu_1)}\sqrt[p]{B^g_3(x;p,\alpha)}\times\sqrt[q]{|f'(mu_1)|^q+|f'(x)|^q}$$

$$+\sqrt[q]{\mathbf{E}_\alpha(u_2-mx)}\sqrt[p]{B^g_4(x;p,\alpha)}\times\sqrt[q]{|f'(mx)|^q+|f'(u_2)|^q}\Bigg\},$$

where

$$B^g_3(x;p,\alpha) = \int_{mu_1}^{mu_1+\mathbf{E}_\alpha(x-mu_1)}\big[g(t)-g(mu_1)\big]^{p\alpha}\,dt \tag{2.17}$$

and

$$B^g_4(x;p,\alpha) = \int_{mx}^{mx+\mathbf{E}_\alpha(u_2-mx)}\big[g(mx+\mathbf{E}_\alpha(u_2-mx))-g(t)\big]^{p\alpha}\,dt. \tag{2.18}$$

Corollary 2.12. *Taking $\lambda = 0$ and $\phi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.3, we obtain*

$$\left|T_{f,\boldsymbol{\Upsilon}^g_m,\boldsymbol{\Theta}^g_m}(0;x,u_1,u_2)\right| \leq \frac{1}{\sqrt[q]{2}\mathbf{E}_\alpha(u_2-mu_1)} \tag{2.19}$$

$$\times\Bigg\{\sqrt[q]{\mathbf{E}_\alpha(x-mu_1)}\sqrt[p]{B^g_5(x;p,\alpha,k)}\times\sqrt[q]{|f'(mu_1)|^q+|f'(x)|^q}$$

$$+\sqrt[q]{\mathbf{E}_\alpha(u_2-mx)}\sqrt[p]{B^g_6(x;p,\alpha,k)}\times\sqrt[q]{|f'(mx)|^q+|f'(u_2)|^q}\Bigg\},$$

where

$$B_5^g(x;p,\alpha,k) = \int_{mu_1}^{mu_1+\mathbf{E}_\alpha(x-mu_1)} \big[g(t) - g(mu_1)\big]^{\frac{p\alpha}{k}} \, dt \tag{2.20}$$

and

$$B_6^g(x;p,\alpha,k) = \int_{mx}^{mx+\mathbf{E}_\alpha(u_2-mx)} \big[g(mx + \mathbf{E}_\alpha(u_2 - mx)) - g(t)\big]^{\frac{p\alpha}{k}} \, dt. \tag{2.21}$$

Corollary 2.13. *Taking* $\lambda = 0$, $\forall\, u \in [0,t]$, $\phi_g(x,t) = t(g(mu_1 + \mathbf{E}_\alpha(x - mu_1)) - t)^{\alpha-1}$ *and* $\forall\, u \in [t,1]$, $\phi_g(x,t) = t(g(mx + \mathbf{E}_\alpha(u_2 - mx)) - t)^{\alpha-1}$ *in Theorem 2.3, we get*

$$\begin{aligned}\big|T_{f,\Upsilon_m^g,\Theta_m^g}(0;x,u_1,u_2)\big| \le{}& \frac{\mathbf{E}_\alpha^{\frac{q+1}{q}}(x - mu_1)}{\sqrt[q]{2}\big[g(mu_1 + \mathbf{E}_\alpha(x - mu_1)) - g(mu_1)\big]\mathbf{E}_\alpha(u_2 - mu_1)} \\ &\times \sqrt[p]{B_7^g(x;p)} \times \sqrt[q]{|f'(mu_1)|^q + |f'(x)|^q} \\ &+ \frac{\mathbf{E}_\alpha^{\frac{q+1}{q}}(u_2 - mx)}{\sqrt[q]{2}\big[g^\alpha(mx + \mathbf{E}_\alpha(u_2 - mx)) - g^\alpha(mx)\big]\mathbf{E}_\alpha(u_2 - mu_1)} \\ &\times \sqrt[p]{B_8^g(x;p,\alpha)} \times \sqrt[q]{|f'(mx)|^q + |f'(u_2)|^q},\end{aligned} \tag{2.22}$$

where

$$B_7^g(x;p) = \int_{mu_1}^{mu_1+\mathbf{E}_\alpha(x-mu_1)} \big[g(t) - g(mu_1)\big]^p \, dt \tag{2.23}$$

and

$$B_8^g(x;p,\alpha) = \int_{mx}^{mx+\mathbf{E}_\alpha(u_2-mx)} \Big[g^\alpha(mx + \mathbf{E}_\alpha(u_2 - mx)) - g^\alpha(t)\Big]^p \, dt. \tag{2.24}$$

Corollary 2.14. *Taking* $\lambda = 0$ *and* $\phi(t) = \frac{t}{\alpha}\exp(-At)$, *where* $A = \frac{1-\alpha}{\alpha}$, *in Theorem 2.3, we have*

$$\begin{aligned}\big|T_{f,\Upsilon_m^g,\Theta_m^g}(0;x,u_1,u_2)\big| \le{}& \frac{\mathbf{E}_\alpha^{\frac{q+1}{q}}(x - mu_1)}{\sqrt[q]{2}\mathbf{E}_\alpha(u_2 - mu_1)} \times \sqrt[p]{B_9^g(x;p,A)} \times \sqrt[q]{|f'(mu_1)|^q + |f'(x)|^q} \\ &+ \frac{\mathbf{E}_\alpha^{\frac{q+1}{q}}(u_2 - mx)}{\sqrt[q]{2}\mathbf{E}_\alpha(u_2 - mu_1)} \sqrt[p]{B_{10}^g(x;p,A)} \times \sqrt[q]{|f'(mx)|^q + |f'(u_2)|^q},\end{aligned} \tag{2.25}$$

where

$$B_9^g(x;p,A) = \int_{mu_1}^{mu_1+\mathbf{E}_\alpha(x-mu_1)} \Big\{1 - \exp\big[A\,(g(mu_1) - g(t))\,\big]\Big\}^p \, dt \tag{2.26}$$

and

$$B_{10}^g(x;p,A) = \int_{mx}^{mx+\mathbf{E}_\alpha(u_2-mx)} \Big\{1 - \exp\big[A\,(g(t) - g(mx + \mathbf{E}_\alpha(u_2 - mx)))\,\big]\Big\}^p \, dt. \tag{2.27}$$

Theorem 2.15. *Let* $f : P \longrightarrow \mathbb{R}$ *be a differentiable mapping on* P° *and* $\lambda \in [0,1]$. *If* $|f'|^q$ *is generalized* m*-convex on* P *and* $\mathbf{E}_\alpha(u_2 - mu_1) > 0$, *then for* $q \ge 1$, *the following inequality for generalized fractional integrals hold:*

$$\big|T_{f,\Upsilon_m^{\phi,g},\Theta_m^{\phi,g}}(\lambda;x,u_1,u_2)\big| \le \frac{\mathbf{E}_\alpha^2(x - mu_1)}{\Upsilon_m^{\phi,g}(x,1)\mathbf{E}_\alpha(u_2 - mu_1)}\Big[B_{\Upsilon_m}^{\phi,g}(x;\lambda,1)\Big]^{1-\frac{1}{q}} \tag{2.28}$$

$$\times\sqrt[q]{\left[B^{\phi,g}_{\mathbf{\Upsilon}_m}(x;\lambda,1)-E^{\phi,g}_{\mathbf{\Upsilon}_m}(x;\lambda)\right]|f'(mu_1)|^q+E^{\phi,g}_{\mathbf{\Upsilon}_m}(x;\lambda)|f'(x)|^q}$$

$$+\frac{\mathbf{E}^2_\alpha(u_2-mx)}{\mathbf{\Theta}^{\phi,g}_m(x,0)\mathbf{E}_\alpha(u_2-mu_1)}\left[B^{\phi,g}_{\mathbf{\Theta}_m}(x;\lambda,1)\right]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{\left[B^{\phi,g}_{\mathbf{\Theta}_m}(x;\lambda,1)-G^{\phi,g}_{\mathbf{\Theta}_m}(x;\lambda)\right]|f'(mx)|^q+G^{\phi,g}_{\mathbf{\Theta}_m}(x;\lambda)|f'(u_2)|^q},$$

where

$$E^{\phi,g}_{\mathbf{\Upsilon}_m}(x;\lambda)=\int_0^1 t\left|\mathbf{\Upsilon}^{\phi,g}_m(x,t)-\lambda\right|dt,\qquad G^{\phi,g}_{\mathbf{\Theta}_m}(x;\lambda)=\int_0^1 t\left|\mathbf{\Theta}^{\phi,g}_m(x,t)-\lambda\right|dt,\tag{2.29}$$

and $B^{\phi,g}_{\mathbf{\Upsilon}_m}(x;\lambda,1)$, $B^{\phi,g}_{\mathbf{\Theta}_m}(x;\lambda,1)$ *are defined as in Theorem 2.3.*

