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Diamond-ϕ_h dynamics on time scales with an Application to Economics

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Abstract

Conventional dynamic models in Economics are usually expressed in discrete or continuous time. A new modelling technique-time scale calculus-unifies both of these approaches into a general framework. In this paper, we present and construct a dynamic Optimization problem from economics in which the utility function is ϕ_h-concave, the value function and constraints are on different time scales. The calculus of variations and optimal control are employed, with the aid of the newly introduced diamond-ϕ_h dynamic calculus by the authors [12] on time scales, to obtain a solution. The Hermite-Hadamard inequality with the diamond-ϕ_h dynamic integral on time scales, follows a proof of the new model. The new diamond-ϕ_h time scale model unify various related existing models involving general and more complex time domains.

Keywords: Time scales, ϕ_h-concave, diamond-ϕ_h, Hermite-Hadamard, dynamic models.
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1. Introduction and Preliminaries

The Hermite-Hadamard inequality is known to be the first fundamental inequality for convex functions. It is stated as:

$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq (b-a)\frac{f(a)+f(b)}{2}, \tag{1.1}$$

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where $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to R$ is a convex function. It was first suggested by Hermite in 1881. Also, Beckenbach [4], a leading expert in the history and theory of complex variables, wrote that the inequality (1.1) was proven by Hadamard in 1893, who apparently was not aware of Hermite's result. In general, (1.1) is now refered to as the Hermite-Hadamard inequality.

When economists design a dynamic model, they usually have to decide whether the model should be expressed in discrete or continuous time. The discrete-time and continuous-time versions to dynamic modelling problems have been studied in literature, see [8]. Recently, a new modelling technique, time scale calculus, was developed to unify both of these approaches into a general framework, see [15]. Because it is a more general approach to dynamic modelling, time scales calculus can be used to model dynamic processes whose time domains are much more complex than sets of integers (difference equations) or real numbers (differential equations).

In time scales calculus models, the time domain $\mathbb{T}$ can be any nonempty closed subset of real numbers $\mathbb{R}$. We refer to the books [6] and [7] for further readings on time scales.

In the theory of time scales, the concepts of the delta and nabla calculus with applications to the Hermite-Hadamard inequality, Calculus of Variations, Optimal control problems and dynamic Optimization Problems in Economics have been introduced, see [1], [3], [5], [6], [14] and [15].

In 2006, Sheng et al. [17] introduced the diamond-α dynamic calculus, a linear combination of these delta and nabla dynamic calculi on time scales, which offer more balanced approximations to the targeted functions and differential equations at satisfactory accuracy than those of Δ and ∇ integrals. This new combined dynamic calculus has generated a lot of interest among mathematicians, particularly in the generalization of the Hermite-Hadamard inequality on time scales theory involving the nabla and delta calculi. see for example, [9] and the references therein.

A more general, combined dynamic calculus, the diamond-ϕ_h calculus, which includes the delta, nabla and diamond-α calculi of Atici et al. [1], Guzowska et al.[14] and Sheng et al.[17] respectively, was recently introduced by the authors [12].

Several generalizations of the Hermite-Hadamard integral inequality for single and double variable-time scales and other related integral inequalities for convex functions and different classes of convex functions on classical intervals and on time scales theory are given in literature, see for example [9], [12],[13], [15], [16] and [18].

In [12], the authors employed the concept of diamond-ϕ_h time scales calculus to establish Hermite-Hadamard integral inequality for the class of ϕ_h-convex functions introduced in [10], by stating the following theorem, among others.

Theorem 1.1. *[12] Let Let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero non negative function with the property that $h(t) > 0$ for all $t \geq 0$ and $f : I_{\mathbb{T}} \to \mathbb{R}$ be a continuous ϕ_h-convex function, $a, b, t \in I_{\mathbb{T}}$, with $a < b$. Then*

$$2^s \left(h(\frac{1}{2})\right)^s f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x) \diamond_{\phi_h} x \leq f(a)\int_0^1 \left(\frac{\lambda}{h(\lambda)}\right)^s \Delta\lambda + f(b)\int_0^1 \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \nabla\lambda. \tag{1.2}$$

In the sequel, we shall need the following definitions.

Definition 1.2. *[12] Let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a nonzero non negative function with the property that $h(t) > 0$ for all $t \geq 0$. The diamond-ϕ_h dynamic derivative of a function $f : \mathbb{T} \to \mathbb{R}$ in $t \in \mathbb{T}$ is defined to be the number denoted by $f^{\diamond_{\phi_h}}(t)$ (when it exists), with the property that for any $\epsilon > 0$, there* *is a neighbourhood U of m such that, for all $n \in U$, $0 \le s \le 1$ and $0 \le \lambda \le 1$, with $\mu_{mn} = \sigma(m) - n$ and $\nu_{mn} = \rho(m) - n$, where $m, n \in \mathbb{T}_k^k$, then,*

$$\left| \left(\frac{\lambda}{h(\lambda)}\right)^s [f(\sigma(m)) - f(n)]\nu_{mn} + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s [f(\rho(m)) - f(n)]\mu_{mn} - f^{\diamond_{\phi_h}}(t)\mu_{mn}\nu_{mn} \right| < \epsilon |\mu_{mn}\nu_{mn}|.$$

Remark 1.3. *(i) In definition 1.2, if $\phi_h = \alpha; h(\lambda) = 1$, $s = 1$, and $\lambda = 1$, we obtain the $\diamond_\alpha$ derivative of [17] on time scales. Thus every diamond-α differentiable function on $\mathbb{T}$ is diamond-ϕ_h differentiable but the converse is not true see [12].*

(ii) The nabla derivative of Atici et al. [1] is obtained for $\phi_h = 0; h(\lambda) = 1$, $s = 1$ and $\lambda = 0$.

(iii) We get the delta derivative [14] when $\phi_h = 1; h(\lambda) = 1$, $s = 1$ and $\lambda = 1$.

