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Scalar Product Graphs of Modules

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Abstract

Let R be a commutative ring with identity and M an R-module. The Scalar-Product Graph of M is defined as the graph $G_R(M)$ with the vertex set M and two distinct vertices x and y are adjacent if and only if there exist r or s belong to R such that $x = ry$ or $y = sx$. In this paper , we discuss connectivity and planarity of these graphs and computing diameter and girth of $G_R(M)$. Also we show some of these graphs is weakly perfect.

Keywords: Scalar Product, Graph, Module.

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1. Introduction

The concept of the zero-divisor graph of a commutative ring, denoted by $\Gamma(R)$, was introduced by Beck [1], where he was mainly interested in coloring.$\Gamma(R)$ is graph with vertices nonzero zero divisors of R and edges those pairs of distinct nonzero zero divisors $\{a, b\}$ such that $ab = 0$. We consider this investigation of coloring of the zero-divisor graph of a commutative ring was then continued by Anderson and Naseer [2].

Let G be an undirected graph with the vertex set $V(G)$. If G contains n vertices then it is said to be an n-vertex graph and we write $|V(G)| = n$. Two graphs G and H are isomorphic if there exists a one-to-one correspondence between their vertex sets which preserves adjacency. A subgraph of G is a graph having all of its vertices and edges in G. The complete graph is a graph in which any two distinct vertices are adjacent.

Throughout this paper all rings are commutative with non-zero identity and all modules unitary. We associate a graph $G_R(M)$ to an R-module M whose vertices are elements of M in these way that two distinct vertices x and y are adjacent if and only if there exists r belong to R that $x = ry$ or $y = rx$. We investigate the relationship between the algebraic properties of an R-module M and the properties of the associated graph $G_R(M)$ namely Scalar-product graph of M.

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Let $G = (V, E)$ be a graph. We say that G is connected if there is a path between any two distinct vertices of G. For vertices x and y of G, we define $d(x, y)$ to be the length of a shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The diameter of G is $diam(G) = sup\{d(x, y) : x, y \in V(G)\}$. The girth of a graph G, denoted by $gr(G)$, is the length of the shortest cycle in G. A graph with no cycle has infinite girth. For a vertex $v \in G$, neighbours of v denotes $N(v)$ is equal $\{u \in V(G)\backslash\{v\} : v \textit{ is adjacent to } u\}$. In a graph G, a set $S \subseteq V(G)$ is an independent set if the subgraph induced by S contains no edge. The independence number $\alpha(G)$ is the maximum size of an independent set in G.

Afkhami and et al. in [3] introduced the cozero-divisor graph of a commutative ring R denoted by $\Gamma'(R)$ as a graph with vertices $W(R)^* = W(R)\backslash\{0\}$ where $W(R)$ is the set of all non-unit elements of R and two distinct vertices x and y are adjacent if and only if $x \notin Ry$ and $y \notin Rx$ where Rc is a ideal generated by $c \in R$.

Let M be a R-module and $W_R(M) = \{x \in M | Rm \neq M\}$. By R as R-module $W_R(R)$ is set of all non-units elements of R. In [4] authors investigate cozero-divisor graphs on R-module M which vertices from $W_R(M)^* = W_R(M)\backslash\{0\}$ and two distinct vertices m and n are adjacent if and only if $m \notin Rn$ and $n \notin Rm$, and they studied girth, independent number, clique number and planarity of this graph.

We use $T(M)$ to denote the set of torsion elements of M; that is,
$T(M) = \{m \in M : rm = 0 \ for \ some \ 0 \neq r \in R\}$. If R is an integral domain, then $T(M)$ is a submodule of M. If $T(M) = 0$, we say that M is torsion-free while if $T(M) = M$ we say that M is torsion. D. Anderson et al. in [5] showed when $T(M)$ is submodule of M and they showed if $T(M) \neq M$ then $T(M)$ is a union of prime sub-modules of M.

In section 2, we compute diameter and girth of $G_R(M)$ and in section 3, we discuss planarity of $G_R(M)$.