Proof . From Lemma 2.1, generalized m-convexity of $|f'|^q$, the well–known power mean inequality and properties of the modulus, we have

$$\left|T_{f,\mathbf{\Upsilon}^{\phi,g}_m,\mathbf{\Theta}^{\phi,g}_m}(\lambda;x,u_1,u_2)\right|$$

$$\leq\frac{\mathbf{E}^2_\alpha(x-mu_1)}{\mathbf{\Upsilon}^{\phi,g}_m(x,1)\mathbf{E}_\alpha(u_2-mu_1)}\times\int_0^1\left|\mathbf{\Upsilon}^{\phi,g}_m(x,t)-\lambda\right|\left|f'\left(mu_1+t\mathbf{E}_\alpha(x-mu_1)\right)\right|dt$$

$$+\frac{\mathbf{E}^2_\alpha(u_2-mx)}{\mathbf{\Theta}^{\phi,g}_m(x,0)\mathbf{E}_\alpha(u_2-mu_1)}\times\int_0^1\left|\mathbf{\Theta}^{\phi,g}_m(x,t)-\lambda\right|\left|f'\left(mx+t\mathbf{E}_\alpha(u_2-mx)\right)\right|dt$$

$$\leq\frac{\mathbf{E}^2_\alpha(x-mu_1)}{\mathbf{\Upsilon}^{\phi,g}_m(x,1)\mathbf{E}_\alpha(u_2-mu_1)}$$

$$\times\left(\int_0^1\left|\mathbf{\Upsilon}^{\phi,g}_m(x,t)-\lambda\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\mathbf{\Upsilon}^{\phi,g}_m(x,t)-\lambda\right|\left|f'\left(mu_1+t\mathbf{E}_\alpha(x-mu_1)\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{\mathbf{E}^2_\alpha(u_2-mx)}{\mathbf{\Theta}^{\phi,g}_m(x,0)\mathbf{E}_\alpha(u_2-mu_1)}$$

$$\times\left(\int_0^1\left|\mathbf{\Theta}^{\phi,g}_m(x,t)-\lambda\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\mathbf{\Theta}^{\phi,g}_m(x,t)-\lambda\right|\left|f'\left(mx+t\mathbf{E}_\alpha(u_2-mx)\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq\frac{\mathbf{E}^2_\alpha(x-mu_1)}{\mathbf{\Upsilon}^{\phi,g}_m(x,1)\mathbf{E}_\alpha(u_2-mu_1)}\sqrt[p]{B^{\phi,g}_{\mathbf{\Upsilon}_m}(x;\lambda,p)}$$

$$\times\left(\int_0^1\left|\mathbf{\Upsilon}^{\phi,g}_m(x,t)-\lambda\right|\left[(1-t)|f'(mu_1)|^q+t|f'(x)|^q\right]dt\right)^{\frac{1}{q}}$$

$$+\frac{\mathbf{E}^2_\alpha(u_2-mx)}{\mathbf{\Theta}^{\phi,g}_m(x,0)\mathbf{E}_\alpha(u_2-mu_1)}\sqrt[p]{B^{\phi,g}_{\mathbf{\Theta}_m}(x;\lambda,p)}$$

$$\times\left(\int_0^1\left|\mathbf{\Theta}^{\phi,g}_m(x,t)-\lambda\right|\left[(1-t)|f'(mx)|^q+t|f'(u_2)|^q\right]dt\right)^{\frac{1}{q}}$$

$$=\frac{\mathbf{E}^2_\alpha(x-mu_1)}{\mathbf{\Upsilon}^{\phi,g}_m(x,1)\mathbf{E}_\alpha(u_2-mu_1)}\left[B^{\phi,g}_{\mathbf{\Upsilon}_m}(x;\lambda,1)\right]^{1-\frac{1}{q}}$$

$$\times \sqrt[q]{\left[B^{\phi,g}_{\mathbf{\Upsilon}_m}(x;\lambda,1) - E^{\phi,g}_{\mathbf{\Upsilon}_m}(x;\lambda)\right]|f'(mu_1)|^q + E^{\phi,g}_{\mathbf{\Upsilon}_m}(x;\lambda)|f'(x)|^q}$$
$$+\frac{\mathbf{E}^2_\alpha(u_2-mx)}{\mathbf{\Theta}^{\phi,g}_m(x,0)\mathbf{E}_\alpha(u_2-mu_1)}\left[B^{\phi,g}_{\mathbf{\Theta}_m}(x;\lambda,1)\right]^{1-\frac{1}{q}}$$
$$\times \sqrt[q]{\left[B^{\phi,g}_{\mathbf{\Theta}_m}(x;\lambda,1) - G^{\phi,g}_{\mathbf{\Theta}_m}(x;\lambda)\right]|f'(mx)|^q + G^{\phi,g}_{\mathbf{\Theta}_m}(x;\lambda)|f'(u_2)|^q}.$$

The proof of Theorem 2.15 is completed. □
We point out some special cases of Theorem 2.15.

Corollary 2.16. *Taking $q = 1$ in Theorem 2.15, we get*

$$\left|T_{f,\mathbf{\Upsilon}^{\phi,g}_m,\mathbf{\Theta}^{\phi,g}_m}(\lambda;x,u_1,u_2)\right| \le \frac{\mathbf{E}^2_\alpha(x-mu_1)}{\mathbf{\Upsilon}^{\phi,g}_m(x,1)\mathbf{E}_\alpha(u_2-mu_1)} \tag{2.30}$$
$$\times\left[\left(B^{\phi,g}_{\mathbf{\Upsilon}_m}(x;\lambda,1) - E^{\phi,g}_{\mathbf{\Upsilon}_m}(x;\lambda)\right)|f'(mu_1)| + E^{\phi,g}_{\mathbf{\Upsilon}_m}(x;\lambda)|f'(x)|\right]$$
$$+\frac{\mathbf{E}^2_\alpha(u_2-mx)}{\mathbf{\Theta}^{\phi,g}_m(x,0)\mathbf{E}_\alpha(u_2-mu_1)}\times\left[\left(B^{\phi,g}_{\mathbf{\Theta}_m}(x;\lambda,1) - G^{\phi,g}_{\mathbf{\Theta}_m}(x;\lambda)\right)|f'(mx)| + G^{\phi,g}_{\mathbf{\Theta}_m}(x;\lambda)|f'(u_2)|\right].$$

Corollary 2.17. *Taking $|f'| \le K$ in Theorem 2.15, we have*

$$\left|T_{f,\mathbf{\Upsilon}^{\phi,g}_m,\mathbf{\Theta}^{\phi,g}_m}(\lambda;x,u_1,u_2)\right| \le \frac{K}{\mathbf{E}_\alpha(u_2-mu_1)} \tag{2.31}$$
$$\times\left[\frac{\mathbf{E}^2_\alpha(x-mu_1)}{\mathbf{\Upsilon}^{\phi,g}_m(x,1)}B^{\phi,g}_{\mathbf{\Upsilon}_m}(x;\lambda,1) + \frac{\mathbf{E}^2_\alpha(u_2-mx)}{\mathbf{\Theta}^{\phi,g}_m(x,0)}B^{\phi,g}_{\mathbf{\Theta}_m}(x;\lambda,1)\right].$$