(iv) If $\phi_h = \frac{1}{2}; h(\lambda) = 1$, $s = 1$ and $\lambda = \frac{1}{2}$, then we have a centralized derivative formula on any uniformly discrete time scale $\mathbb{T}$ see [9].

(v) If f is diamond-ϕ_h differentiable for $0 \le s \le 1$, and $0 \le \lambda \le 1$, then f is both Δ and ∇ differentiable.

Definition 1.4. *[12] Let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a nonzero non negative function with the property that $h(t) > 0$ for all $t \ge 0$. A function is called diamond-ϕ_h differentiable on $\mathbb{T}_k^k$ if $f^{\diamond_{\phi_h}}(t)$ exists for all $t \in \mathbb{T}_k^k$. If $f : \mathbb{T} \to \mathbb{R}$ is differentiable on $\mathbb{T}$ in the sense of Δ and ∇, then f is $\diamond_{\phi_h}$ differentiable at $t \in \mathbb{T}_k^k$ and the $\diamond_{\phi_h}$ derivative $f^{\diamond_{\phi_h}}(t)$ is given by*

$$f^{\diamond_{\phi_h}}(t) = \left(\frac{\lambda}{h(\lambda)}\right)^s f(t) + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f(t), \qquad s \in [0,1],\ 0 \le \lambda \le 1.$$

Definition 1.5. *[12] Let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a nonzero non negative function with the property that $h(t) > 0$ for all $t \ge 0$. The diamond-ϕ_h integral of a function $f : \mathbb{T} \to \mathbb{R}$ from a to b, where $a, b \in \mathbb{T}$ is given by;*

$$\int_a^b f(t) \diamond_{\phi_h} t = \left(\frac{\lambda}{h(\lambda)}\right)^s \int_a^b f(t)\Delta t + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \int_a^b f(t)\nabla t, \tag{1.3}$$

for all $s \in [0,1]$ and $\lambda \in [0,1]$, provided that f has a delta and nabla integral on $I_{\mathbb{T}}$.

Remark 1.6. *(i) The inequality (1.3) reduces to the diamond-α integral defined by Sheng et al. [17], if $\phi_h = \alpha; h(\lambda) = 1$, $s = 1$ and $\lambda = 1$. Thus, every diamond-α integrable function on $\mathbb{T}$ is diamond-ϕ_h integrable but the converse is not true, see [12].*

(ii) If f is diamond-ϕ_h integrable for $0 \le s \le 1$, and $0 \le \lambda \le 1$, then f is both Δ and ∇ integrable.

Obviously, each continuous function has a diamond-ϕ_h integral. The combined derivative $\diamond_{\phi_h}$ is not a dynamic derivative, since we do not have a $\diamond_{\phi_h}$ antiderivative. In General,

$$\left(\int_a^b f(t) \diamond_{\phi_h} t\right)^{\diamond_{\phi_h}} \ne f(t), \quad t \in \mathbb{T}.$$

For more details on the diamond-ϕ_h dynamic calculus, see [11] and [12].

In this paper, the interested reader will find some preliminary results for the different types of existing calculi on time scales in this first section. In the next section, we state some basic Theorems as applications of the diamond-ϕ_h dynamic calculus to the Calculus of variations on time scales and

Optimal control problems that are needed for our purpose. In the third section, we demonstrate the use of our developed theory in section two to solve a simple household consumption model in economics hence unify and extend the recent works of [10] for several kinds of concave(convex) utility functions u, including those of [1] and [14] for the nabla and delta household problems, which includes the classical discrete and continuous time models as special cases. The third section is followed by conclusion.

2. Main result

Time scales calculus theory has found direct applications in many fields such as Engineering, Optimization and Economics in which dynamic processes can be described with the discrete or continuous time systems, variables or models, including applications to mathematical concepts of the calculus of variations and Optimal control on time scales, see [1], [2], [5], [12] and [14].

In [12], the authors presented the simplest variational problem in terms of equation (1.3) as follows:

Let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a nonzero non negative function with the property that $h(t) > 0$ for all $t \geq 0$. The simplest variational problem of finding the function $y = u(t) \in C^1[a, b]$, a weak extremum which minimizes the functional

$$J_{\diamond_{\phi_h}}[u] = \left(\frac{\lambda}{h(\lambda)}\right)^s J_\Delta[u] + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s J_\nabla[u], \tag{2.1}$$

for all $s,\ \lambda \in [0,\, 1]$, where $J_\Delta[u] = \int_a^b L(t, u^\sigma(t), u^\Delta(t))\Delta(t)$ and $J_\nabla[u] = \int_a^b L(t, u^\rho(t), u^\nabla(t))\nabla(t)$, $a, b \in \mathbb{T}$, with $a < b$, $\alpha, \beta \in \mathbb{R}^n, n \in \mathbb{N}$ and $J : \mathbb{T} \times \mathbb{R}^{2n} \to \mathbb{R}$, satisfying the Dirichlet boundary conditions $u(x) = \alpha, u(y) = \beta$, provided the Lagrangian $L(t, u, u^\Delta)$ is a class C^2 function with respect to all its arguments t, u(the state variable), and u^Δ.

For our purpose, we state the following result, whose proof can be found in [12], i.e, the basic, first-order condition in the calculus of variations.

Theorem 2.1. *(Euler's Necessary condition) Let $J_{\diamond_{\phi_h}}[u]$ define a functional of the form (2.1). Then, a necessary condition for $J_{\diamond_{\phi_h}}[u]$ to have a local extremum for a given function $u(t)$ is that it satisfies both the Euler-Lagrange equations*

$$L_{u^\sigma}(t, u^\sigma, u^\Delta) - L_{u^\Delta}{}^\Delta(t, u^\sigma, u^\Delta) = 0, \qquad L_{u^\rho}(t, u^\rho, u^\nabla) - L_{u^\nabla}{}^\nabla(t, u^\rho, u^\nabla) = 0, \tag{2.2}$$

and hence satisfies

$$J_{\diamond_{(\phi_h)_1}}[\eta] = \left(\frac{\lambda}{h(\lambda)}\right)^s \int_a^b \{L_{u^\sigma}(t, u^\sigma, u^\Delta) - L_{u^\Delta}{}^\Delta(t, u^\sigma, u^\Delta)\}\eta^\sigma \Delta t$$
$$+ \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \int_a^b \{L_{u^\rho}(t, u^\rho, u^\nabla) - L_{u^\nabla}{}^\nabla(t, u^\rho, u^\nabla)\}\eta^\rho \nabla t = 0, \tag{2.3}$$

for all $s, \in [0, 1]$, $0 \leq \lambda \leq 1$ with $u(x) = u(y) = 0$ and all admissible variation η.

