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Solving The Optimal Control Problems Using Homotopy Perturbation Transform Method

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Abstract

In this paper, we solve Hamilton-Jocobi-Bellman (HJB) equations arising in optimal control problems using Homotopy Perturbation Transform Method (HPTM). The proposed method is a combined form of the Laplace Transformation Method with the Homotopy Perturbation Method to produce a highly effective method to handle many problems. Applying the HPTM, solution procedure becomes easier, simpler and more straightforward. Some illustrative examples are given to demonstrate the simplicity and efficiency of the proposed method.

Keywords: Homotopy Perturbation Transform Method (HPTM), Homotopy Perturbation Method (HPM), Laplace Transformation, Optimal control problems(OCP), Hamilton-Jocobi-Bellman(HJB).

1. Introduction

Many of equations in physics, applied mathematics and engineering sciences are modeled in terms of nonlinear partial differential equations. The importance of getting the exact or approximate solutions of these equations in physics and mathematics is still an important issue which needs new methods to get exact or approximate solutions. Most nonlinear equations don't have an exact analytical solution; then, numerical methods have extensively been used to handle these equations. Some of the classic analytic methods are Lyapunov's artificial small parameter method [1], perturbation tecniques [2, 3, 4], δ- expansion method [5], Hirota bilinear method [6, 7], Adomian Decomposition Method (ADM) [8], He's semi-inverse method [9], the tah method, the Homotopy Perturbation Method (HPM), the sinh-cosh method, the Differetial Transform Method (DTM) and the Variational Iteration Method (VIM) [10, 11, 12, 13, 14, 15, 16, 17]. Many of techniques including the ADM,

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the VIM and the Laplace decomposition methods have been used to handle physical equations [18, 19, 20, 21, 22, 23, 24]. He [25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38] extended the HPM by combining the standard homotopy and perturbation to solve different physical problems.The Laplace transform is totally incapable to handle nonlinear equations because of the difficulties that are caused by the nonlinear terms. Different ways have been proposed recently to deal with these nonlinearities such as the ADM [39] and the Laplace decomposition algorithm [40, 41, 42, 43, 44]. Furthermore, the HPM is also combined with the well-known Laplace transformation method [45] and the VIM [46] to produce a highly effective method to handle many nonlinear problems.

Theory and application of optimal control have been widely used in different fields such as biomedicine, aircraft systems, robotic, etc. However, optimal control of nonlinear systems is a challenging task which has been studied extensively for decades. It is well-known that the nonlinear OCP leads to a nonlinear two-point boundary value problem (TPBVP) or a Hamilton-Jocobi-Bellman (HJB) partial differential equation, implementing the pontryagin's maximum principle (PMP).

In general, the HJB equation is a nonlinear partial differential equation that is hard to solve in most cases. An excellent literature review on the methods for solving the HJB equation is provided in [47], where a successive Galerkin approximation (SGA) method is also considered. In the SGA a sequence of generalized HJB equations is solved iteratively to obtain a sequence of approximations approaching the solution of HJB equation. However, the proposed sequence may convergevery slowly or even diverge. Another approach is to treat the problem with a Measure theory approach [48]. This changes the nonlinear OCP to a linear programming and gives a piecewise constant control law. Besides nonlinear TPBVP has no analytical solution except for a few simple cases. Thus, many researchers have been devoted to find an approximate solution of the nonlinear equation. There are various efficient methods such as those reported in [49, 50] for the computation of open-loop optimal controls.However, feedback controls are much preferred in many engineering applications. In those, a sequence of nonhomogeneous linear time-varying TPBVPs is solved instead of directly solving the nonlinear TPBVP derived from the maximum principle. However, solving time-varying equations is much more difficult than solving time-invariant ones. Thus, it is required to solve HJB equation by an approximate-analytic method.

In [51], the authors used the basic HPM for examples of HJB equations and then finding optimal control signal u^*. In this paper, we employed Homotopy Perturbation Transform Method (HPTM) to solve Hamilton-Jocobi-Bellman (HJB) equations. Some illustrative examples are given to demonstrate the simplicity and efficiency of the proposed method.