Corollary 2.18. *Taking $m = 1$, $\lambda = 0$, $\mathbf{E}_\alpha(x-mu_1) = x-mu_1$, $\mathbf{E}_\alpha(u_2-mx) = u_2-mx$, $\mathbf{E}_\alpha(u_2-mu_1) = u_2-mu_1$ and $g(t) = \phi(t) = t$ in Theorem 2.15, we get the following Ostrowski type inequality:*

$$\left|T_f(x,u_1,u_2)\right| \le \frac{1}{2\sqrt[q]{3}(u_2-u_1)} \tag{2.32}$$
$$\times\left\{(x-u_1)^2\sqrt[q]{|f'(u_1)|^q + 2|f'(x)|^q} + (u_2-x)^2\sqrt[q]{2|f'(x)|^q + |f'(u_2)|^q}\right\}.$$

Corollary 2.19. *Taking $x = \frac{u_1+u_2}{2}$ in Corollary 2.18 we get the following midpoint type inequality:*

$$\left|T_f(u_1,u_2)\right| \le \frac{(u_2-u_1)}{8\sqrt[q]{3}} \tag{2.33}$$
$$\times\left\{\sqrt[q]{|f'(u_1)|^q + 2\left|f'\left(\frac{u_1+u_2}{2}\right)\right|^q} + \sqrt[q]{2\left|f'\left(\frac{u_1+u_2}{2}\right)\right|^q + |f'(u_2)|^q}\right\}.$$

Corollary 2.20. *Taking $m = 1$, $\lambda = 1$, $\mathbf{E}_\alpha(x-mu_1) = x-mu_1$, $\mathbf{E}_\alpha(u_2-mx) = u_2-mx$, $\mathbf{E}_\alpha(u_2-mu_1) = u_2-mu_1$ and $g(t) = \phi(t) = t$ in Theorem 2.15, we get the following trapezium type inequality:*

$$\left|\overline{T_f}(x,u_1,u_2)\right| \le \frac{1}{2\sqrt[q]{3}(u_2-u_1)} \tag{2.34}$$
$$\times\left\{(x-u_1)^2\sqrt[q]{2|f'(u_1)|^q + |f'(x)|^q} + (u_2-x)^2\sqrt[q]{|f'(x)|^q + 2|f'(u_2)|^q}\right\}.$$

Corollary 2.21. *Taking $m = 1$, $\lambda = \frac{1}{3}$, $\mathbf{E}_\alpha(x - mu_1) = x - mu_1$, $\mathbf{E}_\alpha(u_2 - mx) = u_2 - mx$, $\mathbf{E}_\alpha(u_2 - mu_1) = u_2 - mu_1$ and $g(t) = \phi(t) = t$ in Theorem 2.15, we get the following Simpson type inequality:*

$$\left| T_f\left(\frac{1}{3}; u_1, u_2\right) \right| \le \frac{1}{2\sqrt[q]{243}(u_2 - u_1)} \tag{2.35}$$

$$\times \left\{ (x-u_1)^2 \sqrt[q]{185|f'(u_1)|^q + 58|f'(x)|^q} + (u_2 - x)^2 \sqrt[q]{195|f'(x)|^q + 48|f'(u_2)|^q} \right\}.$$

Corollary 2.22. *Taking $\lambda = 0$ and $\phi(t) = t$ in Theorem 2.15, we get*

$$\left| T_{f,\Upsilon^g_m,\Theta^g_m}(0; x, u_1, u_2) \right| \le \frac{1}{\mathbf{E}_\alpha^{\frac{q+1}{q}}(x - mu_1)\mathbf{E}_\alpha(u_2 - mu_1)} \tag{2.36}$$

$$\times \left[B_1^g(x;1)\right]^{1-\frac{1}{q}} \sqrt[q]{\left[B_1^g(x;1)\mathbf{E}_\alpha(x - mu_1) - C_1^g(x)\right]|f'(mu_1)|^q + C_1^g(x)|f'(x)|^q}$$

$$+ \frac{1}{\mathbf{E}_\alpha^{\frac{q+1}{q}}(u_2 - mx)\mathbf{E}_\alpha(u_2 - mu_1)} \left[B_2^g(x;1)\right]^{1-\frac{1}{q}}$$

$$\times \sqrt[q]{\left[B_2^g(x;1)\mathbf{E}_\alpha(u_2 - mx) - E_1^g(x)\right]|f'(mx)|^q + E_1^g(x)|f'(u_2)|^q},$$

where

$$C_1^g(x) = \int_{mu_1}^{mu_1 + \mathbf{E}_\alpha(x - mu_1)} (t - mu_1)(g(t) - g(mu_1))\, dt, \tag{2.37}$$

$$E_1^g(x) = \int_{mx}^{mx + \mathbf{E}_\alpha(u_2 - mx)} (t - mx)(g(mx + \mathbf{E}_\alpha(u_2 - mx)) - g(t))\, dt, \tag{2.38}$$

and $B_1^g(x;1)$, $B_2^g(x;1)$ are defined as in Corollary 2.10 for value $p = 1$.

Corollary 2.23. *Taking $\lambda = 0$ and $\phi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 2.15, we have*

$$\left| T_{f,\Upsilon^g_m,\Theta^g_m}(0; x, u_1, u_2) \right| \le \frac{1}{\mathbf{E}_\alpha^{\frac{q+1}{q}}(x - mu_1)\mathbf{E}_\alpha(u_2 - mu_1)} \tag{2.39}$$

$$\times \left[B_3^g(x;1,\alpha)\right]^{1-\frac{1}{q}} \sqrt[q]{\left[B_3^g(x;1,\alpha)\mathbf{E}_\alpha(x - mu_1) - C_1^g(x,\alpha)\right]|f'(mu_1)|^q + C_1^g(x,\alpha)|f'(x)|^q}$$

$$+ \frac{1}{\mathbf{E}_\alpha^{\frac{q+1}{q}}(u_2 - mx)\mathbf{E}_\alpha(u_2 - mu_1)} \left[B_4^g(x;1,\alpha)\right]^{1-\frac{1}{q}}$$

$$\times \sqrt[q]{\left[B_4^g(x;1,\alpha)\mathbf{E}_\alpha(u_2 - mx) - E_1^g(x,\alpha)\right]|f'(mx)|^q + E_1^g(x,\alpha)|f'(u_2)|^q},$$

where

$$C_1^g(x,\alpha) = \int_{mu_1}^{mu_1 + \mathbf{E}_\alpha(x - mu_1)} (t - mu_1)\left[g(t) - g(mu_1)\right]^\alpha dt, \tag{2.40}$$

$$E_1^g(x,\alpha) = \int_{mx}^{mx + \mathbf{E}_\alpha(u_2 - mx)} (t - mx)\left[g(mx + \mathbf{E}_\alpha(u_2 - mx)) - g(t)\right]^\alpha dt, \tag{2.41}$$

and $B_3^g(x;1,\alpha)$, $B_4^g(x;1,\alpha)$ are defined as in Corollary 2.11 for value $p = 1$.