By treating the functional (2.1) as a function $J_{\diamond_{\phi_h}}[\eta]$ and setting the delta and nabla derivatives (2.3) equal to zero, we have derived the Euler-Lagrange equation and transversality conditions as first-order necessary conditions for an extremal in the calculus of variations, using the concept of a $\diamond_{\phi_h}$ dynamic calculus on time scale.

Next, we state and proof the following sufficiency result which holds in the calculus of variations.

Theorem 2.2. *For a fixed endpoint problem (2.1), if the integrand functions $L(t, u^\sigma, u^\Delta)$ and $L(t, u^\rho, u^\nabla)$ are ϕ_h-concave jointly in the variables (u^σ, u^Δ) and (u^ρ, u^∇), then the Euler-Lagrange equations (2.2) are sufficient for an absolute maximum of $J_{\diamond_{\phi_h}}[u]$.*

Similarly, if the integrand functions $L(t, u^\sigma, u^\Delta)$ and $L(t, u^\rho, u^\nabla)$ are ϕ_h-convex jointly in the variables (u^σ, u^Δ) and (u^ρ, u^∇) respectively, then the Euler-Lagrange equations (2.3) are sufficient for an absolute minimum of $J_{\diamond_{\phi_h}}[u]$.

Proof . Central to the proof is the defining property of a differentiable ϕ_h-concave function.

The integrand functions $L(t, u^\sigma, u^\Delta)$ and $L(t, u^\rho, u^\nabla)$ are ϕ_h-concave(ϕ_h-convex) jointly in the variables (u^σ, u^Δ) and (u^ρ, u^∇) if and only if they are differentiable.
By [10, Theorem 3.1], a ϕ_h-concave function(ϕ_h-convex) is differentiable and thus has differentiable properties.

Since, $L(t, u^\sigma, u^\Delta)$ and $L(t, u^\rho, u^\nabla)$ satisfy the Euler-Lagrange equations by (2.2) and hence (2.3). Then, $u^\sigma(t)$ and $u^\rho(t)$ are the J-maximizing paths and at the same time, Euler equation (2.3) is a sufficient condition, given the assumption of a ϕ_h-concave function L. The opposite case of a ϕ_h-convex function can be analogously proved for minimizing J. Therefore, if the integrand functions $L(t, u^\sigma, u^\Delta)$ and $L(t, u^\rho, u^\nabla)$ are ϕ_h-concave(ϕ_h-convex) jointly in the variables (u^σ, u^Δ) and (u^ρ, u^∇) in a problem of the form (2.1), then the E-L equation plus the transversality conditions are sufficient for an absolute maximum(minimum) of $J_{\diamond_{\phi_h}}[u]$. $\square$

In order to state the necessary condition for Optimization in the formulation of a dynamic Optimization problem, it is important to state the simplest problem of optimal control by introducing a multiplier $p(t)$ to the state equation (2.1) in the calculus of variations, such that $p^\sigma(t)$ and $p^\rho(t)$ are Δ and ∇ differentiable functions on $I_{\mathbb{T}}$ respectively.

Let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a nonzero non negative function with the property that $h(t) > 0$ for all $t \geq 0$. We are concerned with presenting the simplest form of Optimal control problem in terms of equation (1.3) as;

$$\max\ J_{\diamond_{\phi_h}}[x, u] = \int_a^b L(t, x, u) \diamond_{\phi_h} t$$

$$= \left(\frac{\lambda}{h(\lambda)}\right)^s \int_a^b L(t, x^\sigma, u^\sigma)\Delta t + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \int_a^b L(t, x^\rho, u^\sigma)\nabla t, \tag{2.4}$$

for all $s \in [0, 1]$ and $0 \leq \lambda \leq 1$, among all pairs (x, u) such that $x^\Delta = f(t, x^\sigma, u^\sigma)$ and $x^\Delta = f(t, x^\rho, u^\sigma)$, together with appropriate endpoint conditions
$u^{\diamond'_{\phi_h}}(t) = L(t, u, p)$, $x(0) = u_0$, $u(T)$ free for all $t \in [0, T]$.

Thus, a first order necessary condition to obtain an optimal solution for our variational problem (2.4) is the maximum principle which involves the concepts of the Hamiltonian function and co-state or auxiliary variable defined in terms of our (1.3) is as follows.

Let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a nonzero non negative function with the property that $h(t) > 0$ for all $t \geq 0$. The Hamiltonian $H_{\diamond_{\phi_h}}(t, x, v, p) : [a, b]_{\mathbb{I}_{\mathbb{T}}} \times \mathbb{R}^3 \to R$ of (2.4) is defined by

$$H_{\diamond_{\phi_h}}(t, x, u, p) = \left(\frac{\lambda}{h(\lambda)}\right)^s H_\Delta(t, x^\sigma, u^\sigma, p^\sigma) + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s H_\nabla(t, x^\rho, u^\rho, p^\rho)$$

$$= \left(\frac{\lambda}{h(\lambda)}\right)^s [f(t, x^\sigma, u^\sigma) + p^\sigma g(t, x^\sigma, u^\sigma)] + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s [f(t, x^\rho, u^\rho) + p^\rho g(t, x^\rho, u^\rho)]. \tag{2.5}$$

We can now state the following.