2. Optimal Control Problem

In this section we have a brief describtion of nonlinear optimal control. First consider the following nonlinear system in state space realization:

$$\dot{x}(t) = a(x(t), u(t), t) \tag{2.1}$$

In above system, $x(t)$ is state vector, $u(t)$ is control signal. Our purpose is control of system and finding control signal such that minimize the following cost function:

$$J = h(x(t_f), t_f) + \int_0^{t_f} g(x(\tau), u(\tau), \tau) d\tau \tag{2.2}$$

In this cost function, h and g are arbitrary convex functions and t_f is final time of system operation.With using dynamic programming approach, we introduce a new variable as:

$$J(x(t),t,u(t)) = h(x(t_f),t_f) + \int_0^{t_f} g(x(\tau),u(\tau),\tau)d\tau; t \le \tau \le t_f, 0 \le t \le t_f. \tag{2.3}$$

Suppose that we have:

$$V(x(t),t) = J^*(x(t),t) = \underset{u(\tau)}{Min}\left\{h(x(t_f),t_f) + \int_0^{t_f} g(x(\tau),u(\tau),\tau)d\tau\right\}; t \le \tau \le t_f. \tag{2.4}$$

Therefore, we have:

$$V(x(t),t) = \underset{u(\tau)}{Min}\left\{h(x(t_f),t_f) + \int_0^{t+\Delta t} g(x(\tau),u(\tau),\tau)d\tau + \int_{t+\Delta t}^{t_f} g(x(\tau),u(\tau),\tau)d\tau\right\}; t \le \tau \le t_f. \tag{2.5}$$

According to principle of optimality, we have:

$$V(x(t),t) = \underset{u(\tau)}{Min}\left\{\int_0^{t+\Delta t} g(x(\tau),u(\tau),\tau)d\tau + V(x(t+\Delta t),t+\Delta t)\right\}; t \le \tau \le t+\Delta t. \tag{2.6}$$

Therefore, with using Taylor series, we have:

$$\begin{aligned} V(x(t),t) = \underset{u(\tau)}{Min}\Bigg\{&\int_t^{t+\Delta t} g(x(\tau),u(\tau),\tau)d\tau + V(x(t),t) + \frac{\partial V}{\partial t}\Delta t \\ &+\frac{\partial V}{\partial x}\left[x(t+\Delta t) - x(t)\right] + H.O.T\Bigg\}; t \le \tau \le t+\Delta t. \end{aligned} \tag{2.7}$$

If suppose $\triangle t$ be small enough then $\tau \to t$ and we have:

$$V\left(x,t\right) = \underset{u(\tau)}{Min}\left\{g\Delta t + V\left(x,t\right) + \frac{\partial V}{\partial t}\Delta t + \frac{\partial V}{\partial x}a(x(t),u(t),t)\Delta t + O(\Delta t)\right\}. \tag{2.8}$$

By divided both side of Eq.(2.8) by $\triangle t$, we have:

$$-\frac{\partial V}{\partial t} = \underset{u(\tau)}{Min}\left\{g(x(t),u(t),t) + \frac{\partial V}{\partial x}a(x(t),u(t),t)\right\}. \tag{2.9}$$

This nonlinear time varying differential equation called "H.J.B equation". We have the following boundary condition:

$$J^*\left(x\left(t_f\right),t_f\right) = V\left(x\left(t_f\right),t_f\right) = h\left(x\left(t_f\right),t_f\right). \tag{2.10}$$

By introducing the Hamiltonian function as follows:

$$H\left(x,u,V_x,t\right) = g(x(t),u(t),t) + \frac{\partial V}{\partial x}a(x(t),u(t),t), \tag{2.11}$$

we have:

$$H\left(x,u^*,V_x,t\right) = \underset{u(\tau)}{Min}\, H\left(x,u,V_x,t\right). \tag{2.12}$$

Therefore by substitution of Hamiltonian function (2.12) in Eq.(2.9), we have:

$$-\frac{\partial V}{\partial t} = H\left(x,u^*\left(x,V_x,t\right),V_x,t\right). \tag{2.13}$$