Corollary 2.24. *Taking* $\lambda = 0$ *and* $\phi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *in Theorem 2.15, we obtain*

$$\left|T_{f,\Upsilon_m^g,\Theta_m^g}(0;x,u_1,u_2)\right| \leq \frac{1}{\mathbf{E}_\alpha^{\frac{q+1}{q}}(x-mu_1)\mathbf{E}_\alpha(u_2-mu_1)}\Big[B_5^g(x;1,\alpha,k)\Big]^{1-\frac{1}{q}} \tag{2.42}$$

$$\times\sqrt[q]{\Big[B_5^g(x;1,\alpha,k)\mathbf{E}_\alpha(x-mu_1)-C_1^g(x,\alpha,k)\Big]|f'(mu_1)|^q+C_1^g(x,\alpha,k)|f'(x)|^q}$$

$$+\frac{1}{\mathbf{E}_\alpha^{\frac{q+1}{q}}(u_2-mx)\mathbf{E}_\alpha(u_2-mu_1)}\Big[B_6^g(x;1,\alpha,k)\Big]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{\Big[B_6^g(x;1,\alpha,k)\mathbf{E}_\alpha(u_2-mx)-E_1^g(x,\alpha,k)\Big]|f'(mx)|^q+E_1^g(x,\alpha,k)|f'(u_2)|^q},$$

where

$$C_1^g(x,\alpha,k) = \int_{mu_1}^{mu_1+\mathbf{E}_\alpha(x-mu_1)}(t-mu_1)\big[g(t)-g(mu_1)\big]^{\frac{\alpha}{k}}\,dt, \tag{2.43}$$

$$E_1^g(x,\alpha,k) = \int_{mx}^{mx+\mathbf{E}_\alpha(u_2-mx)}(t-mx)\big[g(mx+\mathbf{E}_\alpha(u_2-mx))-g(t)\big]^{\frac{\alpha}{k}}\,dt, \tag{2.44}$$

and $B_5^g(x;1,\alpha,k)$, $B_6^g(x;1,\alpha,k)$ *are defined as in Corollary 2.12 for value* $p=1$.

Corollary 2.25. *Taking* $\lambda = 0$, $\forall\, u \in [0,t]$, $\phi_g(x,t) = t(g(mu_1+\mathbf{E}_\alpha(x-mu_1))-t)^{\alpha-1}$ *and* $\forall\, u \in [t,1]$, $\phi_g(x,t) = t(g(mx+\mathbf{E}_\alpha(u_2-mx))-t)^{\alpha-1}$ *in Theorem 2.15, we get*

$$\left|T_{f,\Upsilon_m^g,\Theta_m^g}(0;x,u_1,u_2)\right| \leq \frac{1}{\mathbf{E}_\alpha^{\frac{q+1}{q}}(x-mu_1)\mathbf{E}_\alpha(u_2-mu_1)} \tag{2.45}$$

$$\times\Big[B_7^g(x;1,\alpha)\Big]^{1-\frac{1}{q}}\sqrt[q]{\Big[B_7^g(x;1,\alpha)\mathbf{E}_\alpha(x-mu_1)-C_1^g(x)\Big]|f'(mu_1)|^q+C_1^g(x)|f'(x)|^q}$$

$$+\frac{1}{\mathbf{E}_\alpha^{\frac{q+1}{q}}(u_2-mx)\mathbf{E}_\alpha(u_2-mu_1)}\Big[B_8^g(x;1,\alpha)\Big]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{\Big[B_8^g(x;1,\alpha)\mathbf{E}_\alpha(u_2-mx)-L_2^g(x,\alpha)\Big]|f'(mx)|^q+L_2^g(x,\alpha)|f'(u_2)|^q},$$

where

$$L_2^g(x,\alpha) = \int_{mx}^{mx+\mathbf{E}_\alpha(u_2-mx)}(t-mx)\big[g^\alpha(mx+\mathbf{E}_\alpha(u_2-mx))-g^\alpha(t)\big]\,dt, \tag{2.46}$$

and $B_7^g(x;1,\alpha)$, $B_8^g(x;1,\alpha)$ *are defined as in Corollary 2.13 for value* $p=1$ *and* $C_1^g(x)$ *is defined as in Corollary 2.22.*

Corollary 2.26. *Taking* $\lambda = 0$ *and* $\phi(t) = \frac{t}{\alpha}\exp(-At)$, *where* $A = \frac{1-\alpha}{\alpha}$ *in Theorem 2.15, we have*

$$\left|T_{f,\Upsilon_m^g,\Theta_m^g}(0;x,u_1,u_2)\right| \leq \frac{1}{(1-\alpha)\mathbf{E}_\alpha^{\frac{q+1}{q}}(x-mu_1)\mathbf{E}_\alpha(u_2-mu_1)} \tag{2.47}$$

$$\times\Bigg\{\Big[B_9^g(x;1,A)\Big]^{1-\frac{1}{q}}\sqrt[q]{L_3^g(x,A)|f'(mu_1)|^q+L_4^g(x,A)|f'(x)|^q}$$

$$+\frac{1}{(1-\alpha)\mathbf{E}_\alpha^{\frac{q+1}{q}}(u_2-mx)\mathbf{E}_\alpha(u_2-mu_1)}$$

$$\times\Big[B_{10}^g(x;1,A)\Big]^{1-\frac{1}{q}}\sqrt[q]{L_5^g(x,A)|f'(mx)|^q+L_6^g(x,A)|f'(u_2)|^q}\Bigg\},$$

where

$$L_3^g(x,A)=\int_{mu_1}^{mu_1+\mathbf{E}_\alpha(x-mu_1)}(mu_1+\mathbf{E}_\alpha(x-mu_1)-t) \times\Big\{1-\exp\big[A\left(g(mu_1)-g(t)\right)\big]\Big\}\,dt, \tag{2.48}$$

$$L_4^g(x,A)=\int_{mu_1}^{mu_1+\mathbf{E}_\alpha(x-mu_1)}(t-mu_1)\Big\{1-\exp\big[A\left(g(mu_1)-g(t)\right)\big]\Big\}\,dt, \tag{2.49}$$

$$L_5^g(x,A)=\int_{mx}^{mx+\mathbf{E}_\alpha(u_2-mx)}(mx+\mathbf{E}_\alpha(u_2-mx)-t) \times\Big\{1-\exp\big[A\left(g(t)-g(mx+\mathbf{E}_\alpha(u_2-mx))\right)\big]\Big\}\,dt, \tag{2.50}$$

$$L_6^g(x,A)=\int_{mx}^{mx+\mathbf{E}_\alpha(u_2-mx)}(t-mx)\Big\{1-\exp\big[A\left(g(t)-g(mx+\mathbf{E}_\alpha(u_2-mx))\right)\big]\Big\}\,dt, \tag{2.51}$$

and $B_9^g(x;1,A)$, $B_{10}^g(x;1,A)$ *are defined as in Corollary 2.14 for value* $p=1$.

Remark 2.27. *Applying our Theorems 2.3 and 2.15 for special values of parameter* $\lambda\in[0,1]$, *for appropriate choices of function* $g(t)=t$; $g(t)=\ln t, \forall\, t>0;$, $g(t)=e^t$, *etc., where* $\phi(t)=t$, $\frac{t^\alpha}{\Gamma(\alpha)}$, $\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$; $\phi_g(t)=t(g(u_2)-t)^{\alpha-1}$ *for* $\alpha\in(0,1)$; $\phi(t)=\frac{t}{\alpha}\exp\Big[\left(-\frac{1-\alpha}{\alpha}\right)t\Big]$ *for* $\alpha\in(0,1)$, *such that* $|f'|^q$ *to be convex, we can deduce some new general fractional integral inequalities. We omit their proofs and the details are left to the interested readers.*

3. Applications

Consider the following special means for different real numbers u_1,u_2 and $u_1u_2\neq 0$, as follows:

1. the arithmetic mean:
$$A:=A(u_1,u_2)=\frac{u_1+u_2}{2},$$
2. the harmonic mean:
$$H:=H(u_1,u_2)=\frac{2}{\frac{1}{u_1}+\frac{1}{u_2}},$$
3. the logarithmic mean:
$$L:=L(u_1,u_2)=\frac{u_2-u_1}{\ln|u_2|-\ln|u_1|},$$
4. the generalized log–mean:
$$L_r:=L_r(u_1,u_2)=\left[\frac{u_2^{r+1}-u_1^{r+1}}{(r+1)(u_2-u_1)}\right]^{\frac{1}{r}};\quad r\in\mathbb{Z}\setminus\{-1,0\}.$$

It is well known that L_r is monotonic nondecreasing over $r \in \mathbb{Z}$ with $L_{-1} := L$. In particular, we have the following inequality $H \leq L \leq A$. Now, using the theory results in Section 2, we give some applications to special means for different real numbers.