Theorem 2.3. *Let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a nonzero non negative function with the property that $h(t) > 0$ for all $t \geq 0$ and let $(\tilde{x},\tilde{u})$ be a normal extremizer for the problem (2.4), subject to*

$$x^{\Delta}(t) = g(t, x^{\sigma}(t), u^{\sigma}(t)), \qquad x(a) = x_a, \quad x(b) = x_b$$

$$x^{\nabla}(t) = g(t, x^{\rho}(t), u^{\rho}(t)), \qquad x(a) = x_a, \quad x(b) = x_b,$$

where $L_{\diamond_{\phi_h}}$ is continuous in $t, L(t,.,.)$ and $g(t,.,.)$ are C' functions with respect to the second and third variables uniformly in t. Then, for all $t \in [a,b]_{\mathbb{I}_{\mathbb{T}}}$ and $H_{\diamond_{\phi_h}}(t,...)$ satisfying (2.5) above, there exists a function $\tilde{p}$ such that the triple $(\tilde{x},\tilde{u},\tilde{p})$ satisfies the Hamiltonian systems

$$x^{\Delta}(t) = H_{p^{\sigma}}(t, x^{\sigma}(t), u^{\sigma}(t), p^{\sigma}(t)), \qquad (p^{\sigma}(t))^{\Delta} = -H_{x^{\sigma}}(t, x^{\sigma}(t), u^{\sigma}(t), p^{\sigma}(t))$$

and

$$x^{\nabla}(t) = H_{p^{\rho}}(t, x^{\rho}(t), u^{\rho}(t), p^{\rho}(t)), \qquad (p^{\rho}(t))^{\nabla} = -H_{x^{\rho}}(t, x^{\rho}(t), u^{\rho}(t), p^{\rho}(t))$$

and the stationary conditions

$$H_{u^{\sigma}}(t, x^{\sigma}(t), u^{\sigma}(t), p^{\sigma}(t)) = 0, \qquad H_{u^{\rho}}(t, x^{\rho}(t), u^{\rho}(t), p^{\rho}(t)) = 0.$$

Proof . Let the function $f(t,...)$ satisfy the condition of the Theorem 1.1, then Theorem 2.3 satisfies the proof of Theorem 1.1. □

In general, the maximum principle furnishes us with a set of necessary conditions for optimal control but are not sufficient. Hence, we state the following Theorem which guarantees that the conditions stipulated by the maximum principle are sufficient for maximization.

Theorem 2.4. *If the functions f and g are both ϕ_h-concave in (u^{σ}, u^{Δ}) and (u^{ρ}, u^{∇}), and if p is nonnegative, then the Hamiltonian (2.5), being the sum of two ϕ_h-concave functions, must be ϕ_h-concave in (u^{σ}, u^{Δ}) and (u^{ρ}, u^{∇}).*

Proof . Since a simple optimal control problem (2.4) can be translated into an equivalent problem of the calculus of variations (2.1), it follows that the proof of Theorem 2.4 follows the proof of Theorem 2.2 above. Hence, the necessary and sufficient conditions required by the maximum principle are also equivalent to those of the calculus of variations. □

3. The household problem

In this section, we include how a simple utility maximum problem can be set up and solved in time scales settings by employing a more general diamond-ϕ_h dynamic calculus. The model assumes a perfect foresight.

A representative consumer seeks to maximize his lifetime utility u

$$\max U = \sum_{s=0}^{T} \left(\frac{1}{1+\delta} \right)^{s} u(C_s), \tag{3.1}$$

subject to the budget constraints
$A_{s+1} = (1+r)A_s + Y_s - C_s$, for all $s \in [0,T]$ and $A_T \left(\frac{1}{1+\delta}\right)^{T} \geq 0$,
where $0 \leq \delta \leq 1$ is the (constant) discount factor, C_s is consumption during period s, $u(C_s)$ is the utility the consumer derives from consuming C_s units of consumption in periods $s = 0, 1, 2, ..., T$. Utility is assumed to be concave: $u(C_s)$ has $u(C_s)' > 0$ and $u(C_s)'' < 0$. If the consumer consumes

more today, the utility or satisfaction he derives from consumption, is forgone tomorrow as the 'penalty'. The consumer would always like to consume more but each additional unit consumed during the same period generates less utility than the previous unit consumed within the same period. This property of utility function is called the law of diminishing marginal utility (LMDU). This means that the first unit of consumption of a good or service yields more utility than the second or subsequent units, with a continuing reduction for greater amounts.

The individual is constrained by the fact that the value function of his consumption, $u(C)$ must be equal to the value function of his income Y_s, plus the assets/debts, A_s that he might accommodate in a period s. A_{s+1} is the amount of assets held at the beginning of period $t+1$. A could be positive or negative; the consumer might save for the future or borrow against the future at interest rate r in any given period s but the value of A_T, which is the debt accrued with limit or the last period asset holding, has to be nonnegative(the optimal level is naturally zero, we want to spend all the money we have got and we do not care to leave money behind after death).

The same problem above can be solved in a continuous time setting, where lifetime utility is the sum of discounted instantaneous utilities, i.e.

$$U = \int_0^T u(C_s)e^{\delta s}ds. \tag{3.2}$$

This is the utility function in the discrete case (3.1). The consumer's goal is to maximize lifetime utility with respect to the path $\{C_s\}_{s=0}^T$, subject to the budget constraint
$A'_s = A_s r + Y_s - C_s$, where $A_s^{'}$ is the first derivative of A_s, e is the exponential function and $U, u(C_s), \delta, A_s, Y_s$ are as described above.

Thus, consumption and asset holdings are continuous functions of time.

The introduction of delta and nabla calculi on time scales enables mathematicians and economists to combine difference and differential models; (3.1) and (3.2) within the framework of dynamic models on time scales, see [1] and [14].