3. Homotopy Perturbation Method

The Homotopy Perturbation Method is a combination of the classical Perturbation technique and homotopy concept as used in topology. To explain the basic idea of Homotopy Perturbation Method for solving nonlinear differential equations, We consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, r \in \Omega, \tag{3.1}$$

subject to boundary condition:

$$B\left(u, \frac{\partial u}{\partial \eta}\right) = 0, r \in \Gamma, \tag{3.2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function, Γ is the boundary of domain Ω and $\frac{\partial}{\partial \eta}$ denotes differentiation along the normal drawn outwards from Ω. The operator A can be divided into two parts of L and N, where L is the linear part, while N is a nonlinear one. Therefore, (3.1) can be written as follows:

$$L(u) + N(u) - f(r) = 0. \tag{3.3}$$

By the homotopy technique, He constructed a homotopy $v(r, p) : \Omega \times [0, 1]$ which satisfies:

$$H(v, p) = (1 - p)\left[L(v) - L(u_0)\right] + p\left[A(v) - f(r)\right] = 0; p \in [0, 1], r \in \Omega, \tag{3.4}$$

which is equivalent to:

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p\left[N(v) - f(r)\right] = 0, \tag{3.5}$$

where $p \in [0, 1]$ is an embedding parameter and u_0 is an initial guess approximation of (3.1), which satisfies the boundary conditions. It follows from Eq.(3.4) that:

$$H(v, 0) = L(v) - L(u_0) = 0, H(v, 1) = A(v) - f(r).$$

Thus, the changing process of p from 0 to 1 is just that of $v(r, p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic. Here, we use the embedding parameter p as a small parameter and assume that the solution of Eq.(3.4) or (3.5) is as a power series in p:

$$v = \sum_{i=0}^{\infty} p^i v_i. \tag{3.6}$$

Setting $p = 1$, we obtain the approximate solution of Eq.(3.1),

$$u = \lim_{p \to 1} v = \sum_{i=0}^{\infty} v_i \tag{3.7}$$

The coupling of perturbation method and the homotopy method is called homotopy perturbation method, which has eliminated limitations of the traditional perturbation methods. In the other hand, this method can take the full advantage of perturbation techniques. However, the convergence rate depends on the nonlinear operator $A(v)$, has been proved by He [2].

Theorem 1. Suppose $N(v)$ is a nonlinear function and $v = \sum_{i=0}^{\infty} p^i v_i$, then we have

$$\frac{\partial^n}{\partial p^n}N(v)_{p=0} = \frac{\partial^n}{\partial p^n}N\left(\sum_{k=0}^{\infty} p^k v_k\right)_{p=0} = \frac{\partial^n}{\partial p^n}N\left(\sum_{k=0}^{n} p^k v_k\right)_{p=0}. \tag{3.8}$$

proof. See [53].•

Theorem 2. The approximate solution obtained by the homotopy perturbation method can be expressed in He's polynomials:

$$u(x) = f(x) + H_0(v_0) + H_1(v_0, v_1) + \cdots + H_n(v_0, v_1, \cdots, v_n), \tag{3.9}$$

where the He's polynomials are defined as follow:

$$H_r(v_0, v_1, \cdots, v_r) = \frac{1}{r!}\frac{\partial}{\partial p^r}N\left(\sum_{k=0}^{r} p^k v_k\right); r = 0, 1, \cdots, n \tag{3.10}$$

proof. See [53].•

4. Homotopy Perturbation Transform Method

To explain the basic idea of this method[54], we consider a general nonlinear non-homogeneous partial differential equation with initial conditions of the form

$$Du(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), u(x,0) = h(x), u_t(x,0) = f(x), \tag{4.1}$$

