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An Analysis of A Fishing Model with Nonlinear Harvesting Function

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Abstract

In this study, considering the importance of how to exploit renewable natural resources, we analyze a fishing model with nonlinear harvesting function in which the players at the equilibrium point do a static game with complete information that, according to the calculations, will cause a waste of energy for both players and so the selection of cooperative strategies along with the agreement between the players is the result of this research.

Keywords: natural resource management, game theory, bioeconomic models, nonlinear dynamical systems, fishery, harvest function.
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1. Introduction

Since game theory examines situations in which decision-makers interact, this theory has many applications in the commercial competition between individuals, companies, and countries (see [2],[8],[9],[14] and [15]).
For example, using this theory, we can examine how to use renewable natural resources and different strategies. Consider a river, lake or sea exploited by fishermen or companies. If the number of fishermen is increased or more fish are harvested, it will lead to the extinction of the generation of fishes in that source.

If fishering from the source is done only by a fisherman, he(or she) will consider the amount of current harvest because he(or she) knows that the amount of fish may be reduced and harvesting in

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the future will require more effort and cost, so he(or she) considers the effects of fishing today in the effort and cost of future fishing. Now if two individuals or companies fish from a natural resource, then fishing more today that leads to more cost and effort in the future and it will be shared between them. Therefore, there is a motivation for more harvesting in the present by either fisherman.
The importance of exploiting renewable natural resources and paying attention to the above points have led to a widespread examination of the issue of fishing and harvest management strategies that prevent the extinction of species (see [1],[3],[4],[11],[12],[13] and [18]).
Of course, a large number of these studies have been investigated in Dynamic Systems (see [5], [6], [10], [16], [17] and [19]).
The logistic equation with density-dependent harvesting (see [7]) is

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}) - H(t).$$

where N is the population biomass of fish at time t, r is the intrinsic rate of growth of the population, K is the carrying capacity, and $H(t)$ as the harvest function is

$$H(t) = qEN(t)$$

where E is the fishing effort, the intensity of the human activities to extract the fish and $q \geq 0$ is the catchability coefficient which is defined as the fraction of the population fished by a unit of effort.
In the above fishing mdel and many other models, the harvest function is linear in terms of $N(t)$ but we want to examine the effect of the nonlinear harvest function of $H(t) = qE(N(t))^2$ and since that fishermen or companies usually harvest individually and based on their own profits, at the point of equilibrium of the system we consider a static game with complete information and as a result we calculate the amount of the waste of effort and energy.

2. Main results

In this model, we consider the relation between net growth, W, and the carrying capacity, K, a logistic growth function. So, we have

$$W = rN(1 - \frac{N}{K}). \tag{2.1}$$

Assuming a nonlinear harvest function, $H = qEN^2$, the dynamics of this model is

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}) - qEN^2.$$

It can be written as follows

$$\frac{dN}{dt} = rN(1 - \frac{N}{K_0})$$

where $K_0 = \frac{rK}{r+qEK}$. This system has a trivial equilibrium, $N = 0$, and an non-trivial equilibrium, $N = K_0 = \frac{rK}{r+qEK}$.
Considering $f(N) = rN(1 - \frac{N}{K}) - qEN^2$, we have $\frac{dN}{dt} = f(N)$. Since $\frac{df(N)}{dN} = r - 2\frac{rN}{K} - 2qEN$, then $\frac{df(0)}{dN} = r > 0$ so the trivial equilibrium point is unstable, and at the non-trivial equilibrium $\frac{df(K_0)}{dN} = \frac{-r^2-qEK}{r+qEK} < 0$ which shows the stability of this point.

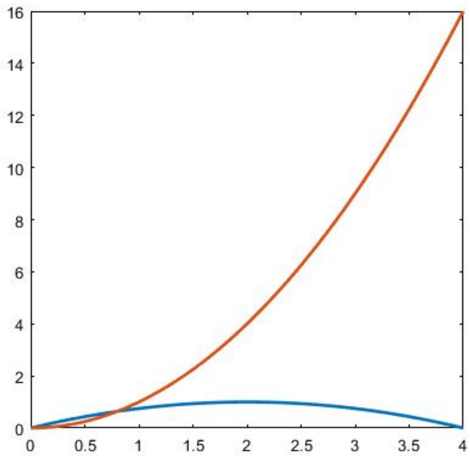

Figure 1: The functins $W = rN(1 - \frac{N}{K})$, $H = qEN^2$ in which $r = 1$, $q = 1$, $E = 1$ and $K = 4$.