An Economic application of the nabla dynamic calculus on time scales, first initiated by Atici and Guseinov [14], can be found in the paper by Atici et al. [1] which is to maximize;

$$U = \int_0^{\sigma(T)} u(C(\rho(s)))e_{-\delta}(\rho(s), 0)\nabla(s), \tag{3.3}$$

subject to the budget constraint
$A^{\nabla}(s) = rA(\rho(s)) + Y(\rho(s)) - C(\rho(s)), \qquad a(0) = a_0, \qquad a(T) = a_T,$
where $U, u(C), \delta, r, A, Y$ are as described in the discrete case but in terms of the backward jump operator ρ in each time scale period s, $e_{-\delta}(\rho(s), 0)$ is the nabla exponential function and $A^{\nabla}(s)$ is the nabla derivative of A. The boundary conditions a_0 can be interpreted as either an inheritance $(a_0 > 0)$ or debt burden $(a_0 < 0)$ passed down from a previous generation, a_T can be either a bequest $(a_T > 0)$ or a debt burden $a_T < 0$ passed down the next generation.

More recently, Guzowska et al.[14] employed the delta notion on time scales to present a simple household consumption model which maximizes;

$$U = \int_0^T e_{-\delta}(\sigma(t),\, 0)u(c^{\sigma}(t)))\Delta(t), \tag{3.4}$$

subject to the budget constraint
$a^{\Delta}(t) = \frac{r}{1+r\mu(t)}a^{\sigma}(t) + \frac{r}{1+r\mu(t)}y^{\sigma}(t) - \frac{r}{1+r\mu(t)}c^{\sigma}(t), \qquad s \in [\sigma(0), T],$

where $U, u(C), \delta, r, A, Y$ are as described in the discrete case but in terms of the forward jump operator σ in each time scale period t, $e_{-\delta}(\sigma(t), 0)$ is the delta exponential function and $A^{\Delta}(t)$ is the delta derivative of A.

Here, the concept of the $\diamond_{\phi_h}$ integral of the authors in [12] is employed, using the same intuition as that of the dynamic optimization problem presented earlier, to state and solve a simple household optimal control model as follows:

Theorem 3.1. *Let $\mathbb{T}$ be a time scale and $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a nonzero non negative function with the property that $h(t) > 0$ for all $t \geq 0$. The value function of the lifetime utility $U_{\diamond_{\phi_h}}$ to be maximized subject to certain constraints is;*

$$Maximize \;\; U_{\diamond_{\phi_h}} = \int_0^T u(C(t))e_{-\delta}(t,\, 0) \diamond_{\phi_h} t, \tag{3.5}$$

subject to the budget constraints

$$a^{\nabla}(t) = (rA + Y - C)(\rho(t)), \quad a^{\Delta}(t) = \frac{r}{1\, +\, r\mu(t)} a^{\sigma}(t) + \frac{1}{1 + r\mu(t)} y^{\sigma}(t) - \frac{1}{1 + r\mu(t)} c^{\sigma}(t), \tag{3.6}$$

$$a(0) = a_0, \qquad a(T) = a_T,$$

where u is ϕ_h-concave ($u^{'}(C) > 0$ and $u^{''}(C) < 0$), $0 \leq \lambda \leq 1$, $s \in [0, 1]$, $A^{\Delta}, A^{\nabla}, r, \delta, A, Y$ and e are as defined in the delta and nabla cases above.

Proof . Let $f(t)$ be a function satisfied by the consumption function path that would maximize lifetime utility $U_{\phi_h} C(t) e_{-\delta}(t,\, 0)$, then the conditions of the Theorem 2.2 hold and hence satisfies the proof of Theorem 2.3, which in turn satisfies Theorem 1.1. □

Therefore, the model (3.5)-(3.6) can be analysed as follows. By writing (3.5) in terms of (1.3) and using Theorem 2.3 for problem (3.5)-(3.6), we state the fundamental first-order necessary condition, i.e, the maximum principle, by giving the Hamiltonian function for the model (3.5)-(3.6) as follows;

$$\begin{aligned} H_{\diamond_{\phi_h}}(t, a, y, c, p) &= \left(\frac{\lambda}{h(\lambda)}\right)^s H_{\Delta}(t, a^{\sigma}, y^{\sigma}, c^{\sigma}, p^{\sigma}) + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s H_{\nabla}(t, a^{\rho}, y^{\rho}, c^{\rho}, p^{\rho}) \\ &= \left(\frac{\lambda}{h(\lambda)}\right)^s \left[e_{-\delta} u_{\phi_h}(C^{\sigma}) + p^{\sigma}\left(\frac{r}{1 + r\mu} a^{\sigma} + \frac{1}{1 + r\mu} y^{\sigma} - \frac{1}{1 + r\mu} c^{\sigma}\right)\right] \\ &\quad + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \left[e_{-\rho} u(C^{\rho}) + p^{\rho}\left(ra^{\rho} + y^{\rho} - C^{\rho}\right)\right], \end{aligned} \tag{3.7}$$

and the necessary optimality conditions:

$$a^{\Delta}(t) = \frac{r}{1 + r\mu} a^{\sigma}(t) + \frac{1}{1 + r\mu} y^{\sigma}(t) - \frac{1}{1 + r\mu} c^{\sigma}(t), \quad a^{\nabla}(t) = rA(\rho(s)) + Y(\rho(s)) - C(\rho(s)), \tag{3.8}$$

with Euler-Lagrange equations;

$$\begin{aligned} &\left(\frac{\lambda}{h(\lambda)}\right)^s \left[e_{-\delta(t,0)} u'(C^{\sigma}(t)) - \frac{1}{1 + r\mu(t)} p^{\sigma}(t)\right] \\ &+ \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \left[e_{-\delta(t,0)} u'(C^{\rho}(t)) + p^{\rho}(t)\right] = 0; \end{aligned} \tag{3.9}$$

$$\left(\frac{\lambda}{h(\lambda)}\right)^s \left[(p^\sigma(t))^\Delta + \frac{r}{1+r\mu(t)}p^\sigma(t)\right] + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \left[(p^\rho(t))^\nabla + rp^\rho(t)\right] = 0, \tag{3.10}$$

where

$$(p^\sigma(t))^\Delta = -\frac{r}{1+r\mu(t)}p^\sigma(t), \qquad (p^\rho(t))^\nabla = -rp^\rho(t), \tag{3.11}$$

and

$$e_{-\delta(t,0)}u'(C^\sigma(t)) - \frac{1}{1+r\mu(t)}p^\sigma(t) = 0, \quad e_{-\delta(t,0)}u'(C^\rho(t)) + p^\rho(t) = 0. \tag{3.12}$$

