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The new results in extended b-metric spaces and applications

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Abstract

In this paper, we give a proof of the results of Miculescu and Mihail [J. Fixed Point Theory Appl., 19 (2017), 2153-2163] and Suzuki [J. Inequal. Appl., 2017:256 (2017)] in extended b-metric spaces. Also we give some applications of our result.

Keywords: Fixed point, Extended b-metric space, Multi-valued mapping.
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1. Introduction and Preliminaries

Kamran et al. [15] introduced the concept of extended b-metric space, which is not necessarily Hausdorff and which generalizes the concepts of metric space and b-metric space.

Definition 1.1. *[15] Let $\mathcal{M}$ be a nonempty set and $e : \mathcal{M} \times \mathcal{M} \to [1, \infty)$. Let the mapping $d_e : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ satisfies:*
(EbM1) $d_e(u, v) = 0$ *if and only if* $u = v$;
(EbM2) $d_e(u, v) = d_e(v, u)$ *for all* $u, v \in \mathcal{M}$;
(EbM3) $d_e(u, z) \leq e(u, z)[d_e(u, v) + d_e(v, z)]$ *for all* $u, v, z \in \mathcal{M}$.
Then d_e is called an extended b-metric on $\mathcal{M}$ with mapping e and $(\mathcal{M}, d_e)$ is called a extended b-metric space.

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Definition 1.2. *[15] Let $(\mathcal{M}, d_e)$ be a extended rectangular b-metric space, $\{u_n\}$ be a sequence in $\mathcal{M}$ and $u \in \mathcal{M}$. Then the sequence $\{u_n\}$ is said to be*

(a) *convergent in $(\mathcal{M}, d_e)$ and converges to u, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d_e(u_n, u) < \varepsilon$ for all $n > n_0$ and this fact is represented by $\lim_{n\to\infty} u_n = u$ or $u_n \to u$ as $n \to \infty$.*

(b) *Cauchy sequence in $(\mathcal{M}, d_e)$ if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d_e(u_m, u_n) < \varepsilon$ for all $m, n \geq n_0$.*

Definition 1.3. *[15] $(\mathcal{M}, d_e)$ is said to be a complete extended b-metric space if every Cauchy sequence in $\mathcal{M}$ converges to some $u \in \mathcal{M}$.*

In the papers of Bakhtin [4] and Czerwik [5] the notion of b-metric space was introduced and some fixed point theorems in those spaces were proved.

Definition 1.4. *Let $\mathcal{M}$ be a nonempty set and let $b \geq 1$ be a given real number. A function $d : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ is said to be a b-metric if for all $u, v, z \in \mathcal{M}$ the following conditions are satisfied:*

(1) $d(u, v) = 0$ *if and only if* $u = v$;

(2) $d(u, v) = d(v, u)$;

(3) $d(u, z) \leq b[d(u, v) + d(v, z)]$.

A triplet $(\mathcal{M}, d, b)$ is called a b-metric space.

Note that a metric space is included in the class of b-metric spaces. In fact, the notions of convergent sequence, Cauchy sequence and complete space are defined as in metric spaces. We have the following diagram where arrows stand for inclusions of various generalizations of metric space. The inverse inclusions do not hold.

metric space, [8]	$\rightarrow$	b-metric space, [4, 5]
		$\downarrow$
		extended b-metric space, [15]

Singh et al. [24] obtained the following result (see also Lemma 3.1 in [12]).

Lemma 1.5. *(Lemma 3.1 in [24]) Let $(\mathcal{M}, d, b)$ be a b-metric space and let $\{u_n\}$ be a sequence in $\mathcal{M}$. Assume that there exists $\lambda \in [0, 1/b)$ satisfying $d(u_{n+1}, u_n) \leq \lambda d(u_n, u_{n-1})$ for any $n \in \mathbb{N}$. Then $\{u_n\}$ is Cauchy.*

Miculescu and Mihail [17] (Lemma 2.2) and Suzuki [26] (Lemma 6) proved that in Lemma 1.5, we can extend the range of λ to the case $0 < \lambda < 1$.