2. Diameter and Girth of $G_R(M)$

Remark 2.1. *Let M be an R-module and $x \in M$, we denote set of vertices that is adjacent to x in $G_R(M)$ by $T_x(M) = \{m \in M : rm = x \ for \ some \ r \in R\}$. The torsion element of M is $T_0(M)$. The $T_x(M)$ is set of neighbours of x or $N(x)$. Note that if $G_R(M)$ is a Scalar product graph of R-module M, then $x, y \in M$ is adjacent if and only if $x \in T_y(M)$ or $y \in T_x(M)$.*

Remark 2.2. *Let M be a finite R-module and $G_R(M)$ a scalar product graph of M. If M is torsion then for every $m \in M$, vertex m is adjacent to 0 and $deg(0) = |M| - 1$. Also, $diam(G_R(M)) \leq 2$. Also, If M is torsion-free then 0 is isolated vertex.*

Proposition 2.3. *Let R be a division ring and M an R-module. If a is adjacent to b in $G_R(M)$, then $N(a) = N(b)$.*

Proof . Assume that a and b are two adjacent vertices of $G_R(M)$. Then $a \in Rb$ or $b \in Ra$. Hence since R is a division ring, we have $Ra = Rb$. First suppose that $x \in N(a)$. Then $x \in Ra$ or $a \in Rx$ hence $x \in Rb$ or $a \in Rx$, Therefore $x \in N(b)$. So $N(a) \subseteq N(b)$. Next if $x \in N(b)$, then $x \in Rb$ or $b \in Rx$. Hence $x \in Ra$ or $b \in Rx$ therefore $x \in N(a)$ so $N(b) \subseteq N(a)$. Thus $N(a) = N(b)$. □

Example 2.4.
- *Let M be a free R-module, then one can see that M is torsion-free, thus 0 is isolated vertex. Also, if V is vector space over field K then V is torsion-free, therefore 0 is isolated vertex.*

- *$\mathbb{Q}$ is torsion-free $\mathbb{Z}$-module. Therefore 0 is isolated vertex.*
- *If R is a integral domain and Q its field of fractions, then $\frac{Q}{R}$ is a torsion R-module. Therefore $diam(G_R(\frac{Q}{R})) \leq 2$.*
- *Consider a linear operator L acting on a finite-dimensional vector space V. If we view V as an $F[L]$-module in the natural way, then, V is a torsion $F[L]$-module. Then $T(V) = V$ as a result by previous proposition we have $deg(0) = |V| - 1$ and $diam(G_{F[L]}(V)) \leq 2$.*

Remark 2.5. *Let $G_R(M)$ be a Scalar product graph of R-module M. If $x, y \in M$ then x is adjacent to y if and only if $< x > \subseteq < y >$ or $< y > \subseteq < x >$ or $Rx \subseteq Ry$ or $Ry \subseteq Rx$.*

Lemma 2.6. *Let M be an R-module and $x, y \in M$. If $< x > = < y >$, then x is adjacent to y in $G_R(M)$ and for all $z \in M$, x is adjacent to z if and only if y is adjacent to z.*

Proof . Suppose $< x > = < y >$ then $< x > \subseteq < y >$. So x is adjacent to y. If z is adjacent to x, then $< z > \subseteq < x >$ or $< x > \subseteq < z >$. Hence $< z > \subseteq < y >$ or $< y > \subseteq < z >$. So z is adjacent to y. Similarly, if y is adjacent to z then x is adjacent to z. □

This concludes that any two vertices that generate the same submodules will have exactly the same set of neighbours.

Corollary 2.7. *Let M be an R-module and $x, y \in M$. If cyclic submodules Rx, Ry are maximal, Then x is not adjacent to y in $G_R(M)$.*

Proof . Suppose x is adjacent to y in $G_R(M)$. Without loss of generality suppose that $Rx \subseteq Ry$ which is contradiction by maximality of Rx. □

Theorem 2.8. *Let $M = M_1 \times M_2 \times \cdots \times M_n$ where M_i is a module $1 \leq i \leq n$. Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in M$, If x_i is not adjacent to y_i in $G_R(M_i)$ for some $i \in \{1, \ldots, n\}$, Then x is not adjacent to y in $G_R(M)$.*

Proof . Suppose x is adjacent to y in $G_R(M)$. Then without loss of generality $x \in Ry$, There exist $z \in R$ such that $zy = x$ or $(z_1y_1, z_2y_2, \ldots, z_ny_n) = (x_1, x_2 \ldots, x_n)$ and for all $i \in \{1, \ldots, n\}$ we have $x_i = z_iy_i$ and hence x_i is adjacent to y_i in $G_R(M_i)$. □

The converse of theorem 2.8 does not hold. Let $M = \mathbb{Z}_{16} \times \mathbb{Z}_{16}$, $R = \mathbb{Z}$. In $G_{\mathbb{Z}}(\mathbb{Z}_{16} \times \mathbb{Z}_{16})$ vertex $(2, 4)$ is not adjacent to vertex $(4, 2)$, but 2 is adjacent to 4 in $G(\mathbb{Z}_{16})$.

