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A Numerical Approach for Fractional Optimal Control Problems by Using Ritz Approximation

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Abstract

In this article, Ritz approximation have been employed to obtain the numerical solutions of a class of the fractional optimal control problems based on the Caputo fractional derivative. Using polynomial basis functions, we obtain a system of nonlinear algebraic equations. This nonlinear system of equation is solved and the coefficients of basis polynomial are derived. The convergence of the numerical solution is investigated. Some numerical examples are presented which illustrate the theoretical results and the performance of the method.

Keywords: Fractional Optimal Control Problems, Caputo fractional derivative, Optimal Control Problems, Polynomial basis functions.

1. Introduction

In the recent years, the dynamic behaviors of fractional order differential systems have received increasing attention. FOCP refers to the minimization of an objective functional subject to dynamic constraints, on state and control variables, which have fractional order models. Some numerical methods for solving some types of FOCPs were recorded [1] and the references cited therein. The general definition of an optimal control problem requires the minimization of a criterion function of the states and control inputs of the system over a set of admissible control functions. The system is subject to constrained dynamics and control variables. Additional constraints such as final time constraints can be considered. This paper introduces an original formulation and a general numerical scheme for a potentially almost unlimited class of FOCPs. An FOCP is an optimal control problem in which the criterion and/or the differential equations governing the dynamics of the system contain at

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least one fractional derivative operator. Integer order optimal controls (IOOCs) have been discussed for a long time and a large collection of numerical techniques have been developed to solve IOOC problems. However, the number of publications on FOCPs is limited. A general formulation and a solution scheme for FOCPs were first introduced in [2] where fractional derivatives were introduced in the Riemann–Liouville sense, and FOCP formulation was expressed using the fractional variational principle and the Lagrange multiplier technique. The state and the control variables were given as a linear combination of test functions, and a virtual work type approach was used to obtain solutions. In [3, 4], the FOCPs are formulated using the definition of fractional derivatives in the sense of Caputo, the FDEs are substituted into Volterra-type integral equations and a direct linear solver helps calculating the solution of the obtained algebraic equations. In [5], the fractional dynamics of the FOCPs are defined in terms of the Riemann–Liouville fractional derivatives. The Grunwald and Letnikov formula is used as an approximation and the resulting equations are solved using a direct scheme. Frederico and Torres [6, 7, 8] using similar definitions of the FOCPs, formulated a Noether-type theorem in the general context of the fractional optimal control in the sense of Caputo and studied fractional conservation laws in FOCPs. However, none of this work has taken advantage of the colossal research achieved in the numerical solutions of IOOCs.

In this section, we briefly give some definitions regarding fractional derivatives allowing us to formulate a general definition of an FOCP. There are many different types of definitions of fractional calculus. For example, the Riemann–Liouville integral operator [9] of order α is defined by

$$(I_x^\alpha f)(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt & \alpha > 0, \\ f(x) & \alpha = 0, \end{cases} \tag{1.1}$$

and its fractional derivative of order α $(\alpha \geq 0)$ is normally used:

$$(\mathcal{D}_x^\alpha f)(x) = (\frac{d}{dx})^m (I^{m-\alpha} f)(x), \qquad (\alpha > 0,\, m-1 < \alpha < m), \tag{1.2}$$

where m is an integer. For Riemann–Liouville's definition, one has

$$I_x^\alpha x^v = \frac{\Gamma(v+1)}{\Gamma(v+1+\alpha)} x^{v+\alpha}. \tag{1.3}$$

The Riemann–Liouville integral operator plays an important role in the development of the theory of fractional derivatives and integrals. However, it has some disadvantages for fractional differential equations with initial and boundary conditions. Therefore, we adopt here Caputo's definition [10], which is a modification of Riemann–Liouville definition:

$$(\mathcal{D}_x^\alpha f)(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(s)}{(x-s)^{\alpha-m+1}} ds & (\alpha > 0,\, m-1 < \alpha < m), \\ \frac{\partial^m f(x)}{\partial x^m} & \alpha = m, \end{cases} \tag{1.4}$$

where m is an integer. Caputo's integral operator has an useful property[10]:

$$(I_x^\alpha \mathcal{D}_x^\alpha f)(x) = f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0^+)}{k!} x^k, \qquad (x \geq 0, m-1 < \alpha < m), \tag{1.5}$$

where m is an integer. For the Caputo derivative we have

$$D_x^\alpha C = 0 \quad (C \ is\, a\, constant), \tag{1.6}$$

$$D_x^\alpha x^\beta = \begin{cases} 0 & \beta \leq \alpha - 1, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha} & \beta > \alpha - 1. \end{cases} \tag{1.7}$$