Equations (3.9) and (3.10) provide a unification of the Euler-Lagrange equations from both the Δ and ∇ approaches for the maximization problem (3.5) and (3.6) respectively.
Combining (3.11) with (3.12) and substituting into (3.9) and (3.10) gives

$$\left(\frac{\lambda}{h(\lambda)}\right)^s \left[[(1+r\mu(t))e_{-\delta}(t,0)u^{'}(C^\sigma(t))]^\Delta - r[-e_{-\delta}(t,0)u^{'}(C^\sigma(t))]\right]$$

$$+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \left[[e_{-\delta}(t,0)u^{'}(C^\rho(t))]^\nabla - r[-e_{-\delta}(t,0)u^{'}(C^\rho(t))]\right] = 0. \tag{3.13}$$

Using product rule with properties of the delta and nabla exponential functions defined by $e_{-\delta}(\sigma(t),0) = (1+\delta\mu(t))e_{-\delta}(t,0)$ and $e_{-\delta}(\rho(t),0) = (1+\delta\nu(t))e_{-\delta}(t,0)$ in the equation (3.13), we obtain the following expression:

$$\left(\frac{\lambda}{h(\lambda)}\right)^s \left[(1+r\mu^\sigma(t))e_{-\delta}(\sigma(t),0)[u^{'}(C^\sigma(t))]^\Delta - [\delta - r - r\mu^\Delta(t)]u^{'}(C^\sigma(t))\right]$$

$$+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \left[e_{-\delta}(\rho(t),0)[u^{'}(C^\rho(t))]^\nabla - [\delta - r]u^{'}(C^\rho(t))\right] = 0, \tag{3.14}$$

where μ, ν are assumed to be delta and nabla differentiable respectively, since the utility function is ϕ_h-concave ($u^{'} > 0, u^{''} < 0$). Thus (3.14) can be thought of as the growth rate of marginal utility.

Remark 3.2. *The new diamond-ϕ_h time scale model unifies both the delta and nabla models within a general framework for different classes of concave utility functions in that;*

(i) If $s = 1, \phi_h = \alpha; h(\lambda) = 1$, (3.14) gives

$$\alpha \left[(1+r\mu^\sigma(t))[u^{'}(C^\sigma(t))]^\Delta - [\delta - r - r\mu^\Delta(t)]u^{'}(C^\sigma(t))\right]$$

$$+ (1-\alpha)\left[(1+\delta\nu(t))[u^{'}(C^\rho(t))]^\nabla - [\delta - r]u^{'}(C^\rho(t))\right] = 0, \tag{3.15}$$

which is the growth rate of marginal utility for a diamond-α model when the utility function is concave $u^{'} > 0, u^{''} < 0$ and λ is chosen as α.

(ii) If $s = 1, \phi_h = \frac{1}{2}; h(\lambda) = 1, \lambda = \frac{1}{2}$, then the growth rate of marginal utility, equation (3.14) can be expressed for a mid-point-concave utility function, that is,

$$\frac{1}{2}\left[(1+r\mu^\sigma(t))[u^{'}(C^\sigma(t))]^\Delta - [\delta - r - r\mu^\Delta(t)]u^{'}(C^\sigma(t))\right]$$

$$+\frac{1}{2}\left[(1+\delta\nu(t))[u^{'}(C^\rho(t))]^\nabla(t) - [\delta - r]u^{'}(C^\rho)\right] = 0. \tag{3.16}$$

(iii) *If $s = 0$, we obtain an expression of the growth rate of marginal utility-(3.14) in terms of a P-concave function*

$$\Big[(1 + r\mu^{\sigma}(t))[u^{'}(C^{\sigma}(t))]^{\Delta} - [\delta - r - r\mu^{\Delta}]u^{'}(C^{\sigma}(t))\Big] + \Big[(1 + \delta\nu(t))[u^{'}(C^{\rho}(t))]^{\nabla} - [\delta - r]u^{'}(C^{\rho}(t))\Big] = 0. \tag{3.17}$$

(iv) *If $h(\lambda) = \lambda^{\frac{s}{s+1}}$, we obtain the growth rate of marginal utility for h-concave utility function on time scales;*

$$h(\lambda)\Big[(1 + r\mu^{\sigma}(t))[u^{'}(C^{\sigma}(t))]^{\Delta} - [\delta - r - r\mu^{\Delta}(t)]u^{'}(C^{\sigma}(t))\Big] + h(1-\lambda)\Big[(1 + \delta\nu(t))[u^{'}(C^{\rho}(t))]^{\nabla} - [\delta - r]u^{'}(C^{\rho}(t))\Big] = 0. \tag{3.18}$$

(v) *If $s = 1, h(\lambda) = 2\sqrt{\lambda(1-\lambda)}$, gives the growth rate of marginal utility (3.14) in terms of an MT-concave utility function on time scales;*

$$\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}\Big[(1 + r\mu^{\sigma}(t))[u^{'}(C^{\sigma}(t))]^{\Delta}(t) - [\delta - r - r\mu^{\Delta}]u^{'}(C^{\sigma}(t))\Big] + \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}\Big[(1 + \delta\nu(t))[u^{'}(C^{\rho})]^{\nabla}(t) - [\delta - r]u^{'}(C^{\rho}(t))\Big] = 0. \tag{3.19}$$

Remark 3.3. *The $\diamond_{\phi_h}$ time scale model unify the delta and nabla time scale models in a more general framework for the class of concave utility functions on time scales as follows;*

(i) *When $h(\lambda) = 1$, $s = 1$ and $\lambda = 1$ in equation (3.14), we obtain*

$$[u'(C^{\sigma}(t))]^{\Delta} = \frac{\delta - r - r\mu^{\Delta}(t)}{1 + r\mu^{\sigma}(t)}u'(C^{\sigma}(t)),$$

which is the growth rate of marginal utility of Guzowska et al. (2015) in terms of the delta derivative on time scale when the utility function is concave. Hence, the delta time scale model is a special case of the $\diamond_{\phi_h}$ time scale model.