We know that any abelian group is a $\mathbb{Z}$-module. If G is a $\mathbb{Z}$-module and $x, y \in G$ then according to definition of scalar product on G, x is adjacent to y if there exist $n \in \mathbb{Z}$ which $x = ny$ or $y = nx$.

Example 2.9. *Let $M = \mathbb{Z}_6$ be $\mathbb{Z}$-module. Scalar product $G_{\mathbb{Z}}(\mathbb{Z}_6)$ have shown in Fig 1.*

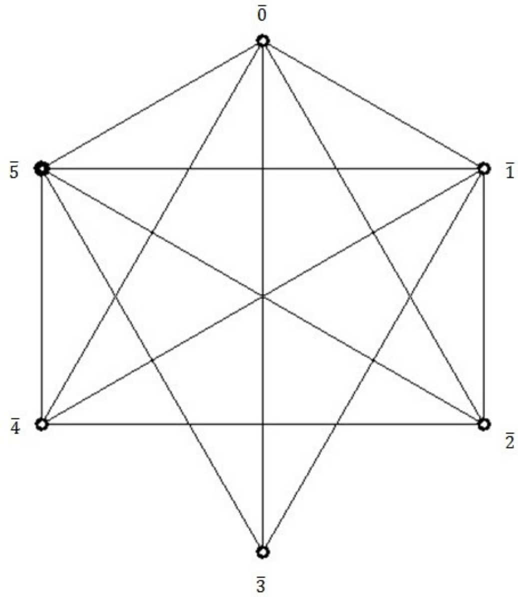

Fig 1. Scalar Product of $\mathbb{Z}$-module $\mathbb{Z}_6$

Proposition 2.10. *Let $\mathbb{Z}_n$ be $\mathbb{Z}$-module. If p, m are prime and positive integer number, then for $n = 1, p, p^m$, Scalar product graph $G_{\mathbb{Z}}(\mathbb{Z}_n)$ is complete.*

Theorem 2.11. *Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_n$ be a $\mathbb{Z}$-module. Then the number of edges e of $G_R(M)$ is given by $2e = \sum_{d|n}\{2d - \phi(d) - 1\}\phi(d)$.*

Proof . In the directed scalar product graph $\overrightarrow{G_R(M)}$, vertex a is adjacent to b if there exist $r \in R$ such that $b = ra$. Therefore, for any vertex $a \in M$, the out-degree of a is $|\{b \in M : b \in Ra, b \neq a\}| = |Ra| - 1$. Also, that the number of arcs in a directed graph is the sum of out-degrees of all the vertices of the graph. Thus the number of arcs of $\overrightarrow{G_R(M)}$ is $\sum_{a\in M} |Ra| - 1$. To counting number of edges in the undirected scalar product graph $G_R(M)$, we have to count the bi-directed arcs only once. The bi-directed arcs occur for some $b \in M, (b \neq a)$ such that, $a \in Rb$ and $b \in Ra$. □

Proposition 2.12. *Let M be an R-module and N submodule of M. Then $G_R(N)$ is an induced subgraph of $G_R(M)$.*

Proof . As $N \subseteq M$, $V(G_R(N)) = N \subseteq M = V(G_R(M))$. Also from the definition of the scalar product graph, it follows that for any $a, b \in N$, a and b are adjacent in $G_R(N)$ if and only if they are adjacent in $G_R(M)$. Thus $G_R(N)$ is an induced subgraph of $G_R(M)$. □

Lemma 2.13. *Let $f : M_1 \longrightarrow M_2$ be a R-module homomorphism. We have:*

1. If vertices x and y are adjacent in $G_R(M_1)$ then $f(x)$ and $f(y)$ are adjacent in $G_R(M_2)$.
2. If $G_R(M_1)$ is complete then $G_R(f(M_1))$ is complete.