Proposition 3.1. *Let $u_1, u_2 \in \mathbb{R} \setminus \{0\}$, where $u_1 < u_2$. Then for $r \in \mathbb{N}$ and $r \geq 2$, where $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality hold:*

$$\left| A^r(u_1, u_2) - L_r^r(u_1, u_2) \right| \leq \frac{r(u_2 - u_1)}{4\sqrt[p]{p+1}} \tag{3.1}$$

$$\times \left\{ \sqrt[q]{A\left(|u_1|^{q(r-1)}, \left|\frac{u_1+u_2}{2}\right|^{q(r-1)}\right)} + \sqrt[q]{A\left(\left|\frac{u_1+u_2}{2}\right|^{q(r-1)}, |u_2|^{q(r-1)}\right)} \right\}.$$

Proof . Taking $m = 1$, $\lambda = 0$, $x = \frac{u_1+u_2}{2}$, $\mathbf{E}_\alpha(x - mu_1) = x - mu_1$, $\mathbf{E}_\alpha(u_2 - mx) = u_2 - mx$, $\mathbf{E}_\alpha(u_2 - mu_1) = u_2 - mu_1$, $f(t) = t^r$ and $g(t) = \phi(t) = t$, in Theorem 2.3, one can obtain the result immediately. □

Proposition 3.2. *Let $u_1, u_2 \in \mathbb{R} \setminus \{0\}$, where $u_1 < u_2$. Then for $r \in \mathbb{N}$ and $r \geq 2$, where $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality hold:*

$$\left| A(u_1^r, u_2^r) - L_r^r(u_1, u_2) \right| \leq \frac{r(u_2 - u_1)}{4\sqrt[p]{p+1}} \tag{3.2}$$

$$\times \left\{ \sqrt[q]{A\left(|u_1|^{q(r-1)}, \left|\frac{u_1+u_2}{2}\right|^{q(r-1)}\right)} + \sqrt[q]{A\left(\left|\frac{u_1+u_2}{2}\right|^{q(r-1)}, |u_2|^{q(r-1)}\right)} \right\}.$$

Proof . Taking $m = 1$, $\lambda = 1$, $x = \frac{u_1+u_2}{2}$, $\mathbf{E}_\alpha(x - mu_1) = x - mu_1$, $\mathbf{E}_\alpha(u_2 - mx) = u_2 - mx$, $\mathbf{E}_\alpha(u_2 - mu_1) = u_2 - mu_1$, $f(t) = t^r$ and $g(t) = \phi(t) = t$, in Theorem 2.3, one can obtain the result immediately. □

Proposition 3.3. *Let $u_1, u_2 \in \mathbb{R} \setminus \{0\}$, where $u_1 < u_2$. Then for $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality hold:*

$$\left| \frac{1}{A(u_1, u_2)} - \frac{1}{L(u_1, u_2)} \right| \leq \frac{(u_2 - u_1)}{4\sqrt[p]{p+1}} \tag{3.3}$$

$$\times \left\{ \frac{1}{\sqrt[q]{H\left(|u_1|^{2q}, \left|\frac{u_1+u_2}{2}\right|^{2q}\right)}} + \frac{1}{\sqrt[q]{H\left(\left|\frac{u_1+u_2}{2}\right|^{2q}, |u_2|^{2q}\right)}} \right\}.$$

Proof . Taking $m = 1$, $\lambda = 0$, $x = \frac{u_1+u_2}{2}$, $\mathbf{E}_\alpha(x - mu_1) = x - mu_1$, $\mathbf{E}_\alpha(u_2 - mx) = u_2 - mx$, $\mathbf{E}_\alpha(u_2 - mu_1) = u_2 - mu_1$, $f(t) = \dfrac{1}{t}$ and $g(t) = \phi(t) = t$, in Theorem 2.3, one can obtain the result immediately. □

Proposition 3.4. *Let $u_1, u_2 \in \mathbb{R} \setminus \{0\}$, where $u_1 < u_2$. Then for $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality hold:*

$$\left| \frac{1}{H(u_1, u_2)} - \frac{1}{L(u_1, u_2)} \right| \leq \frac{(u_2 - u_1)}{4\sqrt[p]{p+1}} \tag{3.4}$$

$$\times\left\{\frac{1}{\sqrt[q]{H\left(|u_1|^{2q},\left|\frac{u_1+u_2}{2}\right|^{2q}\right)}}+\frac{1}{\sqrt[q]{H\left(\left|\frac{u_1+u_2}{2}\right|^{2q},|u_2|^{2q}\right)}}\right\}.$$

Proof . Taking $m=1$, $\lambda=1$, $x=\frac{u_1+u_2}{2}$, $\mathbf{E}_\alpha(x-mu_1)=x-mu_1$, $\mathbf{E}_\alpha(u_2-mx)=u_2-mx$, $\mathbf{E}_\alpha(u_2-mu_1)=u_2-mu_1$, $f(t)=\dfrac{1}{t}$ and $g(t)=\phi(t)=t$, in Theorem 2.3, one can obtain the result immediately. □

Proposition 3.5. *Let $u_1,u_2\in\mathbb{R}\setminus\{0\}$, where $u_1<u_2$. Then for $r\in\mathbb{N}$ and $r\geq 2$, where $q\geq 1$, the following inequality hold:*

$$\left|A^r(u_1,u_2)-L_r^r(u_1,u_2)\right|\leq\sqrt[q]{\frac{2}{3}}\frac{r(u_2-u_1)}{8} \tag{3.5}$$

$$\times\left\{\sqrt[q]{A\left(|u_1|^{q(r-1)},2\left|\frac{u_1+u_2}{2}\right|^{q(r-1)}\right)}+\sqrt[q]{A\left(2\left|\frac{u_1+u_2}{2}\right|^{q(r-1)},|u_2|^{q(r-1)}\right)}\right\}.$$

Proof . Taking $m=1$, $\lambda=0$, $x=\frac{u_1+u_2}{2}$, $\mathbf{E}_\alpha(x-mu_1)=x-mu_1$, $\mathbf{E}_\alpha(u_2-mx)=u_2-mx$, $\mathbf{E}_\alpha(u_2-mu_1)=u_2-mu_1$, $f(t)=t^r$ and $g(t)=\phi(t)=t$, in Theorem 2.15, one can obtain the result immediately. □

Proposition 3.6. *Let $u_1,u_2\in\mathbb{R}\setminus\{0\}$, where $u_1<u_2$. Then for $r\in\mathbb{N}$ and $r\geq 2$, where $q\geq 1$, the following inequality hold:*

$$\left|A(u_1^r,u_2^r)-L_r^r(u_1,u_2)\right|\leq\sqrt[q]{\frac{2}{3}}\frac{r(u_2-u_1)}{8} \tag{3.6}$$

$$\times\left\{\sqrt[q]{A\left(2|u_1|^{q(r-1)},\left|\frac{u_1+u_2}{2}\right|^{q(r-1)}\right)}+\sqrt[q]{A\left(\left|\frac{u_1+u_2}{2}\right|^{q(r-1)},2|u_2|^{q(r-1)}\right)}\right\}.$$

Proof . Taking $m=1$, $\lambda=1$, $x=\frac{u_1+u_2}{2}$, $\mathbf{E}_\alpha(x-mu_1)=x-mu_1$, $\mathbf{E}_\alpha(u_2-mx)=u_2-mx$, $\mathbf{E}_\alpha(u_2-mu_1)=u_2-mu_1$, $f(t)=t^r$ and $g(t)=\phi(t)=t$, in Theorem 2.15, one can obtain the result immediately. □