(ii) *Choosing $h(\lambda) = 1$, $s = 1$ and $\lambda = 0$, in equation (3.14) gives*

$$[u'(C(t))]^{\nabla} = \frac{\delta - r}{1 + \delta\nu^{\rho}(t)}u'(C(t)),$$

the growth rate of marginal utility of Atici et al. (2006) in terms of the nabla derivative on time scale when the utility function is concave. Thus, the nabla time scale model is a special case of the $\diamond_{\phi_h}$ time scale model.

Remark 3.4. *The $\diamond_{\phi_h}$ time scale model unify the conventional discrete and continuous time models in a much more general framework for concave and classes of concave utility functions on classical intervals as follows;*

(i) *When $\mathbb{T} = \mathbb{Z}$, then $\mu(t) = \nu(t) = 1$ and equation (3.14) yields equation*

$$u'(C(t)) = \frac{1+\delta}{1+r}u'(C(t+1)),$$

which is the expression for the growth rate of marginal utility in the discrete model. Hence, the discrete model is a special case of the $\diamond_{\phi_h}$ time scale model.

(ii) When $\mathbb{T} = \mathbb{R}$, then $\mu(t) = \nu(t) = 0$, equation (3.14) gives the growth rate of marginal utility for the continuous time model with a concave utility function, i.e,

$$C'(t) = (\delta - r)\frac{u'(C(t))}{u''(C(t))}.$$

Hence, the continuous model is a special case of the time scale $\diamond_{\phi_h}$ model.

Remark 3.5. *A couple of many other potential advantages that the diamond-ϕ_h time scale calculus brings to economics aside those pointed out above, are discussed below;*

(i) In the two conventional setups, the growth rate of consumption given in Remark 3.3(i&ii) is constant because it is determined by δ and r; when the discount rate, $\delta - r = 0$, the consumption level does not depend on the time scale. Whereas, the new diamond-ϕ_h time scale calculus model implies that the growth rate of consumption (3.14) can change depending on the time scale due to $\mu(t)$ and $\nu(t)$; it is positive if $\delta < r$ and negative when $\delta > r$. Therefore, if the market interest rate r is higher than the internal rate of preference δ, the consumer will wait to consume until later periods. If $\delta > r$, the consumer is impatient and will consume more in the earlier periods and less in the future periods.
So, if consumption data are collected at fixed intervals but the time scale is such that consumption occurs with varying frequency, even if delta and r are constant, we would see fluctuations in the observed growth rate of consumption. Thus, time scale model gives information for a problem with unevenly spaced intervals, for which the standard continuous and discrete models do not.

(ii) The new diamond-ϕ_h model gives unification of existing conventional and time scale models at a glance. Hence its method of analysis is not time consuming as against the existing models.

In what follows, we discuss some other potential contributions of the time scale $\diamond_{\phi_h}$ calculus to economics.

The dynamic optimization problem (3.5)-(3.6), involve a case in which both the value function (3.5) and the constraints (3.6) are on the same time scale. In the following examples, we present dynamic optimization problems in which the value function and constraints can be on different time scales. In such a dynamic optimization problem, a finitely-lived agent seeks to maximize his or her lifetime utility, which is a function of consumption. We suppose that consumption $C(t)$ takes place on some time scales $\mathbb{T}_c$, income $Y(t)$ arrives on a time scale $\mathbb{T}_y$ and asset/debt accrues on time scales $\mathbb{T}_A$ with a joint time scale defined as $\mathbb{T} = \mathbb{T}_c \cup \mathbb{T}_y \cup \mathbb{T}_A$ and $m(t) = \max\{\eta \leq t : \eta \in \mathbb{T}_c\}$, where $t \in \mathbb{T}$.
Since the interest accrues on its own time scale $\mathbb{T}_A$, while t is the point of intersection of the union of all three time scales, then the rate

$$r(t) = \begin{cases} r, & if\ t \in \mathbb{T}_A \\ 0, & otherwise. \end{cases}$$

and the indicator function

$$I(t) = \begin{cases} 1, & if\ t \in \mathbb{T}_y \\ 0, & otherwise. \end{cases}$$

The objective function (3.5) can now be written as

$$\max \quad U_{\diamond_{\phi_h}} = \int_0^T u(C(m(t)))e_{-\delta}(m(t),\,0) \diamond_{\phi_h} t, \tag{3.20}$$

subject to the constraints (3.6), which in the terms of (1.3) takes the form

$$\max\ U_{\diamond_{\phi_h}} = \left(\frac{\lambda}{h(\lambda)}\right)^s \int_0^T u(C^\sigma(m(t)))e_{-\delta}(m(t),\,0)\Delta t + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \int_0^T u(C^\rho(m(t)))e_{-\delta}(m(t),\,0)\nabla t \tag{3.21}$$

subject to the budget constraints

$$a^\nabla(t) = (r(t)A(t) + I(t)Y(t) - C(m(t)))(\rho(t)), \qquad t \in [\sigma(0), T],$$

$$a^\Delta(t) = \frac{r}{1\,+\,r(t)\mu(t)}a^\sigma(t) + \frac{I(t)}{1 + r(t)\mu(t)}y^\sigma(t) - \frac{1}{1 + r(t)\mu(t)}c^\sigma(m(t)), \tag{3.22}$$

$a(0) = a_0, \quad a(T) = a_T,$ where $U(.)$ is the utility function and δ is the subjective rate of preference. Clearly, we can use Theorem 2.3 to derive the necessary optimality conditions for problem (3.21)-(3.22).