On the other hand the differential equation of this system is a separable type and it is solved by a method called *separation of variables*.
By solving this equation and taking $B = \frac{N(0)}{|K_0 - N(0)|}$, it follows that if $K_0 - N > 0$ then $N(t) = \frac{BK_0e^{rt}}{1+Be^{rt}}$ is the solution of this differential equation and since $\lim_{t\to+\infty} N(t) = K_0$, it implies that the non-trivial equilibrium is asymptotically stable and if $K_0 - N < 0$ then $N(t) = \frac{-BK_0e^{rt}}{1-Be^{rt}}$ is the solution of the differential equation and $\lim_{t\to+\infty} N(t) = K_0$ implies that the non-trivial equilibrium is asymptotically stable.

According to the above, when the system reaches the equilibrium point, where $H = W$ (see Figure 1), we have

$$N = \frac{rK}{r + qEK}. \tag{2.2}$$

Which shows the relation between the equilibrium mass and the effort. By substituting (2.2) in $H(t)$, the relation between the level of effort and the harvest is

$$H = qE(\frac{rK}{(r + qEK)})^2. \tag{2.3}$$

We assume that there are two fishing companies, A_1 and A_2, which use this source separately. They as players do a static game with complete information in devoting the amount of effort to harvest. If E_1 and E_2 respectively represent the level of effort of players A_1 and A_2 then the total effort to harvest from this source is $E_T = E_1 + E_2$ and the total harvest of this effort is $H_T = qE_T(\frac{rK}{(r+qE_TK)})^2$. In this model, we assume that the share of each company (player) is equal to its share of total effort in other words

$$H_1 = \frac{E_1}{E_T}H_T = \frac{E_1}{E_T}qE_T(\frac{rK}{(r + qE_TK)})^2 = qE_1(\frac{rK}{(r + q(E_1 + E_2)K)})^2,$$

$$H_2 = \frac{E_2}{E_T}H_T = \frac{E_2}{E_T}qE_T(\frac{rK}{(r + qE_TK)})^2 = qE_2(\frac{rK}{(r + q(E_1 + E_2)K)})^2$$

where $H_T = H_1 + H_2$.
If each unit of the harvest in the market has a value of P and one unit of effort has a cost C, then

the outcome of the players is

$$U_{A_1}(E_1,E_2)=PH_1-CE_1=\frac{qPE_1r^2K^2}{(r+q(E_1+E_2)K)^2}-CE_1,$$

$$U_{A_2}(E_1,E_2)=PH_2-CE_2=\frac{qPE_2r^2K^2}{(r+q(E_1+E_2)K)^2}-CE_2.$$

To find the Nash equilibrium we use the best response method. In this way, we first do the calculations for player 1. So, assuming that $E_2=\overline{E_2}$ is positive and constant, we have

$$U_{A_1}(E_1,\overline{E_2})=0\Longrightarrow E_1=0\vee E_1=\sqrt{\frac{Pr^2}{qC}}-\beta.$$

where $\beta=(\frac{r}{qK}+\overline{E_2})$. But

$$E_1=\sqrt{\frac{Pr^2}{qC}}-\beta>0\Longleftrightarrow \overline{E_2}<r(\sqrt{\frac{P}{qC}}-\frac{1}{qK}). \tag{2.4}$$

On the other hand, if $\frac{\partial U_{A_1}(E_1,\overline{E_2})}{\partial E_1}=\frac{qr^2PK^2(r+q(E_1+\overline{E_2})K)-2r^2q^2PK^3E_1}{(r+q(E_1+\overline{E_2})K)^3}-c=0$ then by a simple calculations we have

$$E_1^3+3E_1^2\beta+(3\beta^2+\frac{r^2P}{qC})E_1=\frac{r^2P}{qC}\beta-\beta^3 \tag{2.5}$$

where $\beta=(\frac{r}{qK}+\overline{E_2})$.