Proof .

1. Let x and y be adjacent in $G_R(M_1)$. By definition there exists $r \in R$ that $x = ry$ or $y = rx$ then $f(x) = f(ry) = rf(y)$ or $f(y) = f(rx) = rf(x)$. Therefore $f(x)$ is adjacent $f(y)$.

2. Let $y_1, y_2 \in f(M_1)$ be two arbitrary vertices in scalar product graph $G_R(f(M_1))$. Then there exist $x_1, x_2 \in M_1$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since the $G_R(M_1)$ is complete x_1 is adjacent x_2. From 1. y_1 is adjacent y_2. Therefore scalar product graph $G_R(f(M_1))$ is complete.

□

Theorem 2.14. *Let M be a R-module. Then scalar product graph $G_R(M)$ is complete if and only if the cyclic submodules of M are linearly ordered by inclusion relation.*

Proof . Let M be a R-module and $N_1 =< a >, N_2 =< b >$ be two cyclic submodules of M that $a \neq b$ in M. Since scalar product graph $G_R(M)$ is complete then a and b is adjacent. We have $< a >\subseteq< b >$ or $< b >\subseteq< a >$ and $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$. Conversely, Let M be R-module which linearly ordered cyclic submodules by inclusion relation. If $a \neq b$ is two vertices of $G_R(M)$ then $< a >\subseteq< b >$ or $< b >\subseteq< a >$. Therefore we have a and b are adjacent in $G_R(M)$. Hence $G_R(M)$ is complete. □

Corollary 2.15. *Let R be a ring and M a finite R-module. If $G_R(M)$ is complete then M is a cyclic R-module.*

Recall that an R-module M is called uniserial if its submodules are linearly ordered by inclusion. Evidently, a valuation ring R is uniserial as a module over itself, and its ring of quotients is likewise a uniserial R-module. It is obvious that submodule and quotients of uniserial modules are again uniserial. As an example, we see that $\mathbb{Z}_4$ is a uniserial $\mathbb{Z}$-module. A right R-module is called a serial module if it is a direct sum of uniserial modules. Note that every uniserial module is serial but serial modules need not be uniserial.

Lemma 2.16. *If M is a R-module, then M is uniserial if and only if the cyclic submodules of M are linearly ordered.*

Proof . According to the definition one side is obvious. Conversely, Let K, L be submodules of M with $K \not\subset L$ and $L \not\subset K$. Choosing $x \in K \setminus L$, $y \in L \setminus K$ we have, $Rx \subset Ry$ or $Ry \subset Rx$. In the first case we have $x \in Ry \subset L$, in the second case $y \in Rx \subset K$. Both are contradiction. □

Corollary 2.17. *If M is an R-module, then the scalar product graph $G_R(M)$ is complete if and only if M is uniserial.*

By this corollary, it's obvious that scalar product graph of uniserial module is complete . So we give some examples of uniserial module and their complete scalar product graph.

Example 2.18.

- *For any prime number p, any cyclic p-group or the quasi-cyclic p-group $C(p^\infty)$ is a uniserial $\mathbb{Z}$-module. So $G_{\mathbb{Z}}(C(p^\infty))$ is complete graph.*
- *Every Simple module is uniserial. So $\mathbb{Z}_p$ is a uniserial $\mathbb{Z}$-module and its scalar product graph is complete.*
- *Every function having finite length is uniserial. So $F[x, y] = \{x^3, x^2y, y^3\}$ is uniserial since its length is 3.*
- *Every semisimple module is serial.*
- $\mathbb{Z}_{p^n} = \frac{1\mathbb{Z}}{p^n\mathbb{Z}} \supset \frac{p\mathbb{Z}}{p^n\mathbb{Z}} \supset \frac{p^2\mathbb{Z}}{p^n\mathbb{Z}} \supset \ldots \supset \frac{p^{n-1}\mathbb{Z}}{p^n\mathbb{Z}} \supset \frac{p^n\mathbb{Z}}{p^n\mathbb{Z}} = 0$, *here $\mathbb{Z}_{p^n}$ is uniserial. So its scalar product graph is complete.*