For m to be the smallest integer that exceeds α, the Caputo space–fractional derivative operator of order $\alpha > 0$ is defined as:

$$(\mathcal{D}_x^{\alpha} f)(x,t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{1}{(x-s)^{\alpha-m+1}} \frac{\partial^m f(s,t)}{\partial x^m} ds & (\alpha > 0,\ m-1 < \alpha < m), \\ \frac{\partial^m f(x,t)}{\partial x^m} & \alpha = m. \end{cases} \tag{1.8}$$

2. Solution of Fractional Optimal Control Problems

In this work we focus on fractional optimal control problems. Let $0 < \alpha < 1$ and let L, f: $[a, +\infty[\, \times R^2 \to R$ be two differentiable functions. Consider the following FOCP [1]:

$$minimize J(x,u,T) = \int_a^T L(t, x(t), u(t))dt, \tag{2.1}$$

subject to the dynamic system

$$M_1 \dot{x}(t) + M_2 D_t^{\alpha} x(t) = f(t, x(t), u(t)), \tag{2.2}$$

where the boundary conditions are as follows:

$$x(a) = x_a, \tag{2.3}$$

where M_1, M_2, T, x_a are fixed real numbers. fractional derivatives are taken in the Caputo sense. The method consists of conversion fractional optimal control problem to optimization problem and expanding the solution by polynomial basis functions with unknown coefficients.

We approximate $x(t)$ as

$$x(t) \cong x_k(t) = \sum_{i=0}^{m} c_i t \phi_i(t) + w(t), \tag{2.4}$$

and $u(t)$ obtain of Eq. (3.2). where $\phi_i(t)$ are polynomial basis functions and c_i are unknown coefficients. In following, we determine $w(t)$ as $w(a) = x_a$.

Now we have the following optimal problem

$$J[c_0, c_1, ..., c_m] = \int_a^T L(t, \tilde{x}(t), \tilde{u}(t))dt, \tag{2.5}$$

If $c_k$are decided by the optimizing function J, then by (2.4), we obtain functions which approximate the optimum value of J in (2.5). To find unknowns c_k, $k = 0, 1, ..., m$ in $\widetilde{x}(t)$, according to the necessary conditions of optimization for (2.5), we have

$$\frac{\partial J}{\partial c_k} = 0, \quad k = 0, ..., m. \tag{2.6}$$

Then by solving the above system of m algebraic equations (2.6), we obtain c_k, $k = 0, 1, ..., m$. The approached demonstrated here relies on the Ritz method. Then with solving this problem by mathematica software, we obtain c_i. The method presented here is based on the Ritz method. We refer the interested reader to [11] for more information.

3. Illustrative examples

To demonstrate the effectiveness of the method, here we consider some fractional optimal control problems. The following examples demonstrate that the desired approximate solution can be determined by solving the resulting system of equations, which can be effectively computed using symbolic computing codes on any personal computer. Illustrative examples show that this method in comparison to other methods has high accuracy and is easily implemented.

Example 3.1. *Consider fractional diffusion equation[1]*

$$\min J(x,u) = \int_0^1 (tu(t) - (\alpha+2)x(t))^2 dt, \tag{3.1}$$

subject to the dynamical system

$$\dot{x}(t) + D_t^\alpha x(t) = u(t) + t^2, \tag{3.2}$$

with initial and boundary conditions: $x(0) = 0, x(1) = \frac{2}{\Gamma(3+\alpha)}$. *The exact solution is given by*

$$(x(t), u(t)) = (\frac{2t^{\alpha+2}}{\Gamma(3+\alpha)}, \frac{2t^{\alpha+1}}{\Gamma(2+\alpha)}). \tag{3.3}$$

We applied the method presented for different values of α *and solved Equation(3.1). We determine*

$$x_m(t) = \sum_{i=0}^{m} c_i t^{i+1}(t-1) + \frac{2t}{\Gamma(3+\alpha)},$$

Fig. 1 shows the absolute error of this problem obtained by the present method with $m = 5, \alpha = \frac{1}{2}$. *From Fig. 1, we can see that the present method provides accurate results.*