We consider $f_1(E_1)=E_1^3+3E_1^2\beta+(3\beta^2+\frac{r^2P}{qC})E_1$ and $f_2(E_1)=\frac{r^2P}{qC}\beta-\beta^3$ that according to (2.4) and $\beta=\frac{r}{qK}+\overline{E_2}>0$ always $f_2(E_1)=\frac{r^2P}{qC}\beta-\beta^3>0$.
On the other hand, $\lim_{E_1\to+\infty}f_1(E_1)=+\infty$ and $\lim_{E_1\to-\infty}f_1(E_1)=-\infty$ and also $f_1(E_1)=E_1(E_1^2+3\beta E_1+(3\beta^2+\frac{r^2P}{qC}))$ that $E_1^2+3\beta E_1+(3\beta^2+\frac{r^2P}{qC})$ has $\Delta=-3\beta^2-4\frac{r^2P}{qC}<0$ then $f_1(E_1)$ only has a real root $E_1=0$.
Since $\frac{df_1(E_1)}{dE_1}=3E_1^2+6\beta E_1+(3\beta^2+\frac{r^2P}{qC})$ then $f_1(E_1)$ for $E_1>0$ is always an increasing function. According to the above, functions f_1 and f_2 for $E_1>0$ intersect each other exactly at one point then $U_{A_1}(E_1,\overline{E_2})$ for $0<\overline{E_2}<r(\sqrt{\frac{P}{qC}}-\frac{1}{qK})$ has exactly a local extremum that we show it with E_1°.

In order to determine the type of this extremum, we use the first derivative test for $U_{A_1}(E_1,\overline{E_2})$ in the neighborhood of this point, E_1°. Since $E_1^\circ>0$, we consider $\epsilon>0$ such that $E_1^\circ-\epsilon>0$ in this case

$$\frac{dU_{A_1}(E_1^\circ,\overline{E_2})}{dE_1}=\frac{-C(qKE_1^\circ+(r+qK\overline{E_2}))^3-q^2r^2PK^3E_1^\circ+qr^2PK^2(r+qK\overline{E_2})}{(qKE_1^\circ+(r+qK\overline{E_2}))^3}=0,$$

$$\frac{dU_{A_1}(E_1^\circ+\epsilon,\overline{E_2})}{dE_1}=\frac{-C(qK(E_1^\circ+\epsilon)+(r+qK\overline{E_2}))^3-q^2r^2PK^3(E_1^\circ+\epsilon)+qr^2PK^2(r+qK\overline{E_2})}{(qK(E_1^\circ+\epsilon)+(r+qK\overline{E_2}))^3}$$

$$=\frac{-Cq^3K^3\epsilon^3-3qKC\theta_1\epsilon-3q^2K^2C\theta_1\epsilon^2-q^2r^2K^3P\epsilon}{(qK(E_1^\circ+\epsilon)+(r+qK\overline{E_2}))^3}<0$$

where $\theta_1=qKE_1^\circ+(r+qK\overline{E_2})>0$,

$$\frac{dU_{A_1}(E_1^\circ-\epsilon,\overline{E_2})}{dE_1}=\frac{Cq^3K^3\epsilon^3+3qKC\theta_1\epsilon(qK(E_1^\circ-\epsilon)+(r+qK\overline{E_2}))+q^2r^2K^3P\epsilon}{(qK(E_1^\circ-\epsilon)+(r+qK\overline{E_2}))^3}>0.$$

Therefore, according to the first derivative test, E_1° is a relative maximal point for $U_{A_1}(E_1, \overline{E_2})$. According to the above, the best response function of player 1 is

$$E_1 = B_1(E_2) = \begin{cases} E_1^\circ & \text{for} \quad E_2 < r(\sqrt{\frac{P}{qC}} - \frac{1}{qK}) \\ 0 & \text{for} \quad E_2 \geq r(\sqrt{\frac{P}{qC}} - \frac{1}{qK}). \end{cases} \tag{2.6}$$