Proposition 3.7. *Let $u_1,u_2\in\mathbb{R}\setminus\{0\}$, where $u_1<u_2$. Then for $q\geq 1$, the following inequality hold:*

$$\left|\frac{1}{A(u_1,u_2)}-\frac{1}{L(u_1,u_2)}\right|\leq\sqrt[q]{\frac{4}{3}}\frac{(u_2-u_1)}{8} \tag{3.7}$$

$$\times\left\{\frac{1}{\sqrt[q]{H\left(2|u_1|^{2q},\left|\frac{u_1+u_2}{2}\right|^{2q}\right)}}+\frac{1}{\sqrt[q]{H\left(\left|\frac{u_1+u_2}{2}\right|^{2q},2|u_2|^{2q}\right)}}\right\}.$$

Proof . Taking $m=1$, $\lambda=0$, $x=\frac{u_1+u_2}{2}$, $\mathbf{E}_\alpha(x-mu_1)=x-mu_1$, $\mathbf{E}_\alpha(u_2-mx)=u_2-mx$, $\mathbf{E}_\alpha(u_2-mu_1)=u_2-mu_1$, $f(t)=\dfrac{1}{t}$ and $g(t)=\phi(t)=t$, in Theorem 2.15, one can obtain the result immediately. □

Proposition 3.8. *Let $u_1, u_2 \in \mathbb{R} \setminus \{0\}$, where $u_1 < u_2$ and $\mathbf{E}_\alpha(u_2 - mu_1) > 0$. Then for $q \geq 1$, the following inequality hold:*

$$\left|\frac{1}{H(u_1,u_2)} - \frac{1}{L(u_1,u_2)}\right| \leq \sqrt[q]{\frac{4}{3}}\frac{(u_2-u_1)}{8} \tag{3.8}$$

$$\times\left\{\frac{1}{\sqrt[q]{H\left(|u_1|^{2q}, 2\left|\frac{u_1+u_2}{2}\right|^{2q}\right)}} + \frac{1}{\sqrt[q]{H\left(2\left|\frac{u_1+u_2}{2}\right|^{2q}, |u_2|^{2q}\right)}}\right\}.$$

Proof . Taking $m = 1$, $\lambda = 1$, $x = \frac{u_1+u_2}{2}$, $\mathbf{E}_\alpha(x - mu_1) = x - mu_1$, $\mathbf{E}_\alpha(u_2 - mx) = u_2 - mx$, $\mathbf{E}_\alpha(u_2 - mu_1) = u_2 - mu_1$, $f(t) = \dfrac{1}{t}$ and $g(t) = \phi(t) = t$, in Theorem 2.15, one can obtain the result immediately. □

Remark 3.9. *Applying our Theorems 2.3 and 2.15 for special values of parameter $\lambda \in [0,1]$, for appropriate choices of function $g(t) = t$; $g(t) = \ln t, \forall t > 0;$, $g(t) = e^t$, etc., where $\phi(t) = t$, $\frac{t^\alpha}{\Gamma(\alpha)}$, $\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$; $\phi_g(t) = t(g(u_2) - t)^{\alpha-1}$ for $\alpha \in (0,1)$; $\phi(t) = \frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ for $\alpha \in (0,1)$, such that $|f'|^q$ to be convex, we can deduce some new general fractional integral inequalities using above special means (and other special means). We omit their proofs and the details are left to the interested readers.*

Next, we provide some new error estimates for the midpoint and trapezium quadrature formula. Let Q be the partition of the points $u_1 = x_0 < x_1 < \ldots < x_k = u_2$ of the interval $[u_1, u_2]$. Let consider the following quadrature formula:

$$\int_{u_1}^{u_2} f(x)dx = M(f,Q) + E(f,Q), \quad \int_{u_1}^{u_2} f(x)dx = T(f,Q) + E^*(f,Q)$$

where

$$M(f,Q) = \sum_{i=0}^{k-1} f\left(\frac{x_i + x_{i+1}}{2}\right)(x_{i+1} - x_i), \quad T(f,Q) = \sum_{i=0}^{k-1}\frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i)$$

are the midpoint and trapezium version and $E(f,Q)$, $E^*(f,Q)$ are denote their associated approximation errors.

Proposition 3.10. *Let $f : [u_1, u_2] \longrightarrow \mathbb{R}$ be a differentiable function on (u_1, u_2), where $u_1 < u_2$. If $|f'|^q$ is convex on $[u_1, u_2]$ for $q > 1$ and $p^{-1} + q^{-1} = 1$, then the following inequality holds:*

$$\big|E(f,Q)\big| \leq \frac{1}{4\sqrt[q]{2}\sqrt[p]{p+1}} \times \sum_{i=0}^{k-1}(x_{i+1} - x_i)^2 \tag{3.9}$$

$$\times\left\{\sqrt[q]{|f'(x_i)|^q + \left|f'\left(\frac{x_i + x_{i+1}}{2}\right)\right|^q} + \sqrt[q]{\left|f'\left(\frac{x_i + x_{i+1}}{2}\right)\right|^q + |f'(x_{i+1})|^q}\right\}.$$

Proof . Applying Theorem 2.3 for $m=1$, $\lambda=0$, $x=\frac{u_1+u_2}{2}$, $\mathbf{E}_\alpha(x-mu_1)=x-mu_1$, $\mathbf{E}_\alpha(u_2-mx)=u_2-mx$, $\mathbf{E}_\alpha(u_2-mu_1)=u_2-mu_1$ and $g(t)=\phi(t)=t$ on the subintervals $[x_i,x_{i+1}]$ $(i=0,\dots,k-1)$ of the partition Q, we have

$$\left|f\left(\frac{x_i+x_{i+1}}{2}\right)-\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}f(x)dx\right|\leq\frac{(x_{i+1}-x_i)}{4\sqrt[q]{2}\sqrt[p]{p+1}} \tag{3.10}$$

$$\times\left\{\sqrt[q]{|f'(x_i)|^q+\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}+\sqrt[q]{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+|f'(x_{i+1})|^q}\right\}.$$

Hence from (3.10), we get

$$\begin{aligned}\big|E(f,Q)\big|&=\left|\int_{u_1}^{u_2}f(x)dx-M(f,Q)\right|\\&\leq\left|\sum_{i=0}^{k-1}\left\{\int_{x_i}^{x_{i+1}}f(x)dx-f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)\right\}\right|\\&\leq\sum_{i=0}^{k-1}\left|\left\{\int_{x_i}^{x_{i+1}}f(x)dx-f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)\right\}\right|\\&\leq\frac{1}{4\sqrt[q]{2}\sqrt[p]{p+1}}\times\sum_{i=0}^{k-1}(x_{i+1}-x_i)^2\end{aligned}$$

$$\times\left\{\sqrt[q]{|f'(x_i)|^q+\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}+\sqrt[q]{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+|f'(x_{i+1})|^q}\right\}.$$

The proof of Proposition 3.10 is completed. □

Proposition 3.11. *Let $f:[u_1,u_2]\longrightarrow\mathbb{R}$ be a differentiable function on (u_1,u_2), where $u_1<u_2$. If $|f'|^q$ is convex on $[u_1,u_2]$ for $q\geq1$, then the following inequality holds:*

$$\big|E(f,Q)\big|\leq\frac{1}{8\sqrt[q]{3}}\times\sum_{i=0}^{k-1}(x_{i+1}-x_i)^2 \tag{3.11}$$

$$\times\left\{\sqrt[q]{|f'(x_i)|^q+2\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}+\sqrt[q]{2\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+|f'(x_{i+1})|^q}\right\}.$$