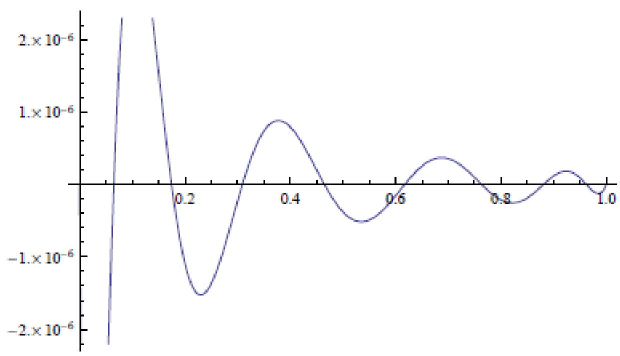

Figure 1: The absolute error between exact and numerical solution for $\alpha = \frac{1}{2}$

The following table shows the values of minimum for different values of approximations and $\alpha = \frac{1}{2}$.

	$m = 3$	$m = 5$	$m = 7$
$\alpha = 0.5$	4.4983×10^{-8}	1.10767×10^{-9}	6.78634×10^{-11}
$\alpha = 0.75$	1.07508×10^{-8}	1.81465×10^{-10}	8.50994×10^{-12}

Example 3.2. *Consider fractional diffusion equation[1]*

$$\min J(x,u) = \int_0^1 (u(t) - x(t))^2 dt, \tag{3.4}$$

subject to the dynamical system:

$$\dot{x}(t) + D_t^\alpha x(t) = u(t) - x(t) + \frac{6t^{\alpha+2}}{\Gamma(\alpha+3)} + t^3.$$

and the boundary conditions: $x(0) = 0, x(1) = \frac{6}{\Gamma(4+\alpha)}$. *The exact solution of the above problem is*

$$(x(t), u(t)) = (\frac{6t^{\alpha+3}}{\Gamma(4+\alpha)}, \frac{6t^{\alpha+3}}{\Gamma(4+\alpha)}). \tag{3.5}$$

We applied the method presented for different values of α *and solved Equation(3.4). We determine*

$$x_m(t) = \sum_{i=0}^{m} c_i t^{i+1}(t-1) + \frac{6t}{\Gamma(4+\alpha)},$$

Fig. 2 shows the absolute error of this problem obtained by the present method with $m = 7, \alpha = \frac{1}{2}$. *From Fig. 2, we can see that the present method provides accurate results. The following table shows the values of minimum* η_m *for different values of approximations.*

	$m=3$	$m=5$	$m=7$
$\alpha = 0.5$	8.17323×10^{-8}	9.07389×10^{-10}	3.67001×10^{-11}
$\alpha = 0.75$	2.58653×10^{-8}	1.78558×10^{-10}	5.31303×10^{-12}

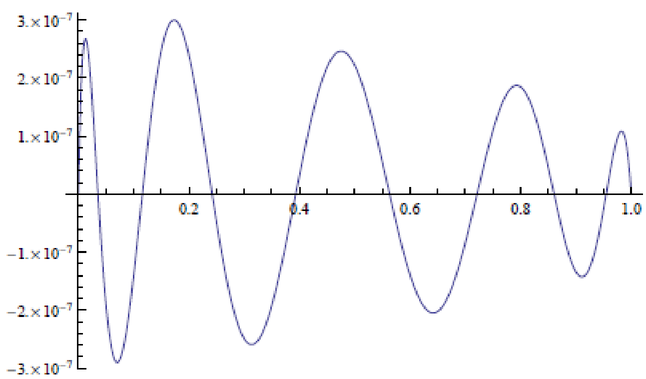

Figure 2: The absolute error between exact and numerical solution for $\alpha = \frac{1}{2}$.