Now we would like to calculate E_1° as the root of Equation (2.5). For

$$E_1^3 + 3E_1^2\beta + (3\beta^2 + \frac{r^2P}{qC})E_1 + (\beta^3 - \frac{r^2P}{qC}\beta) = 0$$

we consider

$$a = 3\beta,\ b = 3\beta^2 + \tfrac{r^2P}{qC},\ \text{and}\ c = \beta^3 - \tfrac{r^2P}{qC}\beta$$

then according to the calculation method of the roots of a third-order polynomial, we have

$$p = b - \tfrac{a^2}{3} = \tfrac{r^2P}{qC},\ q = \tfrac{2a^3}{27} - \tfrac{ab}{3} + c = -2(\tfrac{r^2P\beta}{qC}) \text{ and } \Delta = \tfrac{q^2}{4} + \tfrac{p^3}{27} = \tfrac{r^4P^2}{q^2C^2}(\beta^2 + \tfrac{1}{27}\tfrac{r^2P}{qC}) > 0.$$

Since $\Delta > 0$, then (2.5) has only one real root that we already showed it with E_1° and this root is

$$\begin{aligned} E_1^\circ &= (-\tfrac{q}{2} + \sqrt{\Delta})^{\frac{1}{3}} + (-\tfrac{q}{2} - \sqrt{\Delta})^{\frac{1}{3}} - \tfrac{a}{3} \\ &= ((\tfrac{r^2P}{qC})(\beta + \sqrt{\beta^2 + \tfrac{1}{27}\tfrac{r^2P}{qC}}))^{\frac{1}{3}} + ((\tfrac{r^2P}{qC})(\beta - \sqrt{\beta^2 + \tfrac{1}{27}\tfrac{r^2P}{qC}}))^{\frac{1}{3}} - \beta. \end{aligned}$$

Therefore

$$\begin{aligned} E_1 &= B_1(E_2) \\ &= \begin{cases} ((\tfrac{r^2P}{qC})(\beta + \sqrt{\beta^2 + \tfrac{1}{27}\tfrac{r^2P}{qC}}))^{\frac{1}{3}} + ((\tfrac{r^2P}{qC})(\beta - \sqrt{\beta^2 + \tfrac{1}{27}\tfrac{r^2P}{qC}}))^{\frac{1}{3}} - \beta & \text{for} \quad E_2 < r(\sqrt{\tfrac{P}{qC}} - \tfrac{1}{qK}) \\ 0 & \text{for} \quad E_2 \geq r(\sqrt{\tfrac{P}{qC}} - \tfrac{1}{qK}) \end{cases} \end{aligned}$$

where $\beta = \frac{r}{qK} + E_2$.
According to the symmetry of the game and with a completely similar discussion for the second player, we conclude that

$$\begin{aligned} E_2 &= B_2(E_1) \\ &= \begin{cases} ((\tfrac{r^2P}{qC})(\beta_* + \sqrt{\beta_*^2 + \tfrac{1}{27}\tfrac{r^2P}{qC}}))^{\frac{1}{3}} + ((\tfrac{r^2P}{qC})(\beta_* - \sqrt{\beta_*^2 + \tfrac{1}{27}\tfrac{r^2P}{qC}}))^{\frac{1}{3}} - \beta_* & \text{for} \quad E_1 < r(\sqrt{\tfrac{P}{qC}} - \tfrac{1}{qK}) \\ 0 & \text{for} \quad E_1 \geq r(\sqrt{\tfrac{P}{qC}} - \tfrac{1}{qK}). \end{cases} \end{aligned}$$

where $\beta_* = \frac{r}{qK} + E_1$. According to the functions E_1 and E_2, we can obtain Nash equilibrium for this model from solutions of the following system

$$\begin{cases} E_1 = ((\tfrac{r^2P}{qC})(\beta + \sqrt{\beta^2 + \tfrac{1}{27}\tfrac{r^2P}{qC}}))^{\frac{1}{3}} + ((\tfrac{r^2P}{qC})(\beta - \sqrt{\beta^2 + \tfrac{1}{27}\tfrac{r^2P}{qC}}))^{\frac{1}{3}} - \beta & \text{that} \quad \beta = \tfrac{r}{qK} + E_2 \\ E_2 = ((\tfrac{r^2P}{qC})(\beta_* + \sqrt{\beta_*^2 + \tfrac{1}{27}\tfrac{r^2P}{qC}}))^{\frac{1}{3}} + ((\tfrac{r^2P}{qC})(\beta_* - \sqrt{\beta_*^2 + \tfrac{1}{27}\tfrac{r^2P}{qC}}))^{\frac{1}{3}} - \beta_* & \text{that} \quad \beta_* = \tfrac{r}{qK} + E_1. \end{cases} \tag{2.7}$$