Alqahtani et al. [2] obtained the following result.

Lemma 1.6. *(Lemma 2 in [2]) Let $(\mathcal{M}, d_e)$ be an extended b-metric space and let $\{u_n\}$ be a sequence in $\mathcal{M}$. Assume that there exists $\lambda \in [0, 1)$ satisfying*

$$d_e(u_{n+1}, u_n) \leq \lambda d_e(u_n, u_{n-1}),$$

for any $n \in \mathbb{N}$. If

$$\lim_{m,n\to\infty} e(u_m, u_n) < \frac{1}{\lambda}, \tag{1.1}$$

then $\{u_n\}$ is Cauchy.

Remark 1.7. *Note that if* $\lim_{m,n\to\infty} e(u_m, u_n)$ *does not exists then Lemma 1.6 is valid if, instead of condition (1.1), we use the condition* $\limsup_{m,n\to\infty} e(u_m, u_n) < \frac{1}{\lambda}$.

In this paper, we give a proof result of Suzuki, Miculescu and Mihail framework extended b-metric spaces. We will show that instead of condition (1.1) can be used weaker condition

$$\limsup_{m,n\to\infty} e(u_m, u_n) < \infty. \tag{1.2}$$

Also we give some applications.

2. Main Results

In this section we using some ideas from [19].

Lemma 2.1. *Let* $(\mathcal{M}, d_e)$ *be an extended b-metric space and let* $\{u_n\}$ *be a sequence in* $\mathcal{M}$. *Then for all* $n, p \in \mathbb{N}$,

$$\begin{aligned} d_e(u_n, u_{n+p}) \leq\ & e(u_n, u_{n+p}) e(u_{n+1}, u_{n+p}) \cdots e(u_{n+p-2}, u_{n+p}) \\ & \times [d_e(u_n, u_{n+1}) + d_e(u_{n+1}, u_{n+2}) + \cdots + d_e(u_{n+p-1}, u_{n+p})], \end{aligned}$$

holds.

Proof . Obvious. □

Lemma 2.2. *Let* $(\mathcal{M}, d_e)$ *be an extended b-metric space and let* $\{u_n\}$ *be a sequence in* $\mathcal{M}$ *and let* $\lambda \in [0, 1)$ *and* $C > 0$ *such that*

$$d_e(u_{n+1}, u_n) \leq C\lambda^n,$$

for all $n \in \mathbb{N}$. *If*

$$\lim_{m,n\to\infty} e(u_m, u_n) < \frac{1}{\lambda},$$

then $\{u_n\}$ *is Cauchy.*

Proof . It follows directly from Lemma 1.6 (see the proof of Lemma 2 in [2]). □

Lemma 2.3. *Let* $(\mathcal{M}, d_e)$ *be an extended b-metric space such that the condition (1.2) is fulfilled and let* $\{u_n\}$ *be a sequence in* $\mathcal{M}$. *Assume that there exists* $\lambda \in (0, 1)$ *satisfying*

$$d_e(u_{n+1}, u_n) \leq \lambda d_e(u_n, u_{n-1}), \tag{2.1}$$

for all $n \in \mathbb{N}$. *Let* $n_0 \in \mathbb{N}$ *such that* $n_0 > -\frac{\log \limsup_{m,n\to\infty} e(u_m, u_n)}{\log \lambda}$. *Then*

1. $\{u_{nn_0}\}$ *is Cauchy,*
2. $d_e(u_n, u_{n_0\lfloor \frac{n}{n_0} \rfloor}) \to 0$ *as* $n \to \infty$.

Proof . Let $Q = \lim_{m,n\to\infty} \sup e(u_m, u_n) < \infty$.

1. Using Lemma 2.1 and condition (2.1) we get the following

$$\begin{aligned} d_e(u_{(n+1)n_0}, u_{nn_0}) &\leq Q^{n_0}[d_e(u_{(n+1)n_0}, u_{(n+1)n_0-1}) + \cdots + d_e(u_{nn_0+1}, u_{nn_0})] \\ &\leq Q^{n_0}(\lambda^{(n+1)n_0-1} + \cdots + \lambda^{nn_0})d_e(u_1, u_0) \\ &\leq Q^{n_0}\lambda^{nn_0}\frac{d_e(u_1, u_0)}{1-\lambda} \\ &\leq C\mu^n, \end{aligned}$$

where $C = Q^{n_0}\frac{d_e(u_1,u_0)}{1-\lambda}$ and $\mu = \lambda^{n_0}$. Since $n_0 > -\frac{\log Q}{\log \lambda}$, we have that

$$\limsup_{m,n\to\infty} e(u_m, u_n) < \frac{1}{\mu}.$$

So, from Lemma 2.2 we conclude that $\{u_{nn_0}\}$ is Cauchy.

2.

$$\begin{aligned} d_e(u_n, u_{n_0\lfloor\frac{n}{n_0}\rfloor}) &\leq Q^{n_0}[d_e(u_n, u_{n-1}) + \cdots + d_e(u_{n_0\lfloor\frac{n}{n_0}\rfloor+1}, u_{n_0\lfloor\frac{n}{n_0}\rfloor})] \\ &\leq Q^{n_0}(\lambda^{n-1} + \cdots + \lambda^{n_0\lfloor\frac{n}{n_0}\rfloor})d_e(u_1, u_0) \\ &\leq Q^{n_0}\lambda^{n_0\lfloor\frac{n}{n_0}\rfloor}\frac{d_e(u_1, u_0)}{1-\lambda}. \end{aligned}$$

So, $d_e(u_n, u_{n_0\lfloor\frac{n}{n_0}\rfloor}) \to 0$ as $n \to \infty$. □

Lemma 2.4. *Let $(\mathcal{M}, d_e)$ be an extended b-metric space such that the condition (1.2) is fulfilled and let $\{u_n\}$ be a sequence in $\mathcal{M}$. Assume that there exists $\lambda \in [0, 1)$ satisfying $d_e(u_{n+1}, u_n) \leq \lambda d_e(u_n, u_{n-1})$ for any $n \in \mathbb{N}$. Then $\{u_n\}$ is Cauchy.*

Proof . The case $\lambda = 0$ is obvious. Let $\lambda \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that $n_0 > -\frac{\log \limsup_{m,n\to\infty} e(u_m,u_n)}{\log \lambda}$. Then the proof follows from Lemma 2.3 and the following inequality

$$\begin{aligned} d_e(u_n, u_m) \leq\ & e(u_n, u_{n_0\lfloor\frac{n}{n_0}\rfloor})e(u_{n_0\lfloor\frac{n}{n_0}\rfloor}), u_m)\Big[d_e(u_n, u_{n_0\lfloor\frac{n}{n_0}\rfloor}) \\ &+ d_e(u_{n_0\lfloor\frac{n}{n_0}\rfloor}, u_{n_0\lfloor\frac{m}{n_0}\rfloor}) + d_e(u_{n_0\lfloor\frac{m}{n_0}\rfloor}, u_m)\Big]. \end{aligned}$$

□

Example 2.5. *Let $\mathcal{M} = [0, \infty)$, $e : \mathcal{M} \times \mathcal{M} \to [1, \infty)$ and $d_e : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ such that*

$$e(u, v) = u + v + 2, \quad d_e(u, v) = (u - v)^2.$$

Then $(\mathcal{M}, d_e)$ is extended b-metric space. Define sequence $\{u_n\}$ by $u_0 \in \mathcal{M}$ and $u_n = \frac{3u_{n-1}}{4}$, for all $n \in \mathbb{N}$. Then we have

$$d_e(u_{n+1}, u_n) \leq \lambda d_e(u_n, u_{n-1}),$$

for all $n \in \mathbb{N}$, where $\lambda = \frac{9}{16}$. Then we have $\lim_{m,n\to\infty} e(u_m, u_n) = 2$. So, $\lim_{m,n\to\infty} e(u_m, u_n) > \frac{16}{9}$ and Lemma 1.6 not applicable. In other hand, from Lemma 2.4, we conclued that $\{u_n\}$ is Cauchy.

Remark 2.6. *Note that, $e(u, v) = b$ for any $b \geq 1$, then condition (1.2) is fulfilled and from Lemma 2.4, we obtain the results of Suzuki [26] and Miculescu and Mihail [17] (see, also [19]).*

3. Applications for single-valued mappings

In this section, we prove a theorem of Reich (see [21]) in extended b-metric spaces.

Theorem 3.1. *Let $(\mathcal{M}, d_e)$ be a complete extended b-metric space such that d_e is a continuous functional and the condition (1.2) is fulfilled. Let $\mathcal{T}: \mathcal{M} \to \mathcal{M}$ be a mapping satisfying:*

$$d_e(\mathcal{T}u, \mathcal{T}v) \leq \alpha d_e(u, v) + \beta d_e(u, \mathcal{T}u) + \gamma d_e(v, \mathcal{T}v), \quad \text{for all } u, v \in \mathcal{M}, \tag{3.1}$$

where α, β, γ non-negative real numbers with $\alpha + \beta + \gamma < 1$. Then $\mathcal{T}$ has a unique fixed point $u^* \in \mathcal{M}$. Moreover, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$ converges to u^* such that $u_{n+1} = \mathcal{T}u_n$ for every $n \in \mathbb{N}$.

Proof . Let $u_0 \in \mathcal{M}$ and $\{u_n\}$ be a sequence satisfying (1.2) such that $u_n = \mathcal{T}u_{n-1} = \mathcal{T}^n u_0$. From condition (3.1), we have

$$d_e(u_{n+1}, u_n) \leq \alpha d_e(u_n, u_{n-1}) + \beta d_e(u_n, u_{n+1}) + \gamma d_e(u_{n-1}, u_n).$$

Therefore,

$$d_e(u_{n+1}, u_n) \leq \frac{\alpha + \gamma}{1 - \beta} d_e(u_{n-1}, u_n).$$

Set $\lambda = \frac{\alpha+\gamma}{1-\beta}$. Then, we have that $\lambda \in [0, 1)$. Hence, by Lemma 2.4, we obtain that $\{u_n\}$ is a Cauchy sequence in $\mathcal{M}$. By completeness of $(\mathcal{M}, d_e)$, there exists $z \in \mathcal{M}$ such that

$$\lim_{n\to\infty} u_n = z.$$

Now, we claim that z is the unique fixed point of $\mathcal{T}$. For that,

$$\begin{aligned} d_e(\mathcal{T}z, u_{n+1}) &= d_e(\mathcal{T}z, \mathcal{T}u_n) \\ &\leq \alpha d_e(z, u_n) + \beta d_e(z, \mathcal{T}z) + \gamma d_e(u_n, u_{n+1}) \\ &\leq \alpha d_e(z, u_n) + \beta d_e(z, \mathcal{T}z) + \gamma e(u_n, u_{n+1})[d_e(u_n, z) + d(z, u_{n+1})]. \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we deduce

$$d_e(\mathcal{T}z, z) \leq \beta d_e(\mathcal{T}z, z),$$

which implies that $d_e(\mathcal{T}z, z) = 0$ and so $z = \mathcal{T}z$. Moreover the uniqueness can easily be obtained by using inequality (3.1). □

Remark 3.2. *Note that the Reich result can be obtained without the continuity conditions for d_e, but we need stronger conditions for β and γ (see Theorem 2.3 in [18]).*

If we take $\beta = \gamma = 0$ in (3.1), from Theorem 3.1, we obtain the following result.

Corollary 3.3. *(Theorem 2 in [15]) Let $(\mathcal{M}, d_e)$ be a complete extended b-metric space such that d_e is a continuous functional and the condition (1.2) is fulfilled. Let $\mathcal{T}: \mathcal{M} \to \mathcal{M}$ be a mapping satisfying:*

$$d_e(\mathcal{T}u, \mathcal{T}v) \leq \alpha d_e(u, v), \quad \text{for all } u, v \in \mathcal{M},$$

where $\alpha \in [0, 1)$. Then $\mathcal{T}$ has a unique fixed point. $u^* \in \mathcal{M}$. Moreover, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$ converges to u^* such that $u_{n+1} = \mathcal{T}u_n$ for every $n \in \mathbb{N}$.

If we take $\alpha = 0$ in (3.1), by Theorem 3.1, we obtain the following variant of Kannan theorem [14] in extended b-metric spaces.

Corollary 3.4. *Let $(\mathcal{M}, d_e)$ be a complete extended b-metric space such that d_e is a continuous functional and the condition (1.2) is fulfilled. Let $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be a mapping satisfying:*

$$d_e(\mathcal{T}u, \mathcal{T}v) \leq \beta d_e(u, \mathcal{T}u) + \gamma d_e(v, \mathcal{T}v), \quad \text{for all } u, v \in \mathcal{M},$$

where β, γ non-negative real numbers with $\beta + \gamma < 1$. Then $\mathcal{T}$ has a unique fixed point $u^* \in \mathcal{M}$. Moreover, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$ converges to u^* such that $u_{n+1} = \mathcal{T}u_n$ for every $n \in \mathbb{N}$.

Remark 3.5. *Possible directions for further research in extended b-metric spaces for single valued maps:*
1. The fixed-point theorem of Hardy-Rogers, see [9];
2. The theorem of Ćirić, see [6];
3. The results of the common fixed points, see for example [7] and [13];
4. The Meir-Keeler result of fixed points, see [16];
5. The Sehgal theorem, see [22].

4. Applications for multi-valued mappings

In this section, using Lemma 2.4, we prove the result of Nadler (see [20]) in extended b-metric spaces.

4.1. Fixed point results on the space $K(\mathcal{M})$

Definition 4.1. *[25] Let $(\mathcal{M}, d_e)$ be an extended b-metric space and denote the family of nonempty and compact subsets of $\mathcal{M}$ by $K(\mathcal{M})$. For $A, B \in K(\mathcal{M})$, define $H_e \colon K(\mathcal{M}) \times K(\mathcal{M}) \to \mathbb{R}^+$ by*

$$H_e(A, B) = \max\left\{\sup_{a\in A} d_e(a, B),\ \sup_{b\in B} d_e(b, A)\right\},$$

where $d_e(a, B) = \inf\{d_e(a, b) \colon b \in B\}$. The mapping H_e is called the Pompeiu-Hausdorff metric induced by d_e.

Lemma 4.2. *[25] If $(\mathcal{M}, d_e)$ is a complete extended b-metric space, then also $(K(\mathcal{M}), H_e)$ is complete.*

Now, we can have main theorem of this subsection.

Theorem 4.3. *Let $(\mathcal{M}, d_e)$ be a complete extended b-metric space such that d_e is a continuous functional and the condition (1.2) is fulfilled. Let $\mathcal{T} : \mathcal{M} \to K(\mathcal{M})$ be a mapping satisfying:*

$$H_e(\mathcal{T}u, \mathcal{T}v) \leq \lambda d_e(u, v), \quad \text{for all } u, v \in \mathcal{M}, \tag{4.1}$$

where $\lambda \in [0, 1)$. Then $\mathcal{T}$ has a fixed point $u^* \in \mathcal{M}$. Moreover, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$ converges to u^* such that $u_{n+1} \in \mathcal{T}u_n$ for every $n \in \mathbb{N}$.

Proof . Let u_0 be an arbitrary point of $\mathcal{M}$ and $u_1 \in \mathcal{T}u_0$. If $u_0 = u_1$ or $u_1 \in \mathcal{T}u_1$, then u_1 is a fixed point of $\mathcal{T}$ and so the proof is completed. Because of this, assume that $x_0 \neq u_1$ and $u_1 \notin \mathcal{T}u_1$, then $d_e(u_1, \mathcal{T}u_1) > 0$ and hence $H_e(\mathcal{T}u_0, \mathcal{T}u_1) > 0$. Since $\mathcal{T}u_1$ is compact, there exists $u_2 \in \mathcal{T}u_1$ such that $d_e(u_1, u_2) = d_e(u_1, \mathcal{T}u_1)$. From (4.1), we have

$$d_e(u_1, u_2) = d_e(u_1, \mathcal{T}u_1) \leq H_e(\mathcal{T}u_0, \mathcal{T}u_1) \leq \lambda d_e(u_0, u_1). \tag{4.2}$$

Following the previous procedures, we can assume that $u_1 \neq u_2$ and $u_2 \notin \mathcal{T}u_2$. Then $d_e(u_2, \mathcal{T}u_2) > 0$, and so $H_e(\mathcal{T}u_1, \mathcal{T}u_2) > 0$. Since $\mathcal{T}u_2$ is compact, there exists $u_3 \in \mathcal{T}u_2$ such that $d_e(u_2, u_3) = d_e(u_2, \mathcal{T}u_2)$. Applying (4.1), we get

$$d_e(u_2, u_3) = d_e(u_2, \mathcal{T}u_2) \leq H_e(\mathcal{T}u_1, \mathcal{T}u_2) \leq \lambda d_e(u_1, u_2). \tag{4.3}$$

Repeating this process, we can constitute a sequence $\{u_n\} \subset \mathcal{M}$ such that $u_n \neq u_{n+1} \in \mathcal{T}u_n$ and

$$d_e(u_n, u_{n+1}) \leq \lambda d_e(u_{n-1}, u_n), \qquad \text{for all } n \in \mathbb{N}. \tag{4.4}$$

Hence, by Lemma 2.4, we deduce that $\{u_n\}$ is a Cauchy sequence in $\mathcal{M}$. From the completeness of $(\mathcal{M}, d_e)$, there exists $u \in \mathcal{M}$ such that $u_n \to z$ as $n \to \infty$. We now show that z is a fixed point of $\mathcal{T}$. Using triangle inequality and (4.1), we have

$$\begin{aligned} d_e(z, \mathcal{T}z) &\leq e(z, \mathcal{T}z)[d_e(z, u_{n+1}) + d_e(u_{n+1}, \mathcal{T}z)] \\ &\leq e(z, \mathcal{T}z)[d_e(z, u_{n+1}) + H_e(\mathcal{T}u_n, \mathcal{T}z)] \\ &\leq e(z, \mathcal{T}z)[d_e(z, u_{n+1}) + \lambda d_e(u_n, z)]. \end{aligned}$$

Passing to limit as $n \to \infty$ in the above inequality, we obtain $d_e(z, \mathcal{T}z) \leq 0$ and hence $d_e(z, \mathcal{T}z) = 0$. Since $\mathcal{T}z$ is closed, we deduce that $z \in \mathcal{T}z$. □

4.2. Fixed point results on the space $CB(\mathcal{M})$

Let $(\mathcal{M}, d_e)$ be an extended b-metric space. Denote $CB(\mathcal{M})$ be the collection of nonempty closed bounded subsets of $\mathcal{M}$ and $CL(\mathcal{M})$ be the class of all nonempty closed subsets of $\mathcal{M}$. For each $x \in \mathcal{M}$ and all $A, B \in CL(\mathcal{M})$, we define

$$d_e(x, A) = \inf_{a \in A} d_e(x, a),$$

$$D_e(A, B) = \sup\{d_e(a, B) : a \in A\}.$$

Then the extended Pompeiu-Hausdorff b-metric H_e on $CL(\mathcal{M})$ inducted by extended b-metric d_e is defined as

$$H_e(A, B) = \begin{cases} \max\{D_e(A, B), D_e(B, A)\}, & \text{if the maximum exists,} \\ +\infty, & \text{otherwise,} \end{cases}$$

for all $A, B \in CL(\mathcal{M})$. The following results are useful for the proof main result in the subsection.

Theorem 4.4. *If $(\mathcal{M}, d_e)$ is a complete extended b-metric space, then $(CL(\mathcal{M}), H_e)$ where H_e means the extended Pompeiu-Hausdorff b-metric induced by d_e, is also an extended complete b-metric space.*

Proof . Let $A, B, C \in CL(\mathcal{M})$. Obviously, $H_e(A, B) = 0$ of and only if $A = B$ and $H_e(A, B) = H_e(B, A)$. We define the mapping $E : CL(\mathcal{M}) \times CL(\mathcal{M}) \to [0, \infty)$ with

$$E(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} e(a, b), \sup_{b \in B} \inf_{a \in A} e(b, a)\}. \tag{4.5}$$

We will show that it is worth

$$H_e(A,B) \leq E(A,B)[H_e(A,C) + H_e(C,B)]. \tag{4.6}$$

For all $a \in A, b \in B, c \in C$ we have

$$\begin{aligned} d_e(a,B) &= \inf_{b\in B} d_e(a,b) \\ &\leq d_e(a,b) \\ &\leq e(a,b)[d_e(a,c) + d_e(c,b). \end{aligned}$$

Thus,

$$\begin{aligned} d_e(a,B) &\leq \inf_{b\in B} e(a,b)[d_e(a,c) + d_e(c,B) \\ &\leq \inf_{b\in B} e(a,b)[d_e(a,c) + \sup_{c\in C} d_e(c,B). \end{aligned}$$

So,

$$d_e(a,B) \leq \inf_{b\in B} e(a,b)[d_e(a,C) + d_e(c,B)],$$

from here we conclude

$$\sup_{a\in A} d_e(a,B) \leq \sup_{a\in A} \inf_{b\in B} e(a,b)[\sup_{a\in A} d_e(a,C) + \sup_{c\in C} d_e(c,B)],$$

that is,

$$D_e(A,B) \leq E(A,B)[D_e(A,C) + D_e(C,B)]. \tag{4.7}$$

Similarly, it is obtained

$$D_e(B,A) \leq E(A,B)[D_e(C,A) + D_e(B,C)]. \tag{4.8}$$

From (4.7) and (4.8), we obtain (4.6). □

Lemma 4.5. *Let $(\mathcal{M}, d_e)$ be an extended b-metric space and $A, B \in CB(\mathcal{M})$. Then, for each $a \in A$ and $\epsilon > 0$, there exists $b \in B$ such that*

$$d_e(a,b) \leq H_e(A,B) + \epsilon.$$

Proof . Obviously. □

Definition 4.6. *Let $(\mathcal{M}, d_e)$ be an extended b-metric space. A mapping $\mathcal{T} : \mathcal{M} \to CB(\mathcal{M})$ is called closed, if for all sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ of elements from $\mathcal{M}$ and $u, v \in \mathcal{M}$ such that $\lim_{n\to\infty} u_n = u$, $\lim_{n\to\infty} v_n = v$ and $v_n \in \mathcal{T}(u_n)$ for each $n \in \mathbb{N}$, we have $v \in \mathcal{T}(u)$.*

Definition 4.7. *Let $(\mathcal{M}, d_e)$ be a the extended b-metric spac. The extended b-metric d_e is called $*$-continuous, if for all $A \in CB(\mathcal{M})$, $u \in \mathcal{M}$ and each sequence $(u_n)_{n\in\mathbb{N}}$ of elements from $\mathcal{M}$ such that $\lim_{n\to\infty} u_n = u$, we have $\lim_{n\to\infty} d_e(u_n, A) = d_e(u, A)$.*

Theorem 4.8. *Let $(\mathcal{M}, d_e)$ be an extended b-metric space such that the condition (1.2) is fulfilled and $\mathcal{T} : \mathcal{M} \to CB(\mathcal{M})$ be a mapping satisfying:*

$$H(\mathcal{T}u, \mathcal{T}v) \leq q d_e(u,v), \tag{4.9}$$

for all $u, v \in \mathcal{M}$, where $q \in [0,1)$. Then, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$ converges to some point $x^* \in \mathcal{M}$ such that $u_{n+1} \in \mathcal{T}(u_n)$ for every $n \in \mathbb{N}$. Also, x^* is a fixed point of $\mathcal{T}$ if any of the following conditions are satisfied:

(i) $\mathcal{T}$ *is closed;*

(ii) d_e *is* $*$*-continuous.*

Proof . Let $u_0 \in \mathcal{M}$. Choose $u_1 \in \mathcal{T}u_0$. Let

$$\epsilon = \frac{1-q}{1+q} H(\mathcal{T}u_1, \mathcal{T}u_0).$$

If $H_e(\mathcal{T}u_1, \mathcal{T}u_0) = 0$, then we obtain $\mathcal{T}u_1 = \mathcal{T}u_0$ and $u_1 \in \mathcal{T}u_1$. In this case proof is hold. So, we may assume $\epsilon > 0$. From Lemma 4.5, then there is a point $u_2 \in \mathcal{T}u_1$ such that

$$d_e(u_1, u_2) \leq H_e(\mathcal{T}u_0, \mathcal{T}u_1) + \epsilon = \frac{2}{1+q} H(\mathcal{T}u_0, \mathcal{T}u_1).$$

Similarly, there is a point $u_3 \in \mathcal{T}u_2$ such that

$$d_e(u_2, u_3) \leq H_e(\mathcal{T}u_1, \mathcal{T}u_2) + \epsilon,$$

where

$$\epsilon = \frac{1-q}{1+q} H_e(\mathcal{T}u_2, \mathcal{T}u_1).$$

If $H_e(\mathcal{T}u_2, \mathcal{T}u_1) = 0$, then we obtain $\mathcal{T}u_2 = \mathcal{T}u_1$ and $u_2 \in \mathcal{T}u_2$. In this case proof is hold. So, we may assume $\epsilon > 0$. Hence,

$$d_e(u_2, u_3) \leq \frac{2}{1+q} H_e(\mathcal{T}u_1, \mathcal{T}u_2).$$

Continuing in this process, we produce a sequence $(u_n)_{n\in\mathbb{N}}$ of points of $\mathcal{M}$ such that

$$u_{n+1} \in \mathcal{T}u_n, \quad \text{for all } n \in \mathbb{N}, \tag{4.10}$$

and

$$d_e(u_n, u_{n+1}) \leq \frac{2}{1+q} H_e(\mathcal{T}u_{n-1}, \mathcal{T}u_n), \quad \text{for all } n \in \mathbb{N}. \tag{4.11}$$

It follows from (4.11) that

$$d_e(u_{n+1}, u_n) \leq \frac{2q}{1+q} d_e(u_n, u_{n-1}), \quad \text{for all } n \in \mathbb{N}. \tag{4.12}$$

Now, since $\frac{2q}{1+q} < 1$, Lemma 1.1 implies that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $(\mathcal{M}, d_e)$ is complete, the sequence $(u_n)_{n\in\mathbb{N}}$ converges to some point $u^* \in \mathcal{M}$.
(i) Suppose that $\mathcal{T}$ is closed. From Definition 4.6 and the equation (4.10), we have $u^* \in \mathcal{T}u^*$.
(ii) Suppose that d is $*$-continuous. Then, we have

$$\lim_{n\to\infty} d_e(u_n, \mathcal{T}u^*) = d_e(u^*, \mathcal{T}u^*). \tag{4.13}$$

Also,

$$d_e(u_{n+1}, \mathcal{T}u^*) \leq H_e(\mathcal{T}u_n, \mathcal{T}u^*) \leq q d_e(u_n, u^*).$$

Hence, using (4.13), we obtain

$$d(u^*, \mathcal{T}u^*) \leq q d(u^*, Tu^*). \tag{4.14}$$

Since $q < 1$, we conclude that $d(u^*, \mathcal{T}u^*) = 0$ and hence $u^* \in \mathcal{T}u^*$. $\square$

Remark 4.9. *They remain open problems in obtaining extended b-metric spaces results from papers [1, 3, 10, 11, 23].*

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