Proof . The proof is analogous as to that of Proposition 3.10 taking $m=1$, $\lambda=0$, $x=\frac{u_1+u_2}{2}$, $\mathbf{E}_\alpha(x-mu_1)=x-mu_1$, $\mathbf{E}_\alpha(u_2-mx)=u_2-mx$, $\mathbf{E}_\alpha(u_2-mu_1)=u_2-mu_1$ and $g(t)=\phi(t)=t$ using Theorem 2.15. □

Proposition 3.12. *Let $f:[u_1,u_2]\longrightarrow\mathbb{R}$ be a differentiable function on (u_1,u_2), where $u_1<u_2$. If $|f'|^q$ is convex on $[u_1,u_2]$ for $q>1$ and $p^{-1}+q^{-1}=1$, then the following inequality holds:*

$$\big|E^*(f,Q)\big|\leq\frac{1}{4\sqrt[q]{2}\sqrt[p]{p+1}}\times\sum_{i=0}^{k-1}(x_{i+1}-x_i)^2 \tag{3.12}$$

$$\times\left\{\sqrt[q]{|f'(x_i)|^q+\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}+\sqrt[q]{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+|f'(x_{i+1})|^q}\right\}.$$

Proof . Applying Theorem 2.3 for $m=1$, $\lambda=1$, $x=\frac{u_1+u_2}{2}$, $\mathbf{E}_\alpha(x-mu_1)=x-mu_1$, $\mathbf{E}_\alpha(u_2-mx)=u_2-mx$, $\mathbf{E}_\alpha(u_2-mu_1)=u_2-mu_1$ and $g(t)=\phi(t)=t$ on the subintervals $[x_i,x_{i+1}]$ $(i=0,\dots,k-1)$ of the partition Q, we have

$$\left|\frac{f(x_i)+f(x_{i+1})}{2}-\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}f(x)dx\right|\le\frac{(x_{i+1}-x_i)}{4\sqrt[q]{2}\sqrt[p]{p+1}} \tag{3.13}$$

$$\times\left\{\sqrt[q]{|f'(x_i)|^q+\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}+\sqrt[q]{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+|f'(x_{i+1})|^q}\right\}.$$

Hence from (3.13), we get

$$\begin{aligned}\left|E^*(f,Q)\right|&=\left|\int_{u_1}^{u_2}f(x)dx-T(f,Q)\right|\\&\le\left|\sum_{i=0}^{k-1}\left\{\int_{x_i}^{x_{i+1}}f(x)dx-\frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i)\right\}\right|\\&\le\sum_{i=0}^{k-1}\left|\left\{\int_{x_i}^{x_{i+1}}f(x)dx-\frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i)\right\}\right|\\&\le\frac{1}{4\sqrt[q]{2}\sqrt[p]{p+1}}\times\sum_{i=0}^{k-1}(x_{i+1}-x_i)^2\end{aligned}$$

$$\times\left\{\sqrt[q]{|f'(x_i)|^q+\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}+\sqrt[q]{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+|f'(x_{i+1})|^q}\right\}.$$

The proof of Proposition 3.12 is completed. □

Proposition 3.13. *Let $f:[u_1,u_2]\longrightarrow\mathbb{R}$ be a differentiable function on (u_1,u_2), where $u_1<u_2$. If $|f'|^q$ is convex on $[u_1,u_2]$ for $q\ge1$, then the following inequality holds:*

$$\left|E^*(f,Q)\right|\le\frac{1}{8\sqrt[q]{3}}\times\sum_{i=0}^{k-1}(x_{i+1}-x_i)^2 \tag{3.14}$$

$$\times\left\{\sqrt[q]{2|f'(x_i)|^q+\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}+\sqrt[q]{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+2|f'(x_{i+1})|^q}\right\}.$$

Proof . The proof is analogous as to that of Proposition 3.12 taking $m=1$, $\lambda=1$, $x=\frac{u_1+u_2}{2}$, $\mathbf{E}_\alpha(x-mu_1)=x-mu_1$, $\mathbf{E}_\alpha(u_2-mx)=u_2-mx$, $\mathbf{E}_\alpha(u_2-mu_1)=u_2-mu_1$ and $g(t)=\phi(t)=t$ using Theorem 2.15. □

Remark 3.14. *Applying our Theorems 2.3 and 2.15, where $m=1$, for special values of parameter $\lambda\in[0,1]$, for appropriate choices of function $g(t)=t$; $g(t)=\ln t$, $\forall\, t>0$;, $g(t)=e^t$, etc., where $\phi(t)=t$, $\frac{t^\alpha}{\Gamma(\alpha)}$, $\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$; $\phi_g(t)=t(g(u_2)-t)^{\alpha-1}$ for $\alpha\in(0,1)$; $\phi(t)=\frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ for $\alpha\in(0,1)$, such that $|f'|^q$ to be convex, we can deduce some new bounds for the midpoint and trapezium quadrature formula using above ideas and techniques. We omit their proofs and the details are left to the interested readers.*

4. Conclusion

The new class of functions called generalized m-convex can be applied to obtain several results in convex analysis, related optimization theory and may stimulate further research in different areas of pure and applied sciences.

Acknowledgements

We thank anonymous referee for his/her valuable suggestions for improved our manuscript.

References

[1] Agarwal, R.P., Luo, M.J. and Raina, R.K., *On Ostrowski type inequalities*, Fasc. Math. 204 (2016), 5–27.

[2] Ahmadmir, M. and Ullah, R., *Some inequalities of Ostrowski and Grüss type for triple integrals on time scales*, Tamkang J. Math. 42(4) (2011), 415–426.

[3] Alomari, M., Darus, M., Dragomir, S.S. and Cerone, P., *Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense*, Appl. Math. Lett. 23 (2010), 1071–1076.

[4] Aslani, S.M., Delavar, M.R. and Vaezpour, S.M., *Inequalities of Fejér type related to generalized convex functions with applications*, Int. J. Anal. Appl. 16(1) (2018), 38–49.

[5] Budak, H., *On Fejér type inequalities for convex mappings utilizing fractional integrals of a function with respect to another function*, Result. Math. 74(1) (2019), 29.

[6] Budak, H. and Sarikaya, M. Z, *Hermite–Hadamard type inequalities for s-convex mappings via fractional integrals of a function with respect to another function,*, Fasc. Math. 57(1), (2016): 25-36.

[7] Chen, F.X. and Wu, S.H., *Several complementary inequalities to inequalities of Hermite–Hadamard type for s-convex functions*, J. Nonlinear Sci. Appl. 9(2) (2016), 705–716.

[8] Delavar, M.R. and De La Sen, M. *Some generalizations of Hermite–Hadamard type inequalities*, SpringerPlus 5(1661) (2016).

[9] Dragomir, S.S., *On the Ostrowski's integral inequality for mappings with bounded variation and applications*, Math. Ineq. & Appl. 1(2) (1998).

[10] Dragomir, S.S., *The Ostrowski integral inequality for Lipschitzian mappings and applications*, Comput. Math. Appl. 38 (1999), 33–37.

[11] Dragomir, S.S., *Ostrowski–type inequalities for Lebesgue integral: A survey of recent results*, Aust. J. Math. Anal. Appl. 14(1) (2017), 1–287.

[12] Dragomir, S.S., *Further Ostrowski and trapezoid type inequalities for the generalized Riemann–Liouville fractional integrals of functions with bounded variation*, RGMIA Res. Rep. Coll. 20 (2017), Art. 84.

[13] Dragomir, S.S., *Ostrowski and trapezoid type inequalities for the generalized k-g-fractional integrals of functions with bounded variation*, RGMIA Res. Rep. Coll. 20 (2017), Art. 111.

[14] Dragomir, S.S. and Agarwal, R.P., *Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula*, Appl. Math. Lett. 11(5) (1998), 91–95.

[15] *Dragomir, S.S. and Wang, S., An inequality of Ostrowski–Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules*, Comput. Math. Appl. 13(11) (1997), 15–20.