Taking the Laplace transform on both sides of Eq.(4.1), which denoted by L, we have:

$$L[Du(x,t)] + L[Ru(x,t)] + L[Nu(x,t)] = L[g(x,t)]. \tag{4.2}$$

In view of differentiation property of the Laplace transform, we have:

$$L[Du(x,t)] = s^2L[u(x,t)] - su(x,0) - u_t(x,0) = s^2L[u(x,t)] - sh(x) - f(x), \tag{4.3}$$

then,

$$L[u(x,t)] = \frac{h(x)}{s} + \frac{f(x)}{s^2} - \frac{1}{s^2}L[Ru(x,t)] + \frac{1}{s^2}L[gu(x,t)] - \frac{1}{s^2}L[Nu(x,t)]. \tag{4.4}$$

Now, operating the Laplace inverse on both sides of Eq.(4.4) gives

$$u(x,t) = G(x,t) - L^{-1}\left[\frac{1}{s^2}L[Ru(x,t) + Nu(x,t)]\right], \tag{4.5}$$

where, $G(x,t)$ is

$$G(x,t) = L^{-1}\left[\frac{h(x)}{s} + \frac{f(x)}{s^2} + \frac{1}{s^2}L[gu(x,t)]\right]. \tag{4.6}$$

Now, we apply the HPM for Eq.(4.5) and we have:

$$u(x,t) = G(x,t) - p\left(L^{-1}\left[\frac{1}{s^2}L[Ru(x,t) + Nu(x,t)]\right]\right). \tag{4.7}$$

Substituting Eqs.(3.6) and (3.10) in Eq.(4.7), we have:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) - p\left(L^{-1}\left[\frac{1}{s^2}L\left[R\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right) + \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right), \tag{4.8}$$

which is the coupling of the Laplace transform and the HPM using $He's$ polynomials. Then, by sorting the coefficients with respect to powers of p, we have:

$$\begin{aligned} p^0 &: u_0(x,t) = G(x,t), \\ p^1 &: u_1(x,t) = -L^{-1}\left[\frac{1}{s^2}L\left[Ru_0(x,t) + H_0(u)\right]\right], \\ p^2 &: u_2(x,t) = -L^{-1}\left[\frac{1}{s^2}L\left[Ru_1(x,t) + H_1(u)\right]\right], \\ p^3 &: u_3(x,t) = -L^{-1}\left[\frac{1}{s^2}L\left[Ru_2(x,t) + H_2(u)\right]\right], \\ &\vdots \end{aligned} \tag{4.9}$$

Example 4.1. *Consider the following homogeneous nonlinear partial differential equation.*

$$u_t + uu_x = 0, u(x,0) = -x \tag{4.10}$$

Solution
Taking the Laplace transform on both sides of Eq.(4.10), we have:

$$u(x,s) = -\frac{x}{s} - \frac{1}{s}L[uu_x]. \tag{4.11}$$

The inverse of the Laplace transform Eq.(4.11) gives:

$$u(x,t) = -x - L^{-1}\left[\frac{1}{s}L[uu_x]\right]. \tag{4.12}$$

In view of Eqs.(3.6) and (3.10), we apply the HPM to Eq.(4.12) and we have:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = -x - p\left(L^{-1}\left[\frac{1}{s}L\left[\sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right). \tag{4.13}$$

For example, the first few components of $He's$ polynomials are given by

$$\begin{aligned} H_0(u) &= u_0 u_{0_x}, \\ H_1(u) &= u_0 u_{1_x} + u_1 u_{0_x}, \\ H_2(u) &= u_0 u_{2_x} + u_1 u_{1_x} + u_2 u_{0_x}, \\ &\vdots \end{aligned} \tag{4.14}$$

by sorting the coefficients with respect to powers of p, we have:

$$\begin{aligned} p^0 &: u_0(x,t) = -x, \\ p^1 &: u_1(x,t) = -L^{-1}\left[\frac{1}{s}L\left[H_0(u)\right]\right] = -xt, \\ p^2 &: u_2(x,t) = -L^{-1}\left[\frac{1}{s}L\left[H_1(u)\right]\right] = -xt^2, \\ p^3 &: u_3(x,t) = -L^{-1}\left[\frac{1}{s}L\left[H_2(u)\right]\right] = -xt^3, \\ p^4 &: u_4(x,t) = -L^{-1}\left[\frac{1}{s}L\left[H_3(u)\right]\right] = -xt^4, \\ &\vdots \end{aligned} \tag{4.15}$$

so that the solution $u(x,t)$ is given by

$$u(x,t) = -x\left(1 + t + t^2 + t^3 + t^4 + \ldots\right), \tag{4.16}$$

which, it has closed form as follows:

$$u(x,t) = \frac{x}{t-1}. \tag{4.17}$$

5. Application

In this section, to illustrate the effectiveness of the HPTM, we shall consider three examples.