The following examples show consumption paths for various choices of $\mathbb{T}_c$ and $\mathbb{T}_A$ in the problem (3.21)-(3.22), with the assumption that $\mathbb{T}_y = \mathbb{T}_A$.

Example 3.6. *If we choose $\phi_h = 1; h(\lambda) = 1, s = 1$ and $\lambda = 1$ in (3.21), we derive*

$$u'(C^\sigma(m(t))) = \frac{K}{1 + r\mu(t)} \frac{1}{e_r(\sigma(t),\,0)e_{-\delta}(m(t),\,0)}, \tag{3.23}$$

where K is a constant.

This gives the discrete-time version of the E-L equation by setting $\mathbb{T}_c = \mathbb{T}_y$ and $\mathbb{T}_A = \mathbb{Z}$. Thus equation (3.23) becomes

$$(u'(C(t+1)))\Pi_0^{t+1}\frac{1}{1+\delta} = \frac{1}{\Pi_0^{t+1}(1+r)}$$

From the above expression, we can easily find the Euler equation as

$$\frac{u'(C(t+1))}{u'(C(t))}\frac{\Pi_0^{t+1}\frac{1+\delta}{1+r}}{\Pi_0^{t}\frac{1+\delta}{1+r}} = \frac{1+\delta}{1+r},$$

which is a solution to the conventional discrete maximization problem.

Example 3.7. *If we choose $\phi_h = 1; h(\lambda) = 1, s = 1$ and $\lambda = 1$ in (3.21), we derive*

$$u'(C^\sigma(m(t))) = \frac{K}{1 + r\mu(t)} \frac{1}{e_r(\sigma(t),\,0)e_{-\delta}(m(t),\,0)}, \tag{3.24}$$

where K is a constant.

This gives the discrete-time version of the E-L equation by setting $\mathbb{T}_c = \mathbb{T}_y$ and $\mathbb{T}_A = \mathbb{Z}$. Thus equation (3.23) becomes

$$(u'(C(t+1)))\Pi_0^{t+1}\frac{1}{1+\delta} = \frac{1}{\Pi_0^{t+1}(1+r)}$$

From the above expression, we can easily find the Euler equation as

$$\frac{u'(C(t+1))}{u'(C(t))}\frac{\Pi_0^{t+1}\frac{1+\delta}{1+r}}{\Pi_0^{t}\frac{1+\delta}{1+r}} = \frac{1+\delta}{1+r},$$

$$(u'(C(t+1)))\Pi_0^{t+1}\frac{1}{1+\delta} = \frac{1}{\Pi_0^{t+1}(1+r)}$$

From the above expression, we can easily find the Euler equation as

$$\frac{u'(C(t+1))}{u'(C(t))}\frac{\Pi_0^{t+1}\frac{1+\delta}{1+r}}{\Pi_0^{t}\frac{1+\delta}{1+r}} = \frac{1+\delta}{1+r},$$

which is a solution to the conventional discrete maximization problem.

Example 3.8. *If we choose $\phi_h = 1; h(\lambda) = 1, s = 1$ and $\lambda = 1$ in (3.21), $\mathbb{T}_c = h_c\mathbb{Z}, \mathbb{T}_A = h_A\mathbb{Z}$, consumption takes place at discrete points in time, say $\mathbb{T}_c = 0, 3, 6, 9, ...$ and asset is traded at discrete points in time, say $\mathbb{T}_A = 0, 2, 4, 6, ...$ which do not necessarily coincide with consumption points. The optimal behaviour for the consumer will vary depending on how many 'assets points' there are between 'consumption points'. Thus, we can then compare marginal rate of substitution between points $t = 0$ and $t = 3$ as against $t = 3$ and $t = 6$ first, by finding marginal utilities at each of those points:*

$$u'(0) = \frac{K}{e_r(2,\, 0)e_{-\delta}(0,\, 0)}$$

$$u'(c(3)) = \frac{K}{e_r(4,\, 0)e_{-\delta}(3,\, 0)}$$

$$u'(c(6)) = \frac{K}{e_r(8,\, 0)e_{-\delta}(6,\, 0)},$$

where K is a constant. Hence, by choosing $\delta = 0.1, r = 0.03$, the marginal rate of substitution between $t = 0$ and $t = 3$ is

$$\frac{u'(c(3))}{u'(c(6))} = \frac{1}{e_r(4,\, 2)e_{-\delta}(3,\, 0)} = \frac{1+3\delta}{(1+2r)^2} = 1.16,$$

and by choosing $\delta = 0.1, r = 0.03$, the marginal rate of substitution between $t = 3$ and $t = 6$ is

$$\frac{u'(c(6))}{u'(c(3))} = \frac{1}{e_r(8,\, 4)e_{-\delta}(6,\, 3)} = \frac{1+3\delta}{1+2r} = 2.36.$$

Also, by choosing $\delta = 0.5, r = 0.03$, the marginal rate of substitution between $t = 0$ and $t = 3$ is 2.22 and by choosing $\delta = 0.5, r = 0.03$, the marginal rate of substitution between $t = 3$ and $t = 6$ is 2.36. Thus, the values of the marginal rate of substitution between two points depends on the choices of δ and r. The results are consistent with economic intuition: the trade-off between two periods is a function of interest rate and internal discount rate, which varies from period to period depending on how many times the interest has been added to the principal.

4. Conclusion

A dynamic optimization problem in economics was constructed and analysed with the aid of the new diamond-ϕ_h calculus of variations and optimal control theory, in a more general framework of the time scale theory. The diamond-ϕ_h Optimization model (3.1-3.3) on time scales allows us to handle both the delta and nabla time scale models at the same time, which in turn allows us handle discrete and continuous models as being two pieces of the same framework involving more complex time domains.

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