- *Also $\mathbb{Z}_{p^\infty}$, the $\mathbb{Z}$-injective hull of $\frac{\mathbb{Z}}{p\mathbb{Z}}$, p a prime number, is uniserial. So we have , $\mathbb{Z}_{p^\infty}$ is artinian and uniserial, but not noetherian (not finitely generated).*

Corollary 2.19. *Let M be an R-module, Then $G_R(M)$ is complete, if each of the following condition holds:*

(a) M is uniserial;

(b) the cyclic submodules of M are linearly ordered;

(c) any submodule of N has at most one maximal submodule;

(d) for any finitely generated submodule $0 \neq K \subset N$, $\frac{K}{Rad(K)}$ is simple;

(e) for every factor module L of N, $SocL$ is simple or zero.

Proof . According to previous corollary, if (a) is true, then $G_R(M)$ is complete. Equivalency of next expression to (a) will be discussed:
$(a) \Rightarrow (b)$ is obvious.
$(b) \Rightarrow (a)$ Let K, L be submodules of N with $K \not\subset L$ and $L \not\subset K$. Choosing $x \in K \setminus L$, $y \in L \setminus K$ we have, by (b), $Rx \subset Ry$ or $Ry \subset Rx$. In the first case we conclude $x \in Ry \subset L$, in the second case $y \in Rx \subset K$. Both are contradictions.
$(a) \Rightarrow (c)$ and $(a) \Rightarrow (b) \Rightarrow (e)$ are obvious.
$(d) \Rightarrow (b)$ Let us assume that we can find two cyclic submodules K, $L \subset N$ with $K \not\subset L$ and $L \not\subset K$. Then: $(K + L)/(K \cap L) \simeq K/(K \cap L) \bigoplus L/(K \cap L)$,
and the factor of $(K + L)/(K \cap L)$ by its radical contains at least two simple summands. Therefore the factor of $K + L$ by its radical also contains at least two simple summands. This contradicts (d).
$(e) \Rightarrow (d)$ We show that every non-zero finitely generated submodule $K \subset N$ contains only one maximal submodule: If $V_1, V_2 \subset K$ are different maximal submodules, then $K/(V_1 \cap V_2) \simeq K/V_1 \bigoplus K/V_2$ is contained in the socle of $N/(V_1 \cap V_2)$. This is a contradiction to (e). □

Observation. According to definition of cozero-divisor graph over modules we have the followings:
(1) If M is an R-module, the subgraph of $G_R(M)$ which vertices are $W_R(M)^*$ is complement of cozero-divisors graph of M.
(2) We denote $G_R(M) = \Gamma_1 \vee \Gamma_2$ where Γ_1 is a complete graph with $\mid W_R(M)^* \mid$ vertices and Γ_2 is complement of cozero-divisor graph of M.

3. Planarity

A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph G is a graph resulting from the subdivision of edges in G. The subdivision of some edge e with endpoints $\{u, v\}$ yields a graph containing one new vertex w, and with an edge set replacing e by two new edges, $\{u, w\}$ and $\{w, v\}$. Kuratowski's theorem is a forbidden graph characterization of planar graphs given by Kazimierz Kuratowski in 1930.

Theorem 3.1. *If G is a finite graph, then G is is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$, where K_n is a complete graph with n vertices and $K_{m,n}$ is a complete bipartite graph, for positive integers m, n.*

In this section we discuss planarity of scalar product graph of a module.

Proposition 3.2. *Let M a finite R-module. If $\mid W_R(M)^* \mid \geq 5$, then $G_R(M)$ is not planar.*

Proof . If $\mid W_R(M)^* \mid \geq 5$ then subgraph of $G_R(M)$ which vertices are $W_R(M)^*$ is complete. By Kuratowski's Theorem, we have $G_R(M)$ is not planar. □

Proposition 3.3. *Let M be a Noetherian R-module. If $G_R(M)$ has an clique, then M has a cyclic submodule which contains all vertices of the clique.*