Example 5.1. *Consider a single−input scalar system as follows [55]:*

$$\begin{aligned} \dot{x} &= -x(t) + u(t), \\ J &= \frac{1}{2}\int_0^1 \left(x^2(t) + u^2(t)\right)dt. \end{aligned} \tag{5.1}$$

The corresponding Hamiltonian function will be:

$$H(x,u,V_x,t) = \frac{1}{2}x^2(t) + \frac{1}{2}u^2(t) + \frac{\partial V(x,t)}{\partial x}\left[-x(t) + u(t)\right]. \tag{5.2}$$

For finding u^* , we have:

$$\frac{\partial H}{\partial u} = u(t) + \frac{\partial V}{\partial x} = 0. \tag{5.3}$$

Therefore, we obtain:

$$u^*(t) = -\frac{\partial V}{\partial x}. \tag{5.4}$$

Because $\frac{\partial^2 H}{\partial u^2} = 1 \succ 0$, u^* is a minimum and acceptable. Now, by substituting u^* in HJB equation, we have the following equation:

$$H\left(x,-\frac{\partial V}{\partial x},V_x,t\right)=\frac{1}{2}x^2-\frac{1}{2}\left(\frac{\partial V\left(x,t\right)}{\partial x}\right)^2-x\left(\frac{\partial V\left(x,t\right)}{\partial x}\right). \tag{5.5}$$

That is

$$\begin{aligned}-\frac{\partial V}{\partial t} &= \frac{1}{2}x^2-\frac{1}{2}\left(\frac{\partial V\left(x,t\right)}{\partial x}\right)^2-x\left(\frac{\partial V\left(x,t\right)}{\partial x}\right), \\ V\left(x\left(1\right),1\right) &= 0.\end{aligned} \tag{5.6}$$

or

$$\begin{aligned}\frac{\partial V}{\partial t} &= -\frac{1}{2}x^2+\frac{1}{2}\left(\frac{\partial V\left(x,t\right)}{\partial x}\right)^2+x\left(\frac{\partial V\left(x,t\right)}{\partial x}\right), \\ V\left(x\left(1\right),1\right) &= 0.\end{aligned} \tag{5.7}$$

For this problem, we have the exact solution of state $x(t)$ and the control $u(t)$ as follows [55]:

$$\begin{aligned}x\left(t\right) &= \cosh\left(\sqrt{2}t\right)+\beta\sinh\left(\sqrt{2}t\right), \\ u\left(t\right) &= \left(1+\sqrt{2}\beta\right)\cosh\left(\sqrt{2}t\right)+\left(\sqrt{2}+\beta\right)\sinh\left(\sqrt{2}t\right),\end{aligned}$$

where

$$\beta=\frac{\cosh\left(\sqrt{2}\right)+\sqrt{2}\sinh\left(\sqrt{2}\right)}{\sqrt{2}\cosh\left(\sqrt{2}\right)+\sinh\left(\sqrt{2}\right)}\approx -0.98.$$

For the sake of simplicity, we assume that $u^*\left(x,t\right)=-k\left(t\right)x\left(t\right)$, where

$$k\left(t\right)=\frac{\left(1+\sqrt{2}\beta\right)\cosh\left(\sqrt{2}t\right)+\left(\sqrt{2}+\beta\right)\sinh\left(\sqrt{2}t\right)}{\cosh\left(\sqrt{2}t\right)+\beta\sinh\left(\sqrt{2}t\right)}.$$

To solve Eq.(5.7) by means of HPTM, taking the Laplace transform on both sides of that, we have:

$$sL\left(V\left(x,t\right)\right)-V\left(x\left(0\right),0\right)=-\frac{1}{2}L\left(x^2\right)+\frac{1}{2}L\left[\left(\frac{\partial V\left(x,t\right)}{\partial x}\right)^2\right]+L\left[x\frac{\partial V\left(x,t\right)}{\partial x}\right]. \tag{5.8}$$