Example 3.3. *Consider the following time invariant problem[1]:*

$$\min J(x,u) = \frac{1}{2}\int_0^1 (u^2(t) + x^2(t))dt, \tag{3.6}$$

subject to the dynamical system:

$$\frac{1}{2}\,\dot{x}(t) + D_t^{\alpha}x(t) = u(t) - x(t).$$

and the boundary conditions: $x(0) = 1, x(1) = cosh(\sqrt{2}) + sinh(\sqrt{2})$*. For this problem, we have the exact solution in the case of* $\alpha = 1$ *as follows [1]*

$$\begin{aligned} x(t) &= cosh(\sqrt{2}t) + sinh(\sqrt{2}t), \\ u(t) &= (1 + \sqrt{2}\beta)cosh(\sqrt{2}t) + (\sqrt{2} + \beta)sinh(\sqrt{2}t), \end{aligned}$$

We applied the method presented for different values of α *and solved Equation(3.6). We determine*

$$x_m(t) = \sum_{i=0}^{m} c_i t^{i+1}(t-1) + t(cosh(\sqrt{2}) + sinh(\sqrt{2})) + 1 - t,$$

Figs. 3 display approximate solutions of $x(t)$ *for* $m = 5$ *and* $\alpha = 0.8, 0.9, 0.99$ *and exact solution for* $\alpha = 1$*.*

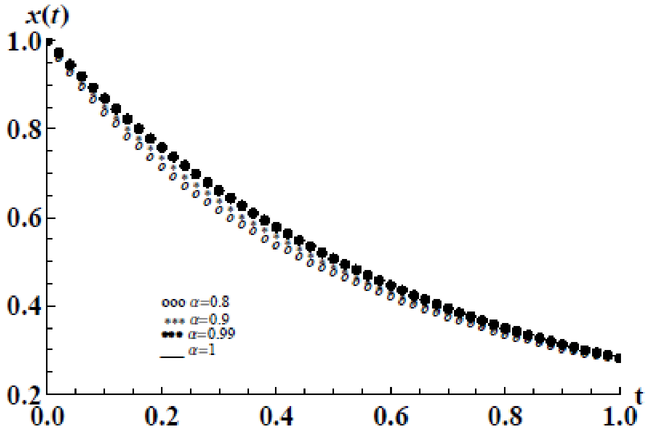

Figure 3: Exact (—) and approximate solution for $\alpha = 0.8, 0.9, 0.99, 1$, and $m = 5$.

4. Conclusion

This paper presents a simple and effective approach to solve a wide class of fractional optimal control problems. The desired approximate solution can be determined by solving the resulting system of equations, which can be effectively computed using symbolic computing codes on any personal computer. Illustrative examples show that this method has high accuracy and is easily implemented. The method will be expected to deal with other fractional problems such as fractional inverse problems, fractional optimal problems and other problems, which will be discussed in a future papers.

References

[1] N. H. Sweilam , T. M. Al-Ajami, Legendre spectral-collocation method for solving some types of fractional optimal control problems, Journal of Advanced Research, (2015) 6, 393–403.

[2] O.P. Agrawal, A general formulation and solution scheme for fractional optimal control problems, Nonlinear Dynamics, 38 (1) (2004) 323–337.

[3] O.P. Agrawal, A quadratic numerical scheme for fractional optimal control problems, ASME Journal of Dynamic Systems, Measurement, and Control, 130 (1) (2008) 011010-1–011010-6.

[4] O.P. Agrawal, Fractional optimal control of a distributed system using eigenfunctions, ASME Journal of Computational and Nonlinear Dynamics 3 (2) (2008) 021204-1- 021204-6.

[5] O.P. Agrawal, D. Baleanu, A hamiltonian formulation and a direct numerical scheme for fractional optimal control problems, Journal of Vibration and Control 13 (9–10) (2007) 1269–1281.

[6] G. Frederico, D. Torres, Noethers theorem for fractional optimal control problems, in: Proc. of the 2nd IFAC Workshop on Fractional Differentiation and its Applications, Porto, Portugal, 2006.

[7] G. Frederico, D. Torres, Fractional conservation laws in optimal control theory, Nonlinear Dynamics 53 (3) (2008) 215–222.

[8] G. Frederico, D. Torres, Fractional optimal control in the sense of caputo and the fractional noethers theorem, International Mathematical Forum 3 (10) (2008) 479–493.

[9] I. Podlubny, Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. New York: Academic Press; 1999.

[10] M. Caputo, Linear models of dissipation whose Q is almost frequency independent. Part II, J. Roy. Austral. Soc. 13 (1967) 529–539.

[11] A. Lotfi, S.A. Yousefi, A numerical technique for solving a class of fractional variational problems, Journal of Computational and Applied Mathematics 237 (2013) 633–643.