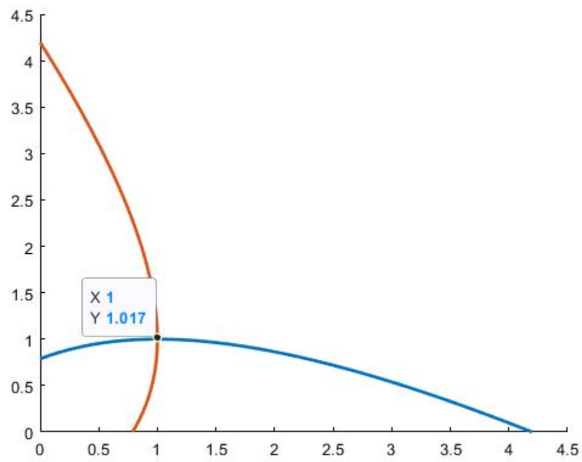

Figure 2: The unique Nash equilibrium is $(E_1^*, E_2^*) = (1,1)$.

Since the above system is complicated based on the parameters r, q, P, C and K, to make a clearer analysis and easier understanding of the problem, we consider the following values for the parameters

$$r = P = 1, q = C = \frac{1}{\sqrt{27}} \text{ and } K = \sqrt{27}.$$

In this case, the system (2.7) is

$$\begin{cases} E_1 = 3(((1+E_2) + \sqrt{(1+E_2)^2+1})^{\frac{1}{3}} + ((1+E_2) - \sqrt{(1+E_2)^2+1})^{\frac{1}{3}}) - 1 - E_2 \\ E_2 = 3(((1+E_1) + \sqrt{(1+E_1)^2+1})^{\frac{1}{3}} + ((1+E_1) - \sqrt{(1+E_1)^2+1})^{\frac{1}{3}}) - 1 - E_1 \end{cases} \tag{2.8}$$

We have solved the above nonlinear system with the iterative method, that, as shown in Figure 2, it has an approximate unique solution $(E_1, E_2) = (1,1)$, with very little error. This unique solution is the Nash equilibrium of this model that we represent with (E_1^*, E_2^*) therefore $(E_1^*, E_2^*) = (1,1)$. The total effort and harvest in the Nash equilibrium, E_T and H_T, are

$$E_T = E_1^* + E_2^* \cong 1 + 1 = 2$$

and

$$H_T = H_1^* + H_2^*$$

where

$$H_1^* = qE_1^*(\frac{rK}{(r + q(E_1^* + E_2^*)K)})^2 = \frac{\sqrt{27}}{9} = 0.5773502691 \cong 0.577$$

and similarly

$$H_2^* \cong 0.577$$

then $H_T \cong 1.15$.
On the other hand, by substituting in (2.3) we have

$$H_T = qE_T(\frac{rK}{(r + qE_TK)})^2 = \frac{\sqrt{27}E_T}{(1+E_T)^2} \cong 1.15.$$

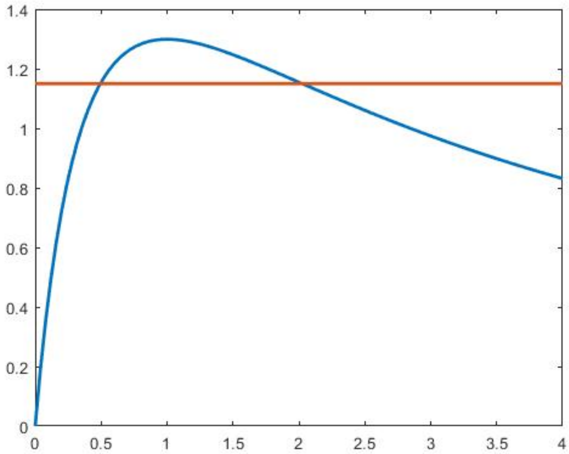

Figure 3: The functins $H_T = \frac{\sqrt{27}E_T}{(1+E_T)^2}$, $H_T = 1.15$ that the intersections of the two functions show E_{T_1}, E_{T_2}.