Then, we have:

$$L\left(V\left(x,t\right)\right)=\frac{-1}{2s}L\left(x^2\right)+\frac{1}{2s}L\left[\left(\frac{\partial V\left(x,t\right)}{\partial x}\right)^2\right]+\frac{1}{s}L\left[x\frac{\partial V\left(x,t\right)}{\partial x}\right]. \tag{5.9}$$

The inverse Laplace transform of Eq.(5.9) gives:

$$V\left(x,t\right)=-\frac{1}{2}x^2t+\frac{1}{2}L^{-1}\left[\frac{1}{s}L\left[\left(\frac{\partial V\left(x,t\right)}{\partial x}\right)^2\right]\right]+L^{-1}\left[\frac{1}{s}L\left[x\frac{\partial V\left(x,t\right)}{\partial x}\right]\right]. \tag{5.10}$$

In view of Eqs.(3.6) and (3.10), we apply the HPM to Eq.(5.10) and we have:

$$\sum_{i=0}^{\infty} v_i p^i = -\frac{1}{2}x^2 t + p\left(\frac{1}{2}L^{-1}\left[\frac{1}{s}L\left[\sum_{i=0}^{\infty} p^i H_i\right]\right] + xL^{-1}\left[\frac{1}{s}L\left[\sum_{i=0}^{\infty}\frac{\partial v_i(x,t)}{\partial x}p^i\right]\right]\right), \tag{5.11}$$

where $p \in [0,1]$ is an embedding parameter.
For example, the first few components of $He's$ polynomials are given by

$$\begin{aligned} H_0(V) &= (v_{0_x})^2, \\ H_1(V) &= 2v_{0_x}v_{1_x}, \\ H_2(V) &= 2v_{0_x}v_{2_x} + (v_{1_x})^2, \\ H_3(V) &= 2v_{0_x}v_{3_x} + 2v_{1_x}v_{2_x}, \\ &\vdots \end{aligned} \tag{5.12}$$

By sorting the coefficients with respect to powers of p and in view of integral property of laplace transform in Eq.(5.11), we have:

$$\begin{aligned} p^0 &: v_0(x,t) = -\frac{1}{2}x^2 t, \\ p^1 &: v_1(x,t) = \left(\frac{1}{6}t^3 - \frac{1}{2}t^2\right)x^2, \\ p^2 &: v_2(x,t) = \left(-\frac{1}{15}t^5 + \frac{1}{3}t^4 - \frac{1}{3}t^3\right)x^2, \\ p^3 &: v_3(x,t) = \left(\frac{17}{630}t^7 - \frac{17}{90}t^6 + \frac{11}{30}t^5 - \frac{1}{6}t^4\right)x^2, \\ &\vdots \end{aligned} \tag{5.13}$$

For simplicity, let $V(x,t) = f(t)x^2(t)$ and in view of Eq.(5.4); then

$$u^*(x,t) = -2f(t)x(t). \tag{5.14}$$

Example 5.2. *Consider the following purely mathematical optimal control problem [56]:*

$$\dot{x} = x(t) + u(t). \tag{5.15}$$

Suppose that, we consider the following cost function for this system:

$$J = x^2(t_f) + \int_0^{t_f} u^2(t)dt. \tag{5.16}$$

The corresponding Hamiltonian function will be:

$$H(x,u,V_x,t) = u^2(t) + \frac{\partial V(x,t)}{\partial x}\left[x(t) + u(t)\right]. \tag{5.17}$$

For finding u^* , we have:

$$\frac{\partial H}{\partial u} = 2u\left(t\right) + \frac{\partial V\left(x,t\right)}{\partial x} = 0, \tag{5.18}$$

which is easily solved to get:

$$u^*\left(x\right) = -\frac{1}{2}\frac{\partial V\left(x,t\right)}{\partial x}. \tag{5.19}$$