[16] Dragomir, S.S. and Wang, S., *A new inequality of Ostrowski's type in L_1-norm and applications to some special means and to some numerical quadrature rules*, Tamkang J. Math. 28 (1997), 239–244.

[17] Du, T.S., Awan, M.U., Kashuri, A. and Zhao, S., *Some k-fractional extensions of the trapezium inequalities through generalized relative semi-(m,h)-preinvexity*, Appl. Anal. (2019), https://doi.org/10.1080/00036811.2019.1616083.

[18] Erdelyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G., *Higher Transcendental Functions*, 3, New York, McGraw–Hill, (1955).

[19] Farid, G., *Existence of an integral operator and its consequences in fractional and conformable integrals*, Open Journal of Mathematical Sciences 3(1) (2019), 210–216.

[20] Farid, G., *Some new Ostrowski type inequalities via fractional integrals*, Int. J. Anal. App. 14(1) (2017), 64–68.

[21] Hristov, J., *Response functions in linear viscoelastic constitutive equations and related fractional operators*, Math. Model. Nat. Phenom. 14(3) (2019), 1–34.

[22] Jleli, M. and Samet, B., *On Hermite–Hadamard type inequalities via fractional integral of a function with respect to another function*, J. Nonlinear Sci. Appl. 9 (2016), 1252–1260.

[23] Kashuri, A. and Liko, R., *Some new Hermite–Hadamard type inequalities and their applications*, Stud. Sci. Math. Hung. 56(1) (2019), 103–142.

[24] Katugampola, U.N., *New approach to a generalized fractional integral*, Appl. Math. Comput., **218**(3) (2011), 860–865.

[25] Khan, M.A., Chu, Y.M., Kashuri, A., Liko, R. and Ali, G., *Conformable fractional integrals versions of Hermite–Hadamard inequalities and their generalizations*, J. Funct. Spaces (2018), Article ID 6928130, pp. 9.

[26] Kilbas, A.A., Marichev, O.I. and Samko, S.G., *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Switzerland, (1993).

[27] Kilbas, A.A., Srivastava, H.M., and Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, (2006).

[28] Khalil, R., Horani, M.A., Yousef, A. and Sababheh, M., *A new definition of fractional derivatives*, J. Comput. Appli. Math. 264 (2014), 65–70.

[29] Ahmad, B., Alsaedi, A., Kirane, M., and Torebek, B.T., *Hermite–Hadamard, Hermite–Hadamard–Fejér, Dragomir–Agarwal and Pachpatti type inequalities for convex functions via fractional integrals*, J. Comput. Appl. Math. 353 (2019), 120–129.

[30] Liu, W.J., *Some Simpson type inequalities for h-convex and (α, m)-convex functions*, J. Comput. Anal. Appl. 16(5) (2014), 1005–1012.

[31] Liu, W., Wen, W. and Park, J., *Hermite–Hadamard type inequalities for MT-convex functions via classical integrals and fractional integrals*, J. Nonlinear Sci. Appl. 9 (2016), 766–777.

[32] Liu, W., Wen, W. and Park, J., *Ostrowski type fractional integral inequalities for MT-convex functions*, Miskolc Math. Notes 16(1) (2015), 249–256.

[33] Liu, Z., *Some Ostrowski–Grüss type inequalities and applications*, Comput. Math. Appl. 53 (2007), 73–79.

[34] Liu, Z., *Some companions of an Ostrowski type inequality and applications*, J. Inequal. in Pure and Appl. Math 10(2) (2009), Art. 52, pp. 12.

[35] Matloka, M., *Ostrowski type inequalities for functions whose derivatives are h-convex via fractional integrals*, Journal of Scientific Research and Reports 3(12) (2014), 1633–1641.

[36] Mihai, M.V., *Some Hermite–Hadamard type inequalities via Riemann–Liouville fractional calculus*, Tamkang J. Math 44(4) (2013), 411–416.

[37] Mubeen, S. and Habibullah, G.M., *k-Fractional integrals and applications*, Int. J. Contemp. Math. Sci. 7 (2012), 89–94.

[38] Noor, M.A., *Some new classes of nonconvex functions*, Nonl. Funct. Anal. Appl. 11 (2006), 165–171.

[39] Omotoyinbo, O. and Mogbodemu, A., *Some new Hermite–Hadamard integral inequalities for convex functions*, Int. J. Sci. Innovation Tech. 1(1) (2014), 1–12.

[40] Özdemir, M.E., Dragomir, S.S. and Yildiz, C., *The Hadamard's inequality for convex function via fractional integrals*, Acta Math. Sci., Ser. B, Engl. Ed. 33(5) (2013), 153–164.

[41] Özdemir, M.E., Kavurmacı, H. and Set, E., *Ostrowski's type inequalities for (α, m)-convex functions*, Kyungpook Math. J. 50 (2010), 371–378.

[42] Pachpatte, B.G., *On an inequality of Ostrowski type in three independent variables*, J. Math. Anal. Appl. 249 (2000), 583–591.

[43] Pachpatte, B.G., *On a new Ostrowski type inequality in two independent variables*, Tamkang J. Math. 32(1) (2001), 45–49.

[44] Qi, F. and Xi, B.Y., *Some integral inequalities of Simpson type for GA-ϵ-convex functions*, Georgian Math. J. 20(5) (2013), 775–788.

[45] Rafiq, A., Mir, N. A. and Ahmad, F., *Weighted Čebyšev–Ostrowski type inequalities*, Applied Math. Mechanics (English Edition) 28(7) (2007), 901–906.

[46] Sarikaya, M.Z., *On the Ostrowski type integral inequality*, Acta Math. Univ. Comenianae 79(1) (2010), 129–134.

[47] Sarikaya, M.Z. and Ertuğral, F., *On the generalized Hermite–Hadamard inequalities*, `https://www.researchgate.net/publication/321760443`.

[48] Sarikaya, M.Z. and Yildirim, H., *On generalization of the Riesz potential*, Indian Jour. of Math. and Mathematical Sci. 3(2), (2007), 231–235.

[49] Set, E., Noor, M.A., Awan, M.U. and Gözpinar, A., *Generalized Hermite–Hadamard type inequalities involving fractional integral operators*, J. Inequal. Appl. 169 (2017), 1–10.

[50] Tunç, M., *Ostrowski type inequalities for functions whose derivatives are MT-convex*, J. Comput. Anal. Appl. 17(4) (2014), 691–696.

[51] Ujević, N., *Sharp inequalities of Simpson type and Ostrowski type*, Comput. Math. Appl. 48 (2004), 145–151.

[52] Wang, H., Du, T.S. and Zhang, Y., *k-fractional integral trapezium–like inequalities through (h, m)-convex and

(α, m)*-convex mappings*, J. Inequal. Appl. 2017(311) (2017), pp. 20.

[53] Xi, B.Y and Qi, F., *Some integral inequalities of Hermite–Hadamard type for convex functions with applications to means*, J. Funct. Spaces Appl. 2012 (2012), Article ID 980438, pp. 14.

[54] Yildiz, Ç., Özdemir, M.E. and Sarikaya, M.Z., *New generalizations of Ostrowski–like type inequalities for fractional integrals*, Kyungpook Math. J. 56 (2016), 161–172.

[55] Zhang, X.M., Chu, Y.M. and Zhang, X.H., *The Hermite–Hadamard type inequality of GA-convex functions and its applications*, J. Inequal. Appl. (2010), Article ID 507560, pp. 11.

[56] Zhang, Y., Du, T.S., Wang, H., Shen, Y.J. and Kashuri, A., *Extensions of different type parameterized inequalities for generalized* (m, h)*-preinvex mappings via* k*-fractional integrals*, J. Inequal. Appl. 2018(49) (2018), pp. 30.

[57] Zhongxue, L., *On sharp inequalities of Simpson type and Ostrowski type in two independent variables*, Comput. Math. Appl. 56 (2008), 2043–2047.