But the recent relation is almost equivalent to

$$1.15E_T^2 - 2.9E_T + 1.15 = 0.$$

For this equation $\Delta = 3.12$, $\sqrt{\Delta} = 1.7663521733 \cong 1.77$, and its roots are $E_{T_1} = 0.4913043478 \cong 0.49$ *and* $E_{T_2} = 2.0304347826 \cong 2.03$. (see Figure 3).
It should be noted that E_{T_2} is approximately the sum of the levels of effort of the players in Nash equilibrium and these two roots indicate that the amount of harvest in the Nash equilibrium can be obtained with E_{T_2} and with less effort E_{T_1}. Since $E_{T_2} - E_{T_1} \cong 1.54$, then in E_{T_2} that players are doing a static game with complete information thay are wasting 1.54 units of effort because the mass of fish is reduced and fishing is hardly done and players should spend more effort to harvest more. Therefore, by considering certain values as parameters, in order to make a clearer analysis, we conclude when the harvest function is nonlinear, doing a static game causes a waste of energy and money of both players. Therefore, doing a cooperative game with the agreement of the parties will be for the benefit of each player.

References

[1] Agnew, TT.,Optimal exploitation of a fishery employing a non-linear harvesting function, Ecological Modelling, Elsevier, 1979.
[2] Bernheim, BD., Rationalizable strategic behavior, Econometrica: Journal of the Econometric Society, JSTOR, 1984.
[3] Bischi, GI., Lamantia, F., Harvesting dynamics in protected and unprotected areas, Journal of Economic Behavior & Organization, Elsevier, 2007.
[4] Clark, CW., Restricted access to common-property fishery resources: a game-theoretic analysis, Dynamic optimization and mathematical economics, Springer, 1980.
[5] Clark, CW., Mathematical models in the economics of renewable resources, Siam Review, SIAM, 1979.
[6] Cohen, Y., A review of harvest theory and applications of optimal control theory in fisheries management, Canadian Journal of Fisheries and Aquatic ..., NRC Research Press, 1987.
[7] Cooke, KL., Witten, M., One-dimensional linear and logistic harvesting models, Mathematical Modelling, Elsevier, 1986.
[8] Dixit, AK., and Skeath, S., Games of Strategy, Fourth International Student Edition, 2015.
[9] Gibbons, R., Game theory for applied economists, Princeton University Press, Harvester Wheatsheaf, 1992.

[10] Dubey, B., Peeyush, C., Sinha, P., A model for fishery resource with reserve area, Nonlinear Analysis: Real World Applications, 2003.
[11] Kaitala, V., Game theory models of fisheries managementa survey, a survey, Dynamic games and applications in economics, Springer, 1986.
[12] Kaitala, V.,Equilibria in a stochastic resource management game under imperfect information, European Journal of Operational Research, Elsevier, 1993.
[13] Kaitala, V., Lindroos, M.,Sharing the benefits of cooperation in high seas fisheries: a characteristic function game approach, Natural Resource Modeling, Wiley Online Library, 1998.
[14] Osborne, MJ., An introduction to game theory, Oxford University Press. New York, 2004.
[15] Osborne, MJ., and Rubinstein, A., A course in game theory, MIT Press, 1994.
[16] Schaefer, MB., Some aspects of the dynamics of populations important to the management of the commercial marine fisheries, American Tropical Tuna Commission Bulletin, 1954.
[17] Schaefer, MB., Some considerations of population dynamics and economics in relation to the management of the commercial marine fisheries, Journal of the Fisheries Board of Canada, NRC Research Press, 1957.
[18] Sumaila, UR., A review of game-theoretic models of fishing, Marine policy, Elsevier, 1999.
[19] Van Long, N., Dynamic games in the economics of natural resources: a survey, Dynamic Games and Applications, Springer, 2011.