Because $\frac{\partial^2 H}{\partial u^2} = 2 \succ 0$, u^* is a minimum and acceptable. Now, by substituting u^* in HJB equation, we have the following equation:

$$\begin{aligned}\frac{\partial V\left(x,t\right)}{\partial t} &= \frac{1}{4}\left(\frac{\partial V\left(x,t\right)}{\partial x}\right)^2 - x\frac{\partial V\left(x,t\right)}{\partial x}, \\ V\left(x(t_f),t_f\right) &= x^2(t_f).\end{aligned} \tag{5.20}$$

In [56], authors the solution of the above HJB equation in form:

$$V\left(x,t,t_f\right) = \frac{2x^2}{1+e^{2\left(t-t_f\right)}}. \tag{5.21}$$

For sake of simplicity, let $u^*\left(x,t\right) = k\left(t\right)x\left(t\right)$, where

$$k\left(t\right) = \frac{-2}{1+e^{2\left(t-t_f\right)}}. \tag{5.22}$$

To solve Eq.(5.20) by means of HPTM, taking the Laplace transform on both sides of that, we have:

$$sL\left(V\left(x,t\right)\right) - V\left(x(0),0\right) = \frac{1}{4}L\left[\left(\frac{\partial V\left(x,t\right)}{\partial x}\right)^2\right] - L\left[x\frac{\partial V\left(x,t\right)}{\partial x}\right]. \tag{5.23}$$

In view of initial conditions Eq.(5.20), we have:

$$L\left(V\left(x,t\right)\right) = \frac{x^2}{s} + \frac{1}{4s}L\left[\left(\frac{\partial V\left(x,t\right)}{\partial x}\right)^2\right] - \frac{1}{s}L\left[x\frac{\partial V\left(x,t\right)}{\partial x}\right]. \tag{5.24}$$

The inverse of the Laplace transform Eq.(5.24) gives:

$$V\left(x,t\right) = x^2 + L^{-1}\left(\frac{1}{s}\right) + \frac{1}{4}L^{-1}\left[\frac{1}{s}L\left[\left(\frac{\partial V\left(x,t\right)}{\partial x}\right)^2\right]\right] - L^{-1}\left[\frac{1}{s}L\left[x\frac{\partial V\left(x,t\right)}{\partial x}\right]\right]. \tag{5.25}$$

In view of Eqs.(3.6) and (3.10), we apply the HPM to Eq.(5.25) and we have:

$$\sum_{i=0}^{\infty} v_i p^i = x^2 + p\left(\frac{1}{4}L^{-1}\left[\frac{1}{s}L\left[\sum_{i=0}^{\infty} p^i H_i\right]\right] - xL^{-1}\left[\frac{1}{s}L\left[\sum_{i=0}^{\infty}\frac{\partial v_i\left(x,t\right)}{\partial x}p^i\right]\right]\right), \tag{5.26}$$

where $p \in [0,1]$ is an embedding parameter.

By sorting the coefficients with respect to powers of p and in view of integral property of laplace transform in Eq.(5.26), we have:

$$\begin{aligned} p^0 &: v_0(x,t) = x^2, \\ p^1 &: v_1(x,t) = (1-t)\,x^2, \\ p^2 &: v_2(x,t) = 0, \\ p^3 &: v_3(x,t) = \frac{(t-1)^3}{3}x^2, \\ p^4 &: v_4(x,t) = 0, \\ &\vdots \end{aligned} \tag{5.27}$$

For simplicity, let $V(x,t) = f(t)\,x^2(t)$ and in view of Eq.(5.19); then

$$u^*(x,t) = -f(t)\,x(t)\,. \tag{5.28}$$

Example 5.3. *Consider the following nonlinear optimal control problem [57]:*

$$\begin{aligned} Min \quad & J = \textstyle\int_0^1 u^2(t)dt, \\ Subjectto \quad & \dot{x} = \tfrac{1}{2}x^2(t)\sin x(t) + u(t)\,, t \in [0,1]\,, \\ & x(0) = 0, x(1) = 0.5. \end{aligned} \tag{5.29}$$