Proof . Let K be an clique in $G_R(M)$ and x_1 be a vertex of K. Assume to the contrary that there is no cyclic submodule in M that contains all vertices of K. Since the cyclic submodule x_1R doesn't contain all vertices of K, there exists a vertex x_2 in K such that $x_2 \notin x_1R$. As x_1 and x_2 are in one clique and are adjacent and $x_2 \notin x_1R$, we have $x_1 \in x_2R$. Therefore, $x_1R \subsetneqq x_2R$. Again since the cyclic submodule x_2R doesn't contain all vertices of K, there exists a vertex x_3 in K such that $x_3 \notin x_2R$. Also, x_2 and x_3 are adjacent. This implies that $x_2 \in x_3R$ and so $x_2R \subsetneqq x_3R$. By continuing this method, we find an increasing sequence of cyclic submodule of M which doesn't stop and this is a contradiction. □

Lemma 3.4. *Let M be a R-module. Assume $x_1 - x_2 - \ldots - x_n$ is a cycle in $G_R(M)$ such that the subgraph induced by vertices $x_1, x_2, \ldots, x_n$ contains no cycle with smaller length. If $Rx_1 \subseteq Rx_n$ then we have $Rx_{2k-1} \subseteq Rx_{2k}$ and $Rx_{2k+1} \subseteq Rx_{2k}$ for $k = 1, \ldots, n$*

Proof . By our assumption, x_2 is not adjacent to x_n thus we have $Rx_2 \nsubseteq Rx_n$ and $Rx_n \nsubseteq Rx_2$. Since x_1 is adjacent to x_2 hence $Rx_1 \subseteq Rx_2$ or $Rx_2 \subseteq Rx_1$. If $Rx_2 \subseteq Rx_1$, by assumption since $Rx_1 \subseteq Rx_n$ then $Rx_2 \subseteq Rx_n$ which is contradiction. Therefore $Rx_1 \subseteq Rx_2$.
Also, x_3 is not adjacent to x_n thus we have $Rx_3 \nsubseteq Rx_n$ and $Rx_n \nsubseteq Rx_3$. Since x_2 is adjacent to x_3 hence $Rx_2 \subseteq Rx_3$ or $Rx_3 \subseteq Rx_2$. If $Rx_2 \subseteq Rx_3$, since $Rx_1 \subseteq Rx_2$ then $Rx_1 \subseteq Rx_3$ which is contradiction. Therefore $Rx_3 \subseteq Rx_2$.
Also, x_4 is not adjacent to x_n thus we have $Rx_4 \nsubseteq Rx_n$ and $Rx_n \nsubseteq Rx_4$. Since x_3 is adjacent to x_4 hence $Rx_3 \subseteq Rx_4$ or $Rx_4 \subseteq Rx_3$. If $Rx_4 \subseteq Rx_3$, since $Rx_3 \subseteq Rx_2$ then $Rx_4 \subseteq Rx_2$ which is contradiction. Therefore $Rx_3 \subseteq Rx_4$.
by similar method we have:$Rx_1 \subseteq Rx_2$, $Rx_3 \subseteq Rx_2$, $Rx_3 \subseteq Rx_4$, $Rx_5 \subseteq Rx_4$, $Rx_5 \subseteq Rx_6$, $Rx_7 \subseteq Rx_6$,... which complete the proof. □

4. Weakly Perfect

For a graph G, a k-colouring of the vertices of G is an assignment of k colors to the vertices of G in such a way that no two adjacent vertices receive the same color. The chromatic number of G, denoted by $\chi(G)$, is the smallest number k such that G admits a k-coloring. A clique of G is a complete sub-graph of G and the number of vertices in a largest clique of G, denoted by $\omega(G)$, is called the clique number of G. It is easy to see that $\chi(G) \geq \omega(G)$, because every vertex of a clique should get a different color. A graph G is called weakly perfect if $\chi(G) = \omega(G)$. If $M = \mathbb{Z}_n$ be an finite $\mathbb{Z}$-module, then $G_{\mathbb{Z}}(M)$ is weakly perfect.

Example 4.1. *Chromatic number and clique number of $G_{\mathbb{Z}}(\mathbb{Z}_n)$ for some n is listed in below (p is prime number):*

n	$\chi(G_{\mathbb{Z}}(\mathbb{Z}_n))$	$\omega(G_{\mathbb{Z}}(\mathbb{Z}_n))$
$n = 1$	1	1
$n = p$	p	p
$n = p^n$	p^n	p^n
$n = 2p$	$2p - 1$	$2p - 1$
$n = 3p$	$3p - 2$	$3p - 2$

Table 1: Clique number, Chromatic of $G_{\mathbb{Z}}(\mathbb{Z}_n)$

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