The corresponding Hamiltonian function will be:

$$H(x,u,V_x,t) = u^2(t) + \frac{\partial V(x,t)}{\partial x}\left[\frac{1}{2}x^2(t)\sin x(t) + u(t)\right]. \tag{5.30}$$

For finding u^*, we have:

$$\frac{\partial H}{\partial u} = 2u(t) + \frac{\partial V}{\partial x} = 0. \tag{5.31}$$

Therefore, we obtain:

$$u^*(t) = -\frac{1}{2}\frac{\partial V}{\partial x}. \tag{5.32}$$

Because $\frac{\partial^2 H}{\partial u^2} = 2 \succ 0$, u^* is a minimum and acceptable. Now, by substituting u^* in HJB equation, we have the following equation:

$$\begin{aligned} \frac{\partial V}{\partial t} &= \frac{1}{4}\left(\frac{\partial V}{\partial x}\right)^2 - \frac{1}{2}x^2(t)\sin x(t)\,\frac{\partial V}{\partial x}, \\ V(x(0),0) &= 0. \end{aligned} \tag{5.33}$$

To solve Eq.(5.33) by means of HPTM, taking the Laplace transform on both sides of that, we have:

$$sL\left(V\left(x,t\right)\right)-V\left(x\left(0\right),0\right)=\frac{1}{4}L\left[\left(\frac{\partial V}{\partial x}\right)^{2}\right]-\frac{1}{2}L\left[x^{2}\sin x\left(t\right)\frac{\partial V}{\partial x}\right]. \quad (5.34)$$

In view of initial conditions Eq.(5.33) and the inverse of the Laplace transform Eq.(5.34) gives:

$$V\left(x,t\right)=\frac{1}{4}L^{-1}\left[\frac{1}{s}L\left[\left(\frac{\partial V}{\partial x}\right)^{2}\right]\right]-\frac{1}{2}L^{-1}\left[\frac{1}{s}L\left[x^{2}\sin x\left(t\right)\frac{\partial V}{\partial x}\right]\right]. \quad (5.35)$$

In view of Eqs.(3.6) and (3.10), we apply the HPM to Eq.(5.35) and we have:

$$\sum_{i=0}^{\infty}v_{i}p^{i}=p\left(\frac{1}{4}L^{-1}\left[\frac{1}{s}L\left[\sum_{i=0}^{\infty}p^{i}H_{i}\right]\right]-\frac{1}{2}x^{2}\sin xL^{-1}\left[\frac{1}{s}L\left[\sum_{i=0}^{\infty}\frac{\partial v_{i}\left(x,t\right)}{\partial x}p^{i}\right]\right]\right), \quad (5.36)$$

where $p\in[0,1]$ is an embedding parameter.

By sorting the coefficients with respect to powers of p and in view of integral property of laplace transform in Eq.(5.36), we have:

$$\begin{aligned}
p^{0} &: v_{0}\left(x,t\right)=0, \quad (5.37)\\
p^{1} &: v_{1}\left(x,t\right)=\frac{1}{4}t+\frac{1}{2}x^{2}t\sin x,\\
p^{2} &: v_{2}\left(x,t\right)=-\frac{1}{8}xt^{2}\left(2\sin x+x^{3}\sin x\cos x-2x^{2}\cos^{2}x+x\cos x+2x^{2}\right),\\
&\vdots
\end{aligned}$$

Since $u^{*}\left(t\right)=-\frac{1}{2}\frac{\partial V}{\partial x}$, use of $V=\sum_{i=0}^{5}v_{i}$ gives $u(x,t)$ as a function of the base period t and the value of the state variable in the base period. In [58], they found that $x(t)=\frac{1}{2}t$ is the optimal state trajectory. By considering this optimal state trajectory, gives $u\left(t\right)\approx\frac{1}{2}-\frac{1}{16}t^{3}+\frac{1}{384}t^{5}$.

6. Conclusion

In this paper, it has been shown that the HPTM provides exactly the same solutions for nonlinear partial differential equations and we have successfully developed that for solving HJB equation. The HPTM was clearly very efficient and powerful technique in finding the solutions of